Determination of a coefficient and kernel in a d-dimensional fractional integrodifferential equation

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Abstract. This paper is devoted to obtaining a unique solution to an inverse problem for a multi-dimensional time-fractional integrodifferential equation. In the case of additional data, we consider an inverse problem. The unknown coefficient and kernel are uniquely determined by the additional data. By using the fixed point theorem in suitable Sobolev spaces, the global in time existence and uniqueness results of this inverse problem are obtained.

Keywords: fractional wave equation; Caputo fractional derivative; Fourier method; Mittag-Leffler function; Bessel inequality.
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1 Introduction and setting up the problem

Fractional calculus plays an important role in mathematical modeling in many scientific and engineering disciplines. They are used in the modeling of many physical and chemical processes and engineering (see, e.g., [1]-[7]). A fractional integrodifferential equation can be used to simulate a wide range of problems in the basic sciences, many scientists have focused their attention on presenting the solutions for these systems. That equation has played a significant role in finding solutions using diverse methods, which is in line with the rapid development in finding the answers to diverse problems originating from the basic sciences. The linear/nonlinear equations fractional integrodifferential equation has various uses in fluid mechanics [8], Stokes flow [9], airfoil [10], quantum mechanics [11], integral models [12], mathematical engineering [13], nuclear physics [14] and the theory of laser [15].

Other studies [16]-[21] demonstrate several interesting features of the fractional diffusion-wave equations, which represent a peculiar union of properties typical for second-order parabolic and wave differential equations. Fractional evolution inclusions are an important form of differential inclusions within nonlinear mathematical analysis. They are generalizations of the much more widely developed fractional evolution equations (such as time-fractional diffusion equations) seen through the lens of multivariate analysis. Compared to fractional evolution equations, research on the theory of fractional differential inclusions is however only in its initial stage of development. This is important because differential models with the fractional derivative provide an excellent instrument for the description of memory and hereditary properties, and have recently been proven valuable tools in the modeling of many physical phenomena (see, [22] and the references therein).

According to the fractional order α, the diffusion process can be specified as sub-diffusion ($\alpha \in (0,1)$) and super-diffusion ($\alpha \in (1,2)$), respectively. There is abundant literature on the studies of fractional equations on various aspects, such as physical backgrounds, weak solutions, and maximum principle and numerical methods (see, [23] and the references therein).

Practical needs often lead to problems in determining the coefficients, kernel, or the right-hand side of a differential equation from certain known information about its solution. Such problems have received the name inverse problems of mathematical physics. Inverse problems arise in various domains of human activity, such as seismology, prospecting for mineral deposits, biology, medical visualization, computer-aided tomography, the remote sounding of Earth, spectral analysis, nondestructive control, etc., (see [24]-[26]). In this paper, we discuss an inverse problem of determining a source term only depending on the time in a fractional-differential equation by the measurement data of time trace at a fixed point $\mathbf{x}_i$.

Let $Q_0^T := \Omega \times (0,T)$ for a given time $T > 0$, where Ω be a bounded domain in $\mathbf{R}^d$ with sufficiently smooth boundary $\partial\Omega$ and $\Sigma_0^T = \partial\Omega \times (0,T)$. We consider a fractional integrodifferential equation with a fractional derivative in time t:

$$\partial_t^\alpha u(\mathbf{x},t) + Au(\mathbf{x},t) = q(t)u_t(\mathbf{x},t) + k * u(\mathbf{x},t) + f(\mathbf{x},t), \quad (\mathbf{x},t) \in Q_0^T, \tag{1.1}$$

where $1 < \alpha < 2$ and $\partial_t^\alpha u(\mathbf{x}, t)$ is the left Caputo fractional derivative with respect to t and defined by [33]

$$\partial_t^\alpha v(t) = \begin{cases} \frac{1}{\Gamma(d-\alpha)} \int_0^t (t-\tau)^{d-\alpha-1} v^{(d)}(\tau)\, d\tau, & d-1 < \alpha < d,\ d \in \mathbf{N}, \\ v^{(d)}(t), & \alpha = d \in \mathbf{N}, \end{cases}$$

$\Gamma(\cdot)$ is the Gamma function and the operator A is a symmetric uniformly elliptic operator defined on $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ given by

$$Av(\mathbf{x}, t) \equiv -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(a_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} v(\mathbf{x}, t) \right) + c(\mathbf{x}) v(\mathbf{x}, t), \quad (\mathbf{x}, t) \in Q_0^T,$$

in which the coefficient satisfy

$$a_{ij} = a_{ji} \in C^1(\Omega), \quad c \in C(\bar{\Omega}), \quad c(\mathbf{x}) \geq 0, \quad x \in \bar{\Omega}$$

and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x}) \xi_i \bar{\xi}_j \geq \mu \sum_{i=1}^{d} |\xi_i|^2, \quad \text{for all} \quad \mathbf{x} \in \bar{\Omega}, \quad \xi \in \mathbf{R}^d,$$

and Laplace convolution

$$f * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau.$$

We supplement the above fractional wave equation with the following initial conditions:

$$u(\mathbf{x}, 0) = a(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = b(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{1.2}$$

and the zero boundary condition:

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \Sigma_0^T. \tag{1.3}$$

If $q(t)$, $k(t)$, $f(\mathbf{x}, t)$, $a(\mathbf{x})$ and $b(\mathbf{x})$ are known, then problem (1.1)-(1.3) is called a direct problem. The inverse problem in this paper is to reconstruct $q(t)$ and $k(t)$ according to the additional data

$$u(\mathbf{x}_i, t) = h_i(t), \quad t \in (0, T) \tag{1.4}$$

where $h_i(t)$, $i = 1, 2$ are given functions and $\mathbf{x}_i \in \Omega\, (i = 1, 2)$ are given numbers.

We investigate the following inverse problem.

Inverse problem. Find $u \in C\left([0, T]; \mathcal{D}(A^{\gamma + \frac{1}{\alpha}})\right) \cap C^1([0, T]; \mathcal{D}(A^\gamma))$, $q \in C^1[0, T]$ and $k \in C[0, T]$ to satisfy (1.1)-(1.3) and the additional measurement (1.4), where $\mathcal{D}(A^\gamma)$ is a Hilbert space with some positive constant γ, see (1.6).

For the convenience of the reader, we present here the necessary definitions from functional analysis and fractional calculus theory.

For integers m, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ (see [28]) and $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of space $H^m(\Omega)$. For a given Banach space V on (Ω), we use the notation $C^m([0, T]; V)$ to denote the following space:

$$C^m([0, T]; V) := \left\{ u : \|\partial_t^j u(t)\|_V \text{ is continuous in } t \text{ on } [0, T] \text{ for all } 0 \leq j \leq m \right\}.$$

We endow $C^m([0, T]; V)$ with the following norm making it to be a Banach space:

$$\|u\|_{C^m([0,T];V)} = \sum_{j=0}^{m} \left(\max_{0 \leq t \leq T} \|\partial_t^j u(t)\|_V \right).$$

In addition, we define Banach space X_0^T by

$$X_0^T := C([0, T]; \mathcal{D}(A^{\gamma + \frac{1}{\alpha}})) \cap C^1([0, T]; \mathcal{D}(A^\gamma)).$$

Furthermore, we set

$$Y_0^T = X_0^T \times C^1[0,T] \times C[0,T]$$

endowed with the norm

$$\|(u,q,k)\|_{Y_0^T} := \|u\|_{X_0^T} + \|q\|_{C^1[0,T]} + \|k\|_{C[0,T]}.$$

It is well-known that the operator A is a symmetric uniformly elliptic operator, the spectrum of A is entirely composed of eigenvalues, and counting according to the multiplicities, we can set: $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lim_{n\to\infty} \lambda_n = \infty$. By $e_n \in H^2(\Omega) \cap H_0^1(\Omega)$, we denote the orthonormal eigenfunction corresponding to λ_n :

$$\begin{cases} Ae_n = \lambda_n e_n, & \text{in } \ \Omega, \\ e_n = 0, & \text{on } \partial\Omega. \end{cases}$$

It is well known that, if the coefficients $a_{ij}(\mathbf{x})$, $c(\mathbf{x})$ are real-valued functions and $a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}) \in L^\infty(\Omega)$, $c(\mathbf{x}) \in L^\infty(\Omega)$, then the eigenfunction sequence $\{e_n\}_{n\in\mathbf{N}}$ is a orthonormal basis in $L^2(\Omega)$. Then for $\gamma \in \mathbf{R}$ we define a Hilbert space $\mathcal{D}(A^\gamma)$ by

$$\mathcal{D}(A^\gamma) := \Big\{ u \in L^2(\Omega) : \sum_{n=1}^\infty \lambda_n^{2\gamma} |(u,e_n)|^2 < \infty \Big\}, \quad A^\gamma u = \sum_{n=1}^\infty \lambda_n^\gamma (u,e_n) e_n$$

equipped with the norm

$$\|u\|_{\mathcal{D}(A^\gamma)} = \left(\sum_{n=1}^\infty \lambda_n^{2\gamma} |(u,e_n)|^2 \right)^{1/2}.$$

We note that the norm $\|u\|_{\mathcal{D}(A^\gamma)}$ is stronger than $\|u\|_{L^2(\Omega)}$ for $\gamma \ge 0$. Since $\mathcal{D}(A^\gamma) \subset L^2(\Omega)$ for $\gamma \ge 0$, identifying the dual of $L^2(\Omega)$ with itself, we have $\mathcal{D}(A^\gamma) \subset L^2(\Omega) \subset (\mathcal{D}(A^\gamma))'$ and $\mathcal{D}(A^{-\gamma}) = (\mathcal{D}(A^\gamma))'$, which consists of bounded linear functionals on $\mathcal{D}(A^\gamma)$. For $u \in \mathcal{D}(A^{-\gamma})$ and $\varphi \in \mathcal{D}(A^\gamma)$, the value obtained by operating u to φ is denoted by ${}_{-\gamma}\langle \cdot,\cdot \rangle_\gamma$. $\mathcal{D}(A^{-\gamma})$ is a Hilbert space with the norm:

$$\|\varphi\|_{\mathcal{D}(A^{-\gamma})} = \left(\sum_{n=1}^\infty \lambda_n^{-2\gamma} |\, {}_{-\gamma}\langle u, e_n\rangle_\gamma \,|^2 \right)^{\frac{1}{2}}.$$

We further note that

$${}_{-\gamma}\langle u,\varphi\rangle_\gamma = (u,\varphi) \quad \text{if } u \in L^2(\Omega) \text{ and } \varphi \in \mathcal{D}(A^\gamma)$$

(see e.g., Chapter V in [30]).

Moreover, we introduce the Mittag-Leffler function in [33]:

$$E_{\rho,\mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z \in \mathbf{C}$$

with $\mathrm{Re}(\rho) > 0$ and $\mu \in \mathbf{C}$. It is known that $E_{\rho,\mu}(z)$ is an entire function in $z \in \mathbf{C}$.

Lemma 1.1. *Let $0 < \rho < 2$ and $\mu \in \mathbf{R}$ be arbitrary and θ satisfy $\frac{\pi\rho}{2} < \theta < \min\{\pi, \pi\rho\}$. Then there exists a constant $c = c(\rho,\mu,\theta) > 0$ such that*

$$|E_{\rho,\mu}(z)| \le \frac{c_1}{1+|z|}, \qquad \theta \le |arg(z)| \le \pi,$$

and the asymptotic behavior of $E_{\rho,\mu}(z)$ at infinity as follows

$$E_{\rho,\mu}(z) = -\sum_{n=1}^N \frac{z^{-n}}{\Gamma(\mu - \rho n)} + O(z^{-n-1}).$$

For the proof, we refer to [31] for example.

Proposition 1.1. (see [33]) *For $\lambda > 0$, $\alpha > 0$ and positive integer $m \in \mathbf{N}$, we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0$$

and

$$\frac{d}{dt}\left(t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\right) = t^{\beta-2}E_{\alpha,\beta-1}(-\lambda t^{\alpha}), \quad \partial_t^{\alpha}(E_{\alpha,1}(-\lambda t^{\alpha})) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), \quad t \geq 0.$$

Also, we mention

$$\max_{y\geq 0} \frac{y^{\theta}}{1+y} = \frac{\left(\frac{\theta}{1-\theta}\right)^{\theta}}{1+\frac{\theta}{1-\theta}}, \quad 0<\theta<1. \tag{1.5}$$

We now give a similar definition of weak solution to (1.1)-(1.3), which is introduced by [32].

Definition 1.1. *We call u a weak solution to (1.1)-(1.3) if (1.1) holds in $L^2(\Omega)$ and $u(\cdot,t) \in H_0^1(\Omega)$ for almost all $t \in (0,T)$, $u, \partial_t u \in C([0,T];\mathcal{D}(A^{-\gamma}))$ and*

$$\lim_{t\to 0} \|u(\cdot,t)-a\|_{\mathcal{D}(A^{-\gamma})} = \lim_{t\to 0} \|u_t(\cdot,t)-b\|_{\mathcal{D}(A^{-\gamma})} = 0$$

with some $\gamma > 0$. Here $\gamma > 0$ may depend on a, b.

Throughout this paper, we set $\gamma_0 > \frac{d}{2}+1$, $\gamma > 0$ and $\frac{1}{\alpha} < \varepsilon < 1$ such that

$$\max\left\{\frac{d}{4}+1, \gamma_0 + \frac{1}{\alpha} - \varepsilon\right\} < \gamma \leq \gamma_0. \tag{1.6}$$

We make the following assumptions:
(C1) $\partial_t^{\alpha} h_i \in C^1[0,T]$, $a \in \mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})$, $b \in \mathcal{D}(A^{\gamma_0})$, $f \in C^1([0,T];\mathcal{D}(A^{\gamma}))$;
(C2) $h_i'(0)q(0) = \partial_t^{\alpha} h_i(0) + Aa(\mathbf{x}_i) - \tilde{f}_i(0)$, where $\tilde{f}_i(t) = f(x_i,t),\ i=1,2$;
(C3) $a(\mathbf{x}_i) = h_i(0),\ b(\mathbf{x}_i) = h_i'(0),\ i=1,2$;
(C4) $p(t) = h_1'(t)h_2(0) - h_2'(t)h_1(0) \neq 0$ and $p(t) \in C^1[0,T]$ satisfies the following inequality:

$$\|p\|_{C^1[0,T]} \geq \frac{1}{p_0} > 0,$$

where p_0 is a given positive constant.

Remark 1.1. By the Sobolev embedding theorem, if $\partial_t^{\alpha} h \in C^1[0,T]$ implies $h_i \in W^{2,1}(0,T) \hookrightarrow H^1(0,T)$ (see [26]) and from this we will be used in Lemma 2.7 below.

Remark 1.2. (C3) is the consistency condition for our problem (1.1)-(1.4), which guarantees that the inverse problem (1.1)-(1.4) is equivalent to (2.30) and (2.32) (see Lemma 3.3).

Our main result in this paper is the following global existence and uniqueness of our inverse problem.

Theorem 1.1. *Let (C1)-(C4) hold. Then, there exists a unique solution $(u,q,k) \in Y_0^T$ of the inverse problem (1.1)-(1.4) for any $T > 0$.*

The outline of the paper is as follows. In Section 2, we give preliminary results in this paper, including the existence and uniqueness of the direct problem (1.1)-(1.3), and also an equivalent problem is presented. In Section 3, the local existence and global uniqueness of the solution of the inverse problem (1.1)-(1.4) is established by using the Fourier method and Banach fixed point theorem. Section 4 contains the proof of Theorem 1.1 (existence global in time). In Section 5, we give an example of the inverse problem (1.1)-(1.4).

2 Preliminary results

This section presents some preliminary results, including the well-posedness for a fractional differential equation, an equivalent lemma for our inverse problem, and a technique result, which will be used to prove our main results.

Let

$$Z_2(t)\eta(\mathbf{x}) = \sum_{n=1}^{\infty}(\eta, e_n)tE_{\alpha,2}(-\lambda_n t^{\alpha})e_n(\mathbf{x}), \quad (\mathbf{x},t) \in Q_0^T$$

for $\eta \in L^2(\Omega)$.

We first consider the following initial and boundary problems:

$$\begin{cases} \partial_t^\alpha u(\mathbf{x},t) + Au(\mathbf{x},t) = F(\mathbf{x},t), & (\mathbf{x},t) \in Q_0^T, \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Sigma_0^T, \\ u(\mathbf{x},0) = a(\mathbf{x}),\ u_t(\mathbf{x},0) = b(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{2.1}$$

Note that if $\alpha = 1$ and $\alpha = 2$, then Eq. (2.1) represents a parabolic equation and a hyperbolic equation respectively. Since we are interested mainly in the fractional cases, we restrict the order α to $1 < \alpha < 2$.

First split (2.1) into the following two initial and boundary value problems:

$$\begin{cases} \partial_t^\alpha v(\mathbf{x},t) + Av(\mathbf{x},t) = 0, & (\mathbf{x},t) \in Q_0^T, \\ v(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Sigma_0^T, \\ v(\mathbf{x},0) = a(\mathbf{x}),\ v_t(\mathbf{x},0) = b(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial_t^\alpha w(\mathbf{x},t) + Aw(\mathbf{x},t) = F(x,t), & (\mathbf{x},t) \in Q_0^T, \\ w(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Sigma_0^T, \\ w(\mathbf{x},0) = 0,\ w_t(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{2.3}$$

Similarly to Theorem 2.3 in [32], it is easy to obtain the following assertion:

Lemma 2.1. *Let $a \in H^2(\Omega) \cap H_0^1(\Omega)$ and $b \in H_0^1(\Omega)$. Let $\gamma > 0$. Then for the unique weak solution $v \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; \mathcal{D}(A^{-\gamma}))$ to (2.2), there exists a constant $c > 0$ satisfying*

$$\|v(\cdot,t)\|_{H^2(\Omega)} + \|v_t(\cdot,t)\|_{\mathcal{D}(A^{-\gamma})} \le c\left(\|a\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)}\right), \tag{2.4}$$

Then we have

$$\begin{cases} v(x,t) = Z_1(t)a(x) + Z_2(t)b(x), & (x,t) \in Q_0^T, \\ v_t(x,t) = -Y(t)a(x) + Z_1(t)b(x), & (x,t) \in Q_0^T \end{cases} \tag{2.5}$$

using

$$Z_1(t)\eta(\mathbf{x}) = \sum_{n=1}^{\infty} (\eta, e_n) E_{\alpha,1}(-\lambda_n t^\alpha) e_n(\mathbf{x}), \quad Y(t)\eta(x) = \sum_{n=1}^{\infty} \lambda_n (\eta, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) e_n(x),$$

the space in $C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; \mathcal{D}(A^{-\gamma}))$.

Proof. The uniqueness of a weak solution is verified similarly to Theorem 2.1 in [32], but smoothness is taken in a different form. Therefore, here we show only (2.4) inequality.

Using the Lemma 1.1 and (1.4), we have

$$\|v(\cdot,t)\|_{H^2(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n^2 \left|(a,e_n) E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2 + \sum_{n=1}^{\infty} \lambda_n^2 \left|(b,e_n) t E_{\alpha,2}(-\lambda_n t^\alpha)\right|^2$$

$$\le c^2 \|a\|_{H^2(\Omega)}^2 + c^2 \sum_{n=1}^{\infty} \lambda_n (b,e_n)^2 \left(\frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1 + \lambda_n t^\alpha}\right)^2 \lambda_n^{1-\frac{2}{\alpha}}.$$

Using $\lambda_n^{1-\frac{2}{\alpha}} \le \lambda_1^{1-\frac{2}{\alpha}}, n = 1, 2, ...,$ we have

$$\|v(\cdot,t)\|_{H^2(\Omega)}^2 \le c^2 \left(\|a\|_{H^2(\Omega)}^2 + \|b\|_{H^1(\Omega)}^2\right). \tag{2.6}$$

Further, as a second equality of (2.5), we have

$$\|v_t(\cdot,t)\|_{\mathcal{D}(A^{-\gamma})}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2\gamma} \left|\lambda_n t^{\alpha-1} (a,e_n) E_{\alpha,\alpha}(-\lambda_n t^\alpha)\right|^2 + \sum_{n=1}^{\infty} \lambda_n^{-2\gamma} \left|(b,e_n) E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2$$

$$\leq \sum_{n=1}^{\infty} \lambda_n^2 (a,e_n)^2 \left(\frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha} \right)^2 \lambda_n^{-2(\gamma+1-\frac{1}{\alpha})} + \sum_{n=1}^{\infty} \lambda_n (b,e_n)^2 \lambda_n^{-2(\gamma+\frac{1}{2})}.$$

In view of $\gamma > 0$, we get $\lambda_n^{-2(\gamma+1-\frac{1}{\alpha})} \leq \lambda_1^{-2(\gamma+1-\frac{1}{\alpha})}$ and $\lambda_n^{-2(\gamma+\frac{1}{2})} \leq \lambda_1^{-2(\gamma+\frac{1}{2})}$. Now, using Lemma 1.1, and (1.4), we have

$$\|v_t(\cdot,t)\|^2_{\mathcal{D}(A^{-\gamma})} \leq c^2 \left(\|a\|^2_{H^2(\Omega)} + \|b\|^2_{H^1(\Omega)} \right). \tag{2.7}$$

Thus the proof of Lemma 2.1 is complete.

We introduce the following auxiliary lemmas to obtain the main results.

Lemma 2.2. *Let $F \in C([0,T];\mathcal{D}(A^{1/\alpha}))$. Then there exists a unique weak solution $w \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ to (2.3) with $\partial_t^\alpha w \in C([0,T];L^2(\Omega))$. In particular, for any $\gamma>0$, we have $w_t \in C([0,T];\mathcal{D}(A^{-\gamma}))$,*

$$\lim_{t\to 0} \|w(\cdot,t)\|_{H^2(\Omega)} = \lim_{t\to 0} \|w_t(\cdot,t)\|_{\mathcal{D}(A^{-\gamma})} = 0,$$

Moreover, there exists a constant $c>0$ such that

$$\|w(\cdot,t)\|_{H^2(\Omega)} + \|w_t(\cdot,t)\|_{\mathcal{D}(A^{-\gamma})} \leq c(t+1)\|F\|_{C([0,t];\mathcal{D}(A^{1/\alpha}))} \tag{2.8}$$

and we have

$$w(\mathbf{x},t) = \int_0^t A^{-1} Y(t-s) F(\mathbf{x},s) ds, \quad (\mathbf{x},t) \in Q_0^T, \tag{2.9}$$

the function (2.9) holds in the $C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];\mathcal{D}(A^{-\gamma}))$.

Proof. The uniqueness of the weak solution is proved similarly to Theorem 2.1 in [32]. Therefore, here we omitted it and we show only regularity, besides (2.8).

We first have

$$\|w(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} \left| \int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds \right|^2$$

$$\leq \sum_{n=1}^{\infty} \max_{0\leq s\leq t} \left| (\lambda_n^{\frac{1}{\alpha}} F, e_n) \right|^2 \left| \int_0^t \lambda_n^{-\frac{1}{\alpha}} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds \right|^2$$

$$\leq \sum_{n=1}^{\infty} \max_{0\leq s\leq t} \left| (\lambda_n^{\frac{1}{\alpha}} F, e_n) \right|^2 \left| \int_0^t \frac{(\lambda_n s^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n s^\alpha} \lambda_n^{-1} ds \right|^2 \leq c\lambda_1^{-2} \|F\|^2_{C([0,t];\mathcal{D}(A^{1/\alpha}))} t^2. \tag{2.10}$$

Furthermore, in a view of the condition of Lemma 2.2, for $F \in C([0,T];\mathcal{D}(A^{1/\alpha}))$ and by Lemma 1.1, we have

$$\|A\omega(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^2 \left| \int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) ds \right|^2$$

$$\leq \sum_{n=1}^{\infty} \lambda_n^2 \int_0^t |(F(\cdot,s),e_n)|^2 ds \int_0^t (t-s)^{2\alpha-2} |E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha)|^2 ds$$

$$\leq \sum_{n=1}^{\infty} \lambda_n^2 \lambda_n^{-\frac{2}{\alpha}} \max_{0\leq s\leq t} \left| (A^{\frac{1}{\alpha}}[F], e_n) \right|^2 \int_0^t \left| \frac{(\lambda_n s^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n s^\alpha} \right|^2 ds \cdot \lambda_n^{-\frac{2\alpha-2}{\alpha}} \cdot t \leq c\|F\|^2_{C([0,t];\mathcal{D}(A^{1/\alpha}))} t^2. \tag{2.11}$$

By (2.3) and (2.10) we can estimate also $\|\partial_t^\alpha \omega(\cdot,t)\|_{C([0,T];L^2(\Omega))}$ and we have $\lim_{t\to 0} \|\omega(\cdot,t)\|_{H^2(\Omega)} = 0$. Next apply Lemma 1.1, Proposition 1.1, and apply Cauchy-Schwarz, and for any $\gamma>0$, we have

$$\|\omega_t(\cdot,t)\|^2_{\mathcal{D}(A^{-\gamma})} = \sum_{n=1}^{\infty} \lambda_n^{-2\gamma} \left| \int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) ds \right|^2$$

$$\leq \sum_{n=1}^{\infty} \lambda_n^{-2\gamma} \lambda_n^{-2/\alpha} \max_{0\leq s\leq t} |(A^{\frac{1}{\alpha}}[F](\cdot,s),e_n)|^2 \left| \int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) ds \right|^2$$

$$\le \sum_{n=1}^{\infty} \lambda_n^{-2\gamma-\frac{2}{\alpha}} \max_{0\le s\le t} |(A^{\frac{1}{\alpha}}[F](\cdot,s),e_n)|^2 \left| \int_0^t \frac{d}{ds}\left(s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\right) ds \right|^2$$

$$\le \sum_{n=1}^{\infty} \lambda_n^{-2\gamma-\frac{2}{\alpha}} \max_{0\le s\le t} |(A^{\frac{1}{\alpha}}[F](\cdot,s),e_n)|^2 \left| t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\right|^2$$

$$\le \sum_{n=1}^{\infty} \lambda_n^{-2\gamma-\frac{2}{\alpha}} \max_{0\le s\le t} |(A^{\frac{1}{\alpha}}[F](\cdot,s),e_n)|^2 \left| \frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha}\right|^2 \lambda_n^{\frac{2-2\alpha}{\alpha}} \le c\lambda_1^{-2\gamma-2}\|F\|^2_{C([0,t];\mathcal{D}(A^{1/\alpha}))}.$$

Therefore $\lim_{t\to 0} \|\omega_t(\cdot,t)\|^2_{\mathcal{D}(A^{-\gamma})} = 0$. Thus the proof of Lemma 2.2 is complete.

By Lemma 2.1 and 2.2, we get the following assertion:

Lemma 2.3. *Let $a \in H^2(\Omega)\cap H_0^1(\Omega)$, $b \in H_0^1(\Omega)$ and $F(x,t) \in C([0,T];D(A^{1/\alpha}))$. Then there exists a unique weak solution $u \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];\mathcal{D}(A^{-\gamma}))$ to (2.1), such that*

$$\|u(\cdot,t)\|_{H^2(\Omega)} + \|u_t(\cdot,t)\|_{\mathcal{D}(A^{-\gamma})} \le c\Big[\|a\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)} + (t+1)\|F\|_{C([0,t];\mathcal{D}(A^{1/\alpha}))}\Big] \qquad (2.12)$$

for all $t \in [0,T]$, where the constant c is dependent on α,Ω and the coefficients of A, but does not depend of T. Furthermore, we have

$$u(\mathbf{x},t) = Z_1(t)a(\mathbf{x}) + Z_2(t)b(\mathbf{x}) + \int_0^t A^{-1}Y(t-s)F(\mathbf{x},s)ds, \quad (\mathbf{x},t) \in Q_0^T, \qquad (2.13)$$

where $Z_j(t)[\cdot],(j=1,2)$ and $Y(t)[\cdot]$ are defined above.

The next two lemmas are regularity results of the solution u of the problem (2.1).

Lemma 2.4. *Let $a \in \mathcal{D}(A^{\gamma+\frac{1}{\alpha}})$, $b \in \mathcal{D}(A^\gamma)$ and $F \in C([0,T];\mathcal{D}(A^\gamma))$. Let $\frac{1}{\alpha} < \varepsilon < 1$. Then $u \in X_0^T$ such that*

$$\|u(\cdot,t)\|_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + \|u_t(\cdot,t)\|_{\mathcal{D}(A^\gamma)} \le c\Big(\|a\|_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^\gamma)}$$
$$+(t^{\alpha-1}+t^{\alpha(1-\varepsilon)})\|F\|_{C([0,t];\mathcal{D}(A^\gamma))}\Big), \qquad (2.14)$$

where c is dependent on α.

Proof., by Lemma 1.1 and Cauchy-Schwarz inequality, we have

$$\|u(\cdot,t)\|^2_{D(A^{\gamma+\frac{1}{\alpha}})} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma+\frac{2}{\alpha}} |(a,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + \sum_{n=1}^{\infty} \lambda_n^{2\gamma+\frac{2}{\alpha}} t^2 |(b,e_n)E_{\alpha,2}(-\lambda_n t^\alpha)|^2$$
$$+\sum_{n=1}^{\infty} \lambda_n^{2\gamma+\frac{2}{\alpha}} \left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2$$
$$\le c^2\sum_{n=1}^{\infty} \lambda_n^{2\gamma+\frac{2}{\alpha}} |(a,e_n)|^2 + \sum_{n=1}^{\infty} \lambda_n^{2\gamma}(b,e_n)^2 \left(\frac{(\lambda_n t^\alpha)^{1/\alpha}}{1+\lambda_n t^\alpha}\right)^2$$
$$+\sum_{n=1}^{\infty} \max_{0\le s\le t} |(A^\gamma[F](\cdot,s),e_n)|^2 \left|\int_0^t \lambda_n^{\frac{1}{\alpha}}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right|^2. \qquad (2.15)$$

From Lemma 1.1, we have

$$E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha) \le \frac{c}{1+\lambda_n(t-s)^\alpha} \le c\lambda_n^{-\varepsilon}(t-s)^{-\alpha\varepsilon} \qquad (2.16)$$

for any $0<\varepsilon<1$. Let $\frac{1}{\alpha} < \varepsilon < 1$. Because of these inequalities, rewrite the inequality (2.15) as follows

$$\|u(\cdot,t)\|^2_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} \le c^2\|a\|^2_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + c^2\|b\|^2_{\mathcal{D}(A^\gamma)}$$
$$+c^2\sum_{n=1}^{\infty} \max_{0\le s\le t} |(A^\gamma[F](\cdot,s),e_n)|^2 \left|\int_0^t \lambda_n^{\frac{1}{\alpha}-\varepsilon}(t-s)^{\alpha-\alpha\varepsilon-1}ds\right|^2$$

$$\leq c^2\|a\|^2_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + c^2\|b\|^2_{\mathcal{D}(A^\gamma)} + c^2\lambda_1^{\frac{2}{\alpha}-2\varepsilon} t^{2\alpha(1-\varepsilon)}\|F\|^2_{C([0,t];\mathcal{D}(A^\gamma))}.$$

As a result, we get

$$\|u(\cdot,t)\|_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} \leq c(\alpha)\left(\|a\|_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^\gamma)} + t^{\alpha(1-\varepsilon)}\|F\|_{C([0,t];\mathcal{D}(A^\gamma))}\right). \tag{2.17}$$

Furthermore, by Lemma 2.3, we have

$$u_t(\mathbf{x},t) = \sum_{n=1}^\infty \left\{-\lambda_n t^{\alpha-1}(a,e_n)E_{\alpha,\alpha}(-\lambda_n t^\alpha) + (b,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right\} e_n(\mathbf{x})$$

$$+\sum_{n=1}^\infty \left\{\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^\alpha)ds\right\} e_n(\mathbf{x}). \tag{2.18}$$

Therefore, applying (1.5), Lemma 1.1 again, and $\lambda_n = O(n^{2/d})$, we have

$$\|u_t(\cdot,t)\|^2_{\mathcal{D}(A^\gamma)} = \sum_{n=1}^\infty \lambda_n^{2\gamma}\lambda_n^2|(a,e_n)|^2|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 + \sum_{n=1}^\infty \lambda_n^{2\gamma}|(b,e_n)|^2|E_{\alpha,1}(-\lambda_n t^\alpha)|^2$$

$$+\sum_{n=1}^\infty \lambda_n^{2\gamma}\left|\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^\alpha)ds\right|^2$$

$$\leq c^2\sum_{n=1}^\infty \lambda_n^{2\gamma+\frac{2}{\alpha}}(a,e_n)^2\left(\frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha}\right)^2 + c^2\sum_{n=1}^\infty \lambda_n^{2\gamma}(b,e_n)^2 + c^2\sum_{n=1}^\infty \max_{0\leq s\leq t}|(A^\gamma[F](\cdot,s),e_n)|^2\times$$

$$\times\left|\int_0^t s^{\alpha-2}\frac{c}{1+\lambda_n s^\alpha}ds\right|^2 \leq c^2\|a\|^2_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + c^2\|b\|^2_{\mathcal{D}(A^\gamma)} + c^2t^{2(\alpha-1)}\|F\|^2_{C([0,t];\mathcal{D}(A^\gamma))}. \tag{2.19}$$

Thus,

$$\|u_t(\cdot,t)\|_{\mathcal{D}(A^\gamma)} \leq c\Big(\|a\|_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^\gamma)} + t^{\alpha-1}\|F\|_{C([0,t];\mathcal{D}(A^\gamma))}\Big) \tag{2.20}$$

for all $t\in[0,T]$. Then we immediately obtain the desired estimate (2.14). This completes the proof of this lemma.

It is easy to see that

$$Au(\mathbf{x}_i,t) = \sum_{n=1}^\infty \lambda_n(a,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)e_n(\mathbf{x}_i) + \sum_{n=1}^\infty \lambda_n(b,e_n)tE_{\alpha,2}(-\lambda_n t^\alpha)e_n(\mathbf{x}_i)$$

$$+\sum_{n=1}^\infty \lambda_n\left(\int_0^t (F(\cdot,s),e_n)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)ds\right)e_n(\mathbf{x}_i), \quad i=1,2. \tag{2.21}$$

The following lemma is valid.

Lemma 2.5. *Let $a\in\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})$, $b\in\mathcal{D}(A^{\gamma_0})$ and $F\in C([0,T];\mathcal{D}(A^\gamma))$. Then there exists a positive constant c such that*

$$\|Au(\mathbf{x}_i,\cdot)\|_{C[0,T]} \leq c\Big(\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})} + T^{\frac{\alpha}{2}}\|F\|_{C([0,T];\mathcal{D}(A^\gamma))}\Big), \quad i=1,2, \tag{2.22}$$

and

$$\|Au_t(\mathbf{x}_i,\cdot)\|_{C[0,T]} \leq c\Big(\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})} + T^{\alpha-1}\|F\|_{C([0,T];\mathcal{D}(A^\gamma))}\Big), \quad i=1,2, \tag{2.23}$$

where c is dependent on Ω, α, γ, γ_0, d, λ_1.

Proof. An inequality similar to the estimate in the form (2.22) was obtained in the work of [27]. However, the smoothness differs from the given ones, so we provide the above inequality (2.22) in detail.

We note that A defines the fractional power A^β with $\beta\in\mathbf{R}$ and

$$\|u\|_{H^{2\beta}(\Omega)} \leq c\|A^\beta u\|_{L^2(\Omega)}$$

(see., [29]).

Let $\varepsilon_0 = \min\{\varepsilon_{01}, \varepsilon_{02}\}$ with $2\varepsilon_{01} = \gamma_0 + \frac{1}{\alpha} - 1 - \frac{d}{2} > 0$ and $2\varepsilon_{02} = \gamma - \frac{d}{4} - \frac{1}{2} > 0$. According to the Sobolev embedding theorem $H^{2\beta}(\Omega) \subset C(\bar{\Omega})$ for $\beta = \frac{d}{4} + \varepsilon_0$, we have

$$\|e_n\|_{C(\bar{\Omega})} \le c(\Omega)\|e_n\|_{H^{2\beta}(\Omega)} \le c(\Omega)\|A^\beta e_n\|_{L^2(\Omega)} \le c(\Omega)\lambda_n^\beta. \tag{2.24}$$

For simplicity, we study $Au(\mathbf{x}_i, t)$ in three parts, namely $Au(\mathbf{x}_i, t) := \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$. For I_1, by Lemma 1.1, and noticing that $\lambda_n = O(n^{2/d})$, we have

$$\begin{aligned}|\mathrm{I}_1| &\le \sum_{n=1}^\infty \lambda_n |(a, e_n)||E_{\alpha,1}(-\lambda_n t^\alpha)||e_n(\mathbf{x}_i)| \le c(\Omega, \alpha) \sum_{n=1}^\infty \lambda_n^{\gamma_0 + \frac{1}{\alpha}} |(a, e_n)| \lambda_n^{-(\gamma_0 + \frac{1}{\alpha} - \beta - 1)} \\ &\le c(\Omega, \alpha) \left(\sum_{n=1}^\infty \lambda_n^{2\gamma_0 + \frac{2}{\alpha}} |(a, e_n)|^2 \right)^{1/2} \left(\sum_{n=1}^\infty \lambda_n^{-2(\gamma_0 + \frac{1}{\alpha} - \beta - 1)} \right)^{1/2} \\ &\le c(\Omega, \alpha) \|a\|_{\mathcal{D}(A^{\gamma_0 + \frac{1}{\alpha}})} \left(\sum_{n=1}^\infty n^{-4(\gamma_0 + \frac{1}{\alpha} - \beta - 1)/d} \right)^{1/2}.\end{aligned}$$

By the choice of β, we have $4(\gamma_0 + \frac{1}{\alpha} - \beta - 1)/d = (d + 8\varepsilon_{01} - 4\varepsilon_0) > 1$, which implies

$$\sum_{n=1}^\infty n^{-4(\gamma_0 + \frac{1}{\alpha} - \beta - 1)/d} < c(\gamma_0, \alpha, d).$$

So, we obtain

$$|\mathrm{I}_1| \le c(\Omega, \alpha, \gamma_0, d) \|a\|_{\mathcal{D}(A^{\gamma_0 + \frac{1}{\alpha}})}. \tag{2.25}$$

Further, by Lemma 1.1 and (1.5), we have the following estimate for I_2:

$$\begin{aligned}|\mathrm{I}_2| &\le \sum_{n=1}^\infty \lambda_n |(b, e_n)||t||E_{\alpha,2}(-\lambda_n t^\alpha)||e_n(\mathbf{x}_i)| \le c(\Omega, \alpha) \sum_{n=1}^\infty |(b, e_n)| \frac{\lambda_n t}{1 + \lambda_n t^\alpha} \lambda_n^\beta \\ &\le c(\Omega, \alpha) \sum_{n=1}^\infty \lambda_n^{\gamma_0} |(b, e_n)| \frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1 + \lambda_n t^\alpha} \lambda_n^{-(\gamma_0 + \frac{1}{\alpha} - \beta - 1)} \le c(\Omega, \alpha) \left(\sum_{n=1}^\infty \lambda_n^{2\gamma_0} |(b, e_n)|^2 \right)^{1/2} \times \\ &\times \left(\sum_{n=1}^\infty \lambda_n^{-2(\gamma_0 + \frac{1}{\alpha} - \beta - 1)} \right)^{1/2} \le c(\Omega, \alpha, \gamma_0, d) \|b\|_{\mathcal{D}(A^{\gamma_0})}.\end{aligned} \tag{2.26}$$

Next we calculate I_3. Here the estimate for I_3 as the same as [27] for $\gamma - \beta - \frac{1}{2} = 2\varepsilon_{02} - \varepsilon_0 > 0$, and we have

$$|\mathrm{I}_3| \le c(\Omega, \alpha) t^{\alpha/2} \|F\|_{C([0,t];\mathcal{D}(A^\gamma))}, \qquad \forall t \in [0, T]. \tag{2.27}$$

According to (2.25)-(2.27), we obtain (2.22).

By directly differentiating (2.21) concerning the variable t and taking into account proposition 1.1, we obtain

$$\begin{aligned}\frac{d}{dt} Au(\mathbf{x}_i, t) &= -\sum_{n=1}^\infty \lambda_n^2 (a, e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) e_n(\mathbf{x}_i) + \sum_{n=1}^\infty \lambda_n (b, e_n) E_{\alpha,1}(-\lambda_n t^\alpha) e_n(\mathbf{x}_i) \\ &+ \sum_{n=1}^\infty \lambda_n \left(\int_0^t (F(\cdot, s), e_n)(t - s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t - s)^\alpha) ds \right) e_n(\mathbf{x}_i) := \tilde{\mathrm{I}}_1 + \tilde{\mathrm{I}}_2 + \tilde{\mathrm{I}}_3.\end{aligned} \tag{2.28}$$

Let $\varepsilon_1 = \min\{\varepsilon_{10}, \varepsilon_{11}\}$ with $2\varepsilon_{10} = \gamma_0 - 1 - \frac{d}{2} > 0$ and $2\varepsilon_{11} = \gamma - \frac{d}{4} - 1 > 0$.

By the asymptotic property of the eigenvalues $\lambda_n = O(n^{2/d})$, for $\tilde{\mathrm{I}}_1$, by Lemma 1.1 and (1.5), we have

$$|\tilde{\mathrm{I}}_1| \le \sum_{n=1}^\infty \lambda_n^2 |(a, e_n)| t^{\alpha-1} |E_{\alpha,\alpha}(-\lambda_n t^\alpha)||e_n(\mathbf{x}_i)| \le c(\Omega, \alpha) \sum_{n=1}^\infty \lambda_n^{\gamma_0 + \frac{1}{\alpha}} |(a, e_n)| \frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1 + \lambda_n t^\alpha} \lambda_n^{-(\gamma_0 - \beta - 1)}$$

$$\leq c(\Omega,\alpha)\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0+\frac{2}{\alpha}}|(a,e_n)|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\lambda_n^{-2(\gamma_0-\beta-1)}\right)^{1/2}$$

$$\leq c(\Omega,\alpha)\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})}\left(\sum_{n=1}^{\infty}n^{-4(\gamma_0-\beta-1)/d}\right)^{1/2}.$$

By choice of β, we have $4(\gamma_0-\beta-1)/d=(d+8\varepsilon_{10}-4\varepsilon_1)/d>1$, which implies

$$\sum_{n=1}^{\infty}n^{-4(\gamma_0-\beta-1)/d}<c(\gamma_0,d).$$

So, we obtain

$$|\tilde{\mathrm{I}}_1|\leq c(\Omega,\alpha,\gamma_0,d)\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})}. \tag{2.29}$$

Similarly, we have the following estimate for $\tilde{\mathrm{I}}_2$:

$$|\tilde{\mathrm{I}}_2|\leq\sum_{n=1}^{\infty}\lambda_n|(b,e_n)||E_{\alpha,1}(-\lambda_n t^{\alpha})||e_n(\mathbf{x}_i)|\leq c(\Omega,\alpha)\sum_{n=1}^{\infty}\lambda_n^{\gamma_0}|(b,e_n)|\frac{\lambda_n^{-(\gamma_0-\beta-1)}}{1+\lambda_n t^{\alpha}}$$

$$\leq c(\Omega,\alpha)\left(\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0}|(b,e_n)|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}n^{-4(\gamma_0-\beta-1)/d}\right)^{1/2}\leq c(\Omega,\alpha,\gamma_0,d)\|b\|_{\mathcal{D}(A^{\gamma_0})}. \tag{2.30}$$

Further, we estimate $\tilde{\mathrm{I}}_3$. By Lemma 1.1 and $\gamma-\beta-1=2\varepsilon_{11}-\varepsilon_1>0$, we have

$$|\tilde{\mathrm{I}}_3|^2\leq\sum_{n=1}^{\infty}\left|\lambda_n\int_0^t(F(\cdot,s),e_n)(t-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(t-s)^{\alpha})ds\cdot e_n(\mathbf{x}_i)\right|^2$$

$$\leq c(\Omega,\alpha)\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\max_{0\leq s\leq t}|(F(\cdot,s),e_n)|^2\left|\int_0^t s^{\alpha-2}\frac{1}{(1+\lambda_n s^{\alpha})^2}ds\right|^2\cdot\lambda_n^{-2(\gamma-\beta-1)}$$

$$\leq c(\Omega,\alpha)\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\max_{0\leq s\leq t}|(F(\cdot,s),e_n)|^2\left|\int_0^t s^{\alpha-2}ds\right|^2\cdot\lambda_1^{-2(\gamma-\beta-1)}.$$

So that

$$|\tilde{\mathrm{I}}_3|\leq c(\Omega,\alpha,\lambda_1)\|F\|_{C([0,t],\mathcal{D}(A^{\gamma}))}t^{\alpha-1},\quad\forall t\in[0,T]. \tag{2.31}$$

Finally, by (2.29)-(2.31), we get (2.23) and so complete the proof of this lemma.

To study the main problem (1.1)-(1.4), we consider the following auxiliary inverse initial and boundary value problem.

Lemma 2.6. *Let (C1)-(C5) be held. Then the problem of finding a solution of (1.1)-(1.4) is equivalent to the problem of determining the functions $u(x,t)\in X_0^T$, $q(t)\in C^1[0,T]$ and $k(t)\in C[0,T]$ satisfying*

$$\begin{cases}(\partial_t^{\alpha}u)(\mathbf{x},t)+Au(\mathbf{x},t)=q(t)u_t(\mathbf{x},t)+(k*u)(t)+f(\mathbf{x},t), & (\mathbf{x},t)\in Q_0^T,\\ u(\mathbf{x},0)=a(\mathbf{x}),\quad u_t(\mathbf{x},0)=b(\mathbf{x}) & \mathbf{x}\in\Omega,\\ u(\mathbf{x},t)=0, & (\mathbf{x},t)\in\Sigma_0^T,\end{cases} \tag{2.32}$$

and

$$q(t)=\frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[u,l](t)-h_1(0)\mathcal{N}_2[u,l](t)\Big),\qquad 0\leq t\leq T, \tag{2.33}$$

$$k(t)=D_t\left[\frac{1}{p(t)}\Big(h_1'(t)\mathcal{N}_2[u,l](t)-h_2'(t)\mathcal{N}_1[u,l](t)\Big)\right],\quad 0\leq t\leq T, \tag{2.34}$$

where $D_t:=(d/dt)$, $\mathcal{N}_i\ (i=1,2)$ are defined by (2.39) below and

$$l(t)=\int_0^t k(\tau)d\tau. \tag{2.35}$$

On the other hand, if (2.32)-(2.34) has a solution and the technical condition (C1)-(C4) holds, then there exists a solution to the inverse problem (1.1)-(1.4).

Remark 2.1. From Lemma 2.6, we know that (2.32)-(2.34) is an equivalent form of the original inverse problem (1.1)-(1.4). So, in the next sections, we discuss (2.32)-(2.34), other than the original one.

Proof. The solution $(u(\mathbf{x},t), q(t), k(t)) \in Y_0^T$ of our inverse problem (1.1)-(1.4) is also a solution to the problem (2.32) in Y_0^T. Because the problem (2.32) is the same as (1.1)-(1.3). Therefore, we should show only (2.33) and (2.34). Let the three $\{u(\mathbf{x},t), q(t), k(t)\}$ functions be a solution of problem (1.1)-(1.4). Taking into account the conditions $\partial_t^\alpha h_i(t) \in C[0,T]$ and implies $h_i \in C^1[0,T]$, and fractional differentiating both sides of (1.4) respect to t gives

$$(\partial_t^\alpha u)(\mathbf{x}_i,t) = (\partial_t^\alpha h_i)(t), \quad u_t(\mathbf{x}_i,t) = h_i'(t), \quad 0 \le t \le T. \tag{2.36}$$

Setting $\mathbf{x} = \mathbf{x}_i$ in Eq. (1.1), the procedure yields

$$\partial_t^\alpha u(\mathbf{x}_i,t) + Au(\mathbf{x}_i,t) = q(t)u_t(\mathbf{x}_i,t) + \int_0^t k(t-\tau)u(\mathbf{x}_i,\tau)d\tau + f(\mathbf{x}_i,t), \; i=1,2. \tag{2.37}$$

We note that $l(t) = \int_0^t k(\tau)d\tau$. Then by integration by parts, we get the following equality:

$$\int_0^t k(\tau)h_i(t-\tau)d\tau = h_i(0)l(t) + \int_0^t l(t-\tau)h_i'(\tau)d\tau. \tag{2.38}$$

With the help of (2.36) and (2.38), we can rewrite (2.37) as

$$\begin{aligned} h_i'(t)q(t) + h_i(0)l(t) &= \partial_t^\alpha h_i(t) + Au(\mathbf{x}_i,t) - (l * h_i')(t) - \tilde{f}_i(t) \\ &:= \mathcal{N}_i[u,l](t), \quad i=1,2. \end{aligned} \tag{2.39}$$

Due to (C4), we can solve this system to get (2.33) and

$$l(t) = \frac{1}{p(t)}\Big(h_1'(t)\mathcal{N}_2[u,l](t) - h_2'(t)\mathcal{N}_1[u,l](t)\Big). \tag{2.40}$$

Furthermore, by differentiating (2.40) concerning t, we get (2.34).

Now we assume that (u,q,k) satisfies (2.32)-(2.34). In order to prove that $\{u,q,k\}$ is the solution to the inverse problem (1.1)-(1.4), it suffices to show that $\{u,q,k\}$ satisfies (1.4).

Setting $\mathbf{x} = \mathbf{x}_i$ to the Eq. in (2.32), we have

$$(\partial_t^\alpha u)(\mathbf{x}_i,t) + Au(\mathbf{x}_i,t) = q(t)u_t(\mathbf{x}_i,t) + (k*u)(t) + \tilde{f}_i(t). \tag{2.41}$$

On the other hand, from (C2), we easily see that

$$\frac{1}{p(0)}\Big(h_1'(0)\mathcal{N}_2[u,l](0) - h_2'(0)\mathcal{N}_1[u,l](0)\Big) = 0.$$

We get (2.40) by integrating (2.34) over $[0,t]$. From (2.33) and (2.40), we conclude that

$$\begin{aligned} h_i'(t)q(t) &= -h_i(0)l(t) + \partial_t^\alpha h_i(t) + Au(\mathbf{x}_i,t) - (l*h_i')(t) - \tilde{f}_i(t) \\ &= \partial_t^\alpha h_i(t) + Au(\mathbf{x}_i,t) - (k*h_i)(t) - \tilde{f}_i(t) \end{aligned}$$

or

$$\tilde{f}_i(t) = -h_i'(t)q(t) + \partial_t^\alpha h_i(t) + Au(\mathbf{x}_i,t) - (k*h_i)(t). \tag{2.42}$$

Then substituting (2.42) into (2.41), and using (C3), we have that $P_i(t) := u(\mathbf{x}_i,t) - h_i(t)\,(i=1,2)$ satisfy

$$\begin{cases} \partial_t^\alpha P_i(t) = q(t)P_i'(t) + (k*P_i)(t), & t>0, \\ P_i(0) = P_i'(0) = 0. \end{cases} \tag{2.43}$$

Then, the fractional initial value problem (2.43) is equivalent to the integral equation (see, [33], pp. 199)

$$P_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\int_s^t (t-\tau)^{\alpha-1}k(\tau-s)d\tau\right) P_i(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}q'(s)P_i(s)ds$$

$$+\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}q(s)P_i(s)ds, \quad i=1,2. \tag{2.44}$$

This is a weakly singular homogeneous integral equation, and it has only a trivial solution for $q(t) \in C^1[0,T]$ and $k(t) \in C[0,T]$ (see, [33], pp. 205). Then, $u(\mathbf{x}_i,t) - h_i(t) = 0$, $0 \le t \le T$, i.e., the condition (1.4) is satisfied. This completes the proof of Lemma 2.6

At the end of this section, we give a lemma that will be used to estimate q and k.

Lemma 2.7. *Let (C1) hold. Then for all $(u,q,k) \in Y_0^T$ and $l \in C^1[0,T]$, there exists a constant $c>0$ depending on f, a, b, h_i, but independent of T, such that*

$$\|\mathcal{N}_i[u,l]\|_{C^1[0,T]} \le c\Big[1 + (T^{\frac{\alpha}{2}} + T^{\alpha-1})(1 + \|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(A^\gamma))})$$

$$+(T^{\frac{\alpha}{2}+1} + T^{\alpha})\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}}))} + T^{\frac{1}{2}}\|l\|_{C^1[0,T]}\Big], \tag{2.45}$$

where $\mathcal{N}_i$ $(i=1,2)$ are the same as those in (2.39) and $l(t)$ is in (2.35).

Proof. By Lemma 2.5 and condition (C1), we see that

$$\|\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + \|Au(x_i,t)\|_{C[0,T]} + \|l * h_i'\|_{C[0,T]}$$

$$+\|f_i\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + c\Big(\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})}$$

$$+T^{\frac{\alpha}{2}}\|F\|_{C([0,T];\mathcal{D}(A^\gamma))}\Big) + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i\|_{C[0,T]}.$$

By the definition of F, the last inequality becomes

$$\|\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|\partial_t^\alpha h_i\|_{C[0,T]} + c\Big[\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})}$$

$$+T^{\frac{\alpha}{2}}\Big(\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(A^\gamma))} + \lambda_1^{-\frac{1}{\alpha}}T\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}}))}$$

$$+\|f\|_{C([0,T];\mathcal{D}(A^\gamma))}\Big)\Big] + T^{\frac{1}{2}}\|l\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i\|_{C[0,T]}, \tag{2.46}$$

where we have used

$$\|v\|^2_{D(A^\gamma)} = \sum_{n=1}^{\infty}\lambda_n^{2\gamma+\frac{2}{\alpha}}(v,e_n)^2\lambda_n^{-\frac{2}{\alpha}} \le \lambda_1^{-\frac{2}{\alpha}}\|v\|^2_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})}.$$

On the other hand, direct calculations imply

$$D_t\mathcal{N}_i[u,l](t) = (\partial_t^\alpha h_i)' + Au_t(\mathbf{x}_i,t) - (l' * h_i')(t) - \tilde{f}_i'(t). \tag{2.47}$$

Here we note that $l(0)=0$. By Lemma 2.5, we have

$$\|D_t\mathcal{N}_i[u,l]\|_{C[0,T]} \le \|(\partial_t^\alpha h_i)'\|_{C[0,T]} + c\Big[\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})}$$

$$+T^{\alpha-1}\Big(\|q\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(A^\gamma))} + \lambda_1^{-\frac{1}{\alpha}}T\|k\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}}))}$$

$$+\|f\|_{C([0,T];\mathcal{D}(A^\gamma))}\Big)\Big] + T^{\frac{1}{2}}\|l'\|_{C[0,T]}\|h_i'\|_{L^2(0,T)} + \|\tilde{f}_i'\|_{C[0,T]}. \tag{2.48}$$

(2.46) and (2.48) bring the desired estimate (2.45). This is complete proof of this lemma.

3 Existence of the solution to an inverse problem

We can now prove the existence of a solution to our inverse problem, i.e. Theorem 1.1, which proceeds by a fixed point argument. First, we define the function set

$$B_{\rho,T} = \Big\{(\bar{u},\bar{q},\bar{k}) \in Y_0^T : \ \bar{u}(\mathbf{x},0) = a(\mathbf{x}), \bar{u}_t(\mathbf{x},0) = b(\mathbf{x}), \ \bar{u}(\mathbf{x},t) = 0, \ (\mathbf{x},t) \in \Sigma_0^T,$$

$$\|\bar{u}\|_{X_0^T} + \|\bar{q}\|_{C^1[0,T]} + \|\bar{k}\|_{C[0,T]} \le \rho\Big\}.$$

Here r is a large constant depending on the initial data a, b, f measurement data h_i. For given $(\bar{u},\bar{q},\bar{k}) \in B_{\rho,T}$, we consider

$$\begin{cases} (\partial_t^\alpha u)(\mathbf{x},t) + Au(\mathbf{x},t) = F(\mathbf{x},t), & (\mathbf{x},t) \in Q_0^T, \\ u(\mathbf{x},0) = a(\mathbf{x}), \quad u_t(\mathbf{x},0) = b(\mathbf{x}) & \mathbf{x} \in \Omega, \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Sigma_0^T, \end{cases} \tag{3.1}$$

where

$$F(\mathbf{x},t) = \bar{q}(t)\bar{u}_t(\mathbf{x},t) + (\bar{k} * \bar{u})(t) + f(\mathbf{x},t),$$

and

$$q(t) = \frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[u,\bar{l}](t) - h_1(0)\mathcal{N}_2[u,\bar{l}](t)\Big), \tag{3.2}$$

$$k(t) = \frac{d}{dt}\left(\frac{h_1'(t)\mathcal{N}_2[u,\bar{l}](t) - h_2'(t)\mathcal{N}_1[u,\bar{l}](t)}{p(t)}\right) \tag{3.3}$$

to generate (u,q,k), where $\bar{l}(t) = \int_0^t \bar{k}(\tau)d\tau$, $\mathcal{N}_i$ $(i = 1,2)$ are the same as those in (2.33).

By Hölder's inequality, we have

$$\|(\bar{k} * \bar{u})(t)\|^2_{\mathcal{D}(A^\gamma)} \le \int_0^t |\bar{k}(t-\tau)|^2 d\tau \int_0^t \|u(\cdot,\tau)\|^2_{\mathcal{D}(A^\gamma)} d\tau \le \lambda_1^{-\frac{2}{\alpha}} t^2 \|\bar{k}\|^2_{C[0,t]} \|\bar{u}\|^2_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} \tag{3.4}$$

which implies

$$\|(\bar{k} * \bar{u})(t)\|_{C([0,T];\mathcal{D}(A^\gamma))} \le \lambda_1^{-\frac{1}{\alpha}} \rho T.$$

Furthermore

$$\|\bar{q}\bar{u}_t\|^2_{C([0,T];\mathcal{D}(A^\gamma))} = \max_{0\le t\le T} \left|\sum_{n=1}^\infty \lambda_n^{2\gamma} (\bar{q}(t)\bar{u}_t(\cdot,t),e_n)^2\right| \le \|\bar{q}\|^2_{C[0,T]} \|\bar{u}_t\|^2_{C([0,T];\mathcal{D}(A^\gamma))} \le \rho^2. \tag{3.5}$$

Using these results together with $f \in C^1([0,T];\mathcal{D}(A^\gamma))$, we have

$$\bar{q}(t)\bar{u}_t(\mathbf{x},t) + \bar{k} * \bar{u} + f(\mathbf{x},t) \in C([0,T];\mathcal{D}(A^\gamma)).$$

By Lemma 2.4, the unique solution $u \in X_0^T$ of the problem (3.1), given by (2.13) satisfies

$$\|u\|_{X_0^T} \le c\left(\|a\|_{\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^\gamma)} + (T^{\alpha-1} + T^{\alpha(1-\varepsilon)})\|F\|_{C([0,T];\mathcal{D}(A^\gamma))}\right). \tag{3.6}$$

Further, (3.2)-(3.3) define the functions $q(t)$ and $k(t)$ in terms of u. Furthermore, by Lemma 2.7, we have

$$\begin{aligned} \|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} \le{}& c\|1/p\|_{C^1[0,T]} \left(|h_1(0)| + |h_2(0)| + \|h_1'\|_{C^1[0,T]} + \|h_2'\|_{C^1[0,T]}\right) \\ &\times\Big(1 + (T^{\frac{\alpha}{2}} + T^{\alpha-1})(1 + \|\bar{q}\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(A^\gamma))}) \\ &+ (T^{\frac{\alpha}{2}+1} + T^\alpha)\|\bar{k}\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}}))} + T^{\frac{1}{2}}\|\bar{l}\|_{C^1[0,T]}\Big). \end{aligned} \tag{3.7}$$

Note $\bar{l}(t) = \int_0^t \bar{k}(\tau)d\tau$. We obtain

$$\|\bar{l}\|_{C^1[0,T]} = \left\|\int_0^t \bar{k}(\tau)d\tau\right\|_{C[0,T]} + \|\bar{k}\|_{C[0,T]} \le (1+T)\|\bar{k}\|_{C[0,T]}. \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$\|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} \le c(T)\Big[1 + \|\bar{q}\|_{C[0,T]}\|u_t\|_{C([0,T];\mathcal{D}(A^\gamma))}$$
$$+\|\bar{k}\|_{C[0,T]}\|u\|_{C([0,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}})} + \|\bar{k}\|_{C[0,T]}\Big]. \tag{3.9}$$

This implies that $q(t) \in C^1[0,T]$ and $k(t) \in C[0,T]$.

Thus the mapping

$$Z : B_{r,T} \to Y_0^T, \qquad (\bar{u}, \bar{q}, \bar{k}) \mapsto (u, q, k) \tag{3.10}$$

given by (3.1)-(3.3) is well defined.

The next lemma shows that Z is a contraction map on $B_{\rho,T}$ for sufficiently small $T > 0$. More precisely, we have the following result:

Lemma 3.1. *Let (C1)-(C5) be hold. For $(\bar{u}, \bar{q}, \bar{k})$, $(\bar{U}, \bar{Q}, \bar{K}) \in B_{r,T}$, define*

$$(u, q, k) = Z(\bar{u}, \bar{q}, \bar{k}), \quad (U, Q, K) = Z(\bar{U}, \bar{Q}, \bar{K}).$$

Then for properly small $\tau > 0$, we have

$$\|(u, q, k)\|_{Y_0^T} \le \rho$$

and

$$\|(u-U, q-Q, k-K)\|_{Y_0^T} \le \frac{1}{2}\|(\bar{u}-\bar{U}, \bar{q}-\bar{Q}, \bar{k}-\bar{K})\|_{Y_0^T} \tag{3.11}$$

for all $T \in (0, \tau]$.

Everywhere the following proof, we use c_j to denote a constant which depends on $\Omega, \alpha, \gamma, \gamma_0, \lambda_1$ and the known functions a, b, f and measurement data $h_i, i = 1, 2$, but independent of ρ and T.

Proof. First we prove that the operator $Z(B_{\rho,T}) \subset B_{\rho,T}$ for sufficiently small T and suitable larger ρ. To simplify the calculations, we restrict $T \in (0, 1]$. From Lemma 2.4, (3.4)-(3.6), we have

$$\|u\|_{X_0^T} \le c\lambda_1^{-(\gamma_0-\gamma)}(\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})})$$
$$+c(T^{\alpha-1} + T^{\alpha(1-\varepsilon)})(\|\bar{q}(t)\bar{u}_t\|_{C([0,T],\mathcal{D}(A^\gamma))} + \|(\bar{k} * \bar{u})\|_{C([0,T],\mathcal{D}(A^\gamma))} + \|f\|_{C([0,T],\mathcal{D}(A^\gamma))}))$$
$$\le c_1\left[1 + \left(T^{\alpha-1} + T^{\alpha(1-\varepsilon)}\right)\left(1 + \rho + \rho T\right)\right]. \tag{3.12}$$

On the other hand, by (3.2)-(3.3), together with Lemma 2.7 and (3.8), we have

$$\|q\|_{C^1[0,T]} + \|k\|_{C[0,T]} \le c_2\Big(\|\mathcal{N}_1[u, \bar{l}]\|_{C^1[0,T]} + \|\mathcal{N}_2[u, \bar{l}]\|_{C^1[0,T]}\Big)$$
$$\le c_3\Big(1 + T^{\frac{\alpha}{2}} + T^{\alpha-1} + \rho(T^{\frac{\alpha}{2}} + T^{\alpha-1})\|u_t\|_{C([0,T];\mathcal{D}(A^\gamma))}$$
$$+\rho(T^{\frac{\alpha}{2}+1} + T^\alpha)\|u\|_{C([0,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}}))} + \rho T^{\frac{1}{2}}(1+T)\Big)$$
$$\le c_3\Big(1 + T^{\frac{\alpha}{2}} + T^{\alpha-1} + \rho(T^{\frac{\alpha}{2}} + T^{\alpha-1})\|u\|_{X_0^T} + \rho T^{\frac{1}{2}}(1+T)\Big), \tag{3.13}$$

where we have used the assumption $T \in (0, 1]$. Then, adding up (3.12) and (3.13) leads to

$$\|(u, q, k)\|_{Y_0^T} \le c_4(1 + T^{\frac{\alpha}{2}} + T^{\alpha-1}) + c_4\rho(T^{\frac{\alpha}{2}} + T^{\alpha-1})\Big(1 + T^{\alpha-1} + T^{\alpha(1-\varepsilon)}$$
$$+\rho(1+T)(T^{\alpha-1} + T^{\alpha(1-\varepsilon)}) + T^{1/2} + T^{3/2}\Big). \tag{3.14}$$

We choose sufficiently small τ_1 such that

$$c_4(1 + T^{\frac{\alpha}{2}} + T^{\alpha-1}) + c_4\rho(T^{\frac{\alpha}{2}} + T^{\alpha-1})\Big(1 + T^{\alpha-1} + T^{\alpha(1-\varepsilon)}$$
$$+\rho(1+T)(T^{\alpha-1} + T^{\alpha(1-\varepsilon)}) + T^{1/2} + T^{3/2}\Big) \le \rho. \tag{3.15}$$

and therefore, for all $T < \min\{1, \tau_1\}$ we have

$$\|(\bar{u}, \bar{q}, \bar{k})\|_{Y_0^T} \le \rho. \tag{3.16}$$

That is, Z maps $B_{\rho,T}$ into itself for each fixed $T \in (0, \min\{1, \tau_1\}]$.

Next, we check the second condition of contractive mapping Z. Let $(u, q, k) = Z(\bar{u}, \bar{q}, \bar{k})$ and $(U, Q, K) = Z(\bar{U}, \bar{Q}, \bar{K})$. Then we obtain that $(u - U, q - Q, k - K)$ satisfies that

$$u(\mathbf{x}, t) - U(\mathbf{x}, t) = \int_0^t A^{-1} Y(t-s) \bar{F}(\mathbf{x}, s) ds, \quad (\mathbf{x}, t) \in Q_0^T, \tag{3.17}$$

and

$$q(t) - Q(t) = \frac{1}{p(t)} \Big(h_2(0)(\mathcal{N}_1[u, \bar{l}](t) - \mathcal{N}_1[U, \bar{L}](t)) - h_1(0)(\mathcal{N}_2[u, \bar{l}](t) - \mathcal{N}_2[U, \bar{L}](t)) \Big), \tag{3.18}$$

$$k(t) - K(t) = \frac{d}{dt} \left(\frac{h_1'(t)(\mathcal{N}_2[u, \bar{l}](t) - \mathcal{N}_2[U, \bar{L}](t)) - h_2'(t)(\mathcal{N}_1[u, \bar{l}](t) - \mathcal{N}_1[U, \bar{L}](t))}{p(t)} \right) \tag{3.19}$$

where $\bar{L}(t) = \int_0^t \bar{K}(\tau) d\tau$ and

$$\bar{F} := q(u_t - U_t) + (q - Q)U_t + k * (u - U) + (k - K) * U.$$

Using Lemma 2.4, (3.5), and (3.6), we get

$$\begin{aligned} \|u - U\|_{X_0^T} &\le c(T^{\alpha-1} + T^{\alpha(1-\varepsilon)}) \Big[\|(\bar{q} - \bar{Q})\bar{u}_t\|_{C([0,T], \mathcal{D}(A^\gamma))} + \|(\bar{u}_t - \bar{U}_t)\bar{q}\|_{C([0,T], \mathcal{D}(A^\gamma))} \\ &\quad + \|(\bar{k} - \bar{K}) * \bar{u})\|_{C([0,T], \mathcal{D}(A^\gamma))} + \|\bar{k} * (\bar{u} - \bar{U})\|_{C([0,T], \mathcal{D}(A^\gamma))} \Big] \\ &\le c(T^{\alpha-1} + T^{\alpha(1-\varepsilon)}) \Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} \|\bar{u}_t\|_{C([0,T]; \mathcal{D}(A^\gamma))} + \|\bar{u}_t - \bar{U}_t\|_{C([0,T], \mathcal{D}(A^\gamma))} \|\bar{q}\|_{C[0,T]} \\ &\quad + T^2 \lambda_1^{-\frac{1}{\alpha}} \|\bar{k} - \bar{K}\|_{C[0,T]} \|\bar{u}\|_{C([0,T], \mathcal{D}(A^{\gamma + \frac{1}{\alpha}}))} + T^2 \lambda_1^{-\frac{1}{\alpha}} \|\bar{u} - \bar{U}\|_{C([0,T], \mathcal{D}(A^{\gamma + \frac{1}{\alpha}}))} \|\bar{k}\|_{C[0,T]} \Big] \\ &\le rc \left(T^\alpha + T^{\alpha(1-\varepsilon)} \right) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} \Big[\|\bar{q} - \bar{Q}\|_{C[0,T]} + \|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{k} - \bar{K}\|_{C[0,T]} \Big]. \end{aligned} \tag{3.20}$$

Similarly, by (3.18)-(3.19) and Lemma 2.7, we have

$$\begin{aligned} \|q - Q\|_{C^1[0,T]} + \|k - K\|_{C[0,T]} &\le rc(T^{\alpha/2+1} + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \Big[\|q - Q\|_{C[0,T]} \\ &\quad + \|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{k} - \bar{K}\|_{C[0,T]} \Big]. \end{aligned} \tag{3.21}$$

Therefore, by (3.20) and (3.21), we have

$$\begin{aligned} \|(u - U, q - Q.k - K)\|_{Y_0^T} &\le cr \Big[\left(T^\alpha + T^{\alpha(1-\varepsilon)} \right) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} \\ &\quad + (T^{\alpha/2+1} + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \Big] \|(\bar{u} - \bar{U}, \bar{q} - \bar{Q}, \bar{k} - \bar{K})\|_{Y_0^T}. \end{aligned} \tag{3.22}$$

Hence we can choose sufficiently small τ_2 such that

$$cr \Big[\left(T^\alpha + T^{\alpha(1-\varepsilon)} \right) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}\} + (T^{\alpha/2+1} + T^{\alpha-1}) \max\{1, T^2 \lambda_1^{-\frac{1}{\alpha}}, T^{\frac{3}{2}}\} \Big] \le 1/2 \tag{3.23}$$

for all $T \in (0, \tau_2]$ to obtain

$$\|(u - U, q - Q.k - K)\|_{Y_0^T} \le \frac{1}{2} \|(\bar{u} - \bar{U}, \bar{q} - \bar{Q}.\bar{k} - \bar{K})\|_{Y_0^T}. \tag{3.24}$$

Estimates (3.16) and (3.24) show that Z is a contraction map on $B_{r,T}$ for all $T \in (0, \tau]$, if we choose $\tau \le \min\{1, \tau_1, \tau_2\}$.

To prove the main result, we should prove the following assertion.

Lemma 3.2. *Under conditions (C1)-(C5), for given measurement data $h_i(t)$ for $i = 1, 2$ in (1.4), if the inverse problem (1.1)-(1.4) has two solutions $(u_j, q_j, k_j) \in Y_0^T$ $(j = 1, 2)$ for any time, then $(u_1, q_1, k_1) = (u_2, q_2, k_2)$ in $[0, T]$.*

According to Remark 2.1, we know that (2.32)-(2.34) is equivalent to (1.1)-(1.4). So, in Lemma 3.2 we discuss the global uniqueness of the inverse problem (2.32)-(2.34).

Proof Lemma 3.2. Given any time T, let $(u_i, q_i, k_i)\,(i = 1, 2)$ be two solutions to the inverse problem (2.32)-(2.34) in $[0, T]$ with the regularity $(u_i, q_i, k_i) \in Y_0^T$. This implies

$$\|u_i, q_i, k_i\|_{Y_0^T} \le C^*, \quad i = 1, 2, \tag{3.25}$$

where C^* is depending on α, T, initial data φ and ψ, the known function f and measurement data h_i.

Let

$$\tilde{u} = u_1 - u_2, \quad \tilde{q} = q_1 - q_2, \quad \tilde{k} = k_1 - k_2.$$

Then $(\tilde{u}, \tilde{q}, \tilde{k})$ satisfies

$$\begin{cases} \partial_t^\alpha \tilde{u} + A\tilde{u} = q_1 \tilde{u}_t + \tilde{q} u_{2t} + k_1 * \tilde{u} + \tilde{k} * u_2, & (\mathbf{x}, t) \in Q_0^T, \\ \tilde{u}(\mathbf{x}, 0) = \tilde{u}_t(\mathbf{x}, 0) = 0, & \mathbf{x} \in \Omega, \\ \tilde{u}(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Sigma_0^T, \end{cases} \tag{3.26}$$

and

$$\tilde{q}(t) = \frac{1}{p(t)} \Big(h_2(0) A\tilde{u}(\mathbf{x}_1, t) - h_1(0) A\tilde{u}(\mathbf{x}_2, t) - \tilde{l} * p \Big), \tag{3.27}$$

$$\tilde{k}(t) = \frac{d}{dt} \left(\frac{h_1'(t) \left(A\tilde{u}(\mathbf{x}_2, t) - \tilde{l} * h_2' \right) - h_2'(t) \left(A\tilde{u}(\mathbf{x}_1, t) - \tilde{l} * h_1' \right)}{p(t)} \right), \tag{3.28}$$

where $\tilde{l}(t) = l_1 - l_2$ and the functions $l_i\,(i = 1, 2)$ satisfy $l_i(t) = \int_0^t k_i(s) ds$. We have to show

$$\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^T} = 0. \tag{3.29}$$

Define

$$\sigma = \inf \Big\{ t \in [0, T] : \|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_0^T} > 0 \Big\}. \tag{3.30}$$

It suffices to prove that $\sigma = T$. If (3.30) is not true, then it is obvious that σ is well-defined and satisfies $\sigma < T$. Choose ϵ such that $0 < \epsilon < T - \sigma$.

Further, by (2.13), we can write the solution $\tilde{u}$ as

$$\tilde{u}(\mathbf{x}, t) = \int_0^t A^{-1} Y(t - s) \tilde{F}(\mathbf{x}, s) ds, \quad (\mathbf{x}, t) \in Q_\sigma^{\sigma+\epsilon}, \tag{3.31}$$

where

$$\tilde{F}(\mathbf{x}, t) = q_1 \tilde{u}_t + \tilde{q} u_{2t} + k_1 * \tilde{u} + \tilde{k} * u_2.$$

Then similar to the proofs of Lemma 2.4 and 2.5, we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \le c_5 (\epsilon^{\alpha-1} + \epsilon^{\alpha(1-\varepsilon)}) \|\tilde{F}\|_{C([\sigma, \sigma+\epsilon]; \mathcal{D}(\mathcal{L}^\gamma))}, \tag{3.32}$$

and

$$\begin{cases} \|A\tilde{u}(\mathbf{x}_i, \cdot)\|_{C[\sigma, \sigma+\epsilon]} \le c_6 \epsilon^{\frac{\alpha}{2}} \|\tilde{F}\|_{C([\sigma, \sigma+\epsilon]; \mathcal{D}(\mathcal{L}^\gamma))}, \\ \|A\tilde{u}_t(\mathbf{x}_i, \cdot)\|_{C[\sigma, \sigma+\epsilon]} \le c_7 \epsilon^{\alpha-1} \|\tilde{F}\|_{C([\sigma, \sigma+\epsilon]; \mathcal{D}(\mathcal{L}^\gamma))}. \end{cases} \tag{3.33}$$

From the definition of σ, we see that

$$\tilde{u} = \tilde{q} = \tilde{k} = 0 \quad \text{in} \quad [0, \sigma]. \tag{3.34}$$

By the definition of $\tilde{F}$, and using (3.4), (3.5) and (3.25), we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \le c_8 \left(\epsilon^{\alpha-1} + \epsilon^{\alpha(1-\varepsilon)} \right) \Big(\|q_1 \tilde{u}_t\|_{C([\sigma, \sigma+\epsilon]; \mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{q} u_{2t}\|_{C([\sigma, \sigma+\epsilon]; \mathcal{D}(\mathcal{L}^\gamma))}$$

$$+\|k_1 * \tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{k} * u_2\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}\Big)$$

$$\leq c_8 C^* \left(\epsilon^{\alpha-1} + \epsilon^{\alpha(1-\varepsilon)}\right) \Big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \|\tilde{q}\|_{C[\sigma,\sigma+\epsilon]} + \lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}$$

$$+\lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}\Big). \tag{3.35}$$

Due to $\tilde{q}(\sigma) = 0$, then implies

$$\|\tilde{q}\|_{C[\sigma,\sigma+\epsilon]} = \max_{\sigma\leq t\leq\sigma+\epsilon} \left|\int_\sigma^t \tilde{q}'(s)ds\right| \leq \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]}. \tag{3.36}$$

Substituting (3.36) into (3.35), we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \leq c_8 C^* \left(\epsilon^{\alpha-1} + \epsilon^{\alpha(1-\varepsilon)}\right) \max\{1, \epsilon, \lambda_1^{-\frac{1}{\alpha}}\epsilon\}\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}}. \tag{3.37}$$

Note $\|\tilde{q}\|_{C^1[0,\sigma]} = \|\tilde{k}\|_{C[0,\sigma]} = 0$. On the other hand, by (3.27), and using (3.33), we have the following estimate for $\tilde{q}$

$$\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]} \leq c_9(\epsilon^{\alpha/2} + \epsilon^{\alpha-1})\Big(\|h_2(0)/p(t)\|_{C^1[\sigma,\sigma+\epsilon]} + \|h_2(0)/p(t)\|_{C^1[\sigma,\sigma+\epsilon]}\Big)\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))}$$

$$+\epsilon^{1/2}\|p\|_{C[\sigma,\sigma+\epsilon]}\|\tilde{l}\|_{C[\sigma,\sigma+\epsilon]} \leq c_9 C(h_i)(\epsilon^{\alpha/2} + \epsilon^{\alpha-1})\Big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]}$$

$$+\lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\Big) + C(h_i)\epsilon^{3/2}\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}, \tag{3.38}$$

where we have used that

$$\|\tilde{l}\|_{C[\sigma,\sigma+\epsilon]} = \max_{\sigma\leq t\leq\sigma+\epsilon} \left|\int_\sigma^t \tilde{k}(s)ds\right| \leq \epsilon\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}.$$

Similar to (3.38), by (3.28) we can easily estimate for $\tilde{k}$ as follows

$$\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]} \leq C(h_i)\Big[c_{10}(\epsilon^{\alpha/2} + \epsilon^{\alpha-1})\big(\|\tilde{u}_t\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^\gamma))} + \epsilon\|\tilde{q}\|_{C^1[\sigma,\sigma+\epsilon]}$$

$$+\lambda_1^{-\frac{1}{\alpha}}\epsilon\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];\mathcal{D}(\mathcal{L}^{\gamma+\frac{1}{\alpha}}))}\big) + \epsilon^{3/2}\|\tilde{k}\|_{C[\sigma,\sigma+\epsilon]}\Big]. \tag{3.39}$$

From (3.37)-(3.39), we obtain

$$\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_{\sigma,\sigma+\epsilon}} \leq C(h_i, C^*)\eta(\epsilon)\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_{\sigma,\sigma+\epsilon}} \tag{3.40}$$

with

$$\lim_{\epsilon\to+0}\eta(\epsilon) = \lim_{\epsilon\to+0}\left(\epsilon^{\alpha/2} + 2\epsilon^{\alpha-1} + \epsilon^{\alpha(1-\varepsilon)}\right)\max\{1, \epsilon, \lambda_1^{-\frac{1}{\alpha}}\epsilon, \epsilon^{3/2}\} = 0,$$

and implying

$$\|(\tilde{u}, \tilde{q}, \tilde{k})\|_{Y_{\sigma,\sigma+\epsilon}} = 0$$

for some sufficiently small positive constant ϵ. This means that $(u_1 - u_2, q_1 - q_2, k_1 - k_2)$ vanishes in $[0, \sigma + \epsilon]$, which contradicts with the definition of σ. Therefore (3.29) is proved. From here, we can conclude that

$$(u_1, q_1, k_1) = (u_2, q_2, k_2) \quad \text{in} \quad [0, T]$$

for any time T.

4 Proof of the main result

In this section, we give proof of the global solubility of the solution to our inverse problem, i.e., Theorem 1.1.

Lemma 3.1 ensures that there exists a unique solution $(u, q, k) \in Y_0^\tau$ of the inverse problem (2.32)-(2.34) for sufficiently small $\tau > 0$. In this section, we show that the unique solution (u, q, k) in $[0, \tau]$ can be extended to a large time interval $[0, 2\tau]$.

To do this, we consider

$$\begin{cases} (\partial_t^\alpha v)(\mathbf{x}, t) + Av(\mathbf{x}, t) = y(t)v_t(\mathbf{x}, t) + \int_0^\tau k(t-s)u(\mathbf{x}, s)ds \\ \qquad\qquad + \int_\tau^t r(t-s)v(\mathbf{x}, s)ds + f(\mathbf{x}, t), & (\mathbf{x}, t) \in Q_\tau^T, \\ v(\mathbf{x}, \tau) = u(\mathbf{x}, \tau), \quad v_t(\mathbf{x}, \tau) = u_t(\mathbf{x}, \tau) & \mathbf{x} \in \Omega, \\ v(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Sigma_\tau^T, \end{cases} \tag{4.1}$$

and

$$y(t) = \frac{1}{p(t)}\Big(h_2(0)\mathcal{N}_1[v, \hat{l}^\tau](t) - h_1(0)\mathcal{N}_2[v, \hat{l}^\tau](t)\Big), \quad \tau \le t \le T, \tag{4.2}$$

$$r(t) = \frac{d}{dt}\left(\frac{h_1'(t)\mathcal{N}_2[v, \hat{l}^\tau](t) - h_2'(t)\mathcal{N}_1[v, \hat{l}^\tau](t)}{p(t)}\right), \quad \tau \le t \le T, \tag{4.3}$$

where

$$\mathcal{N}_i[v, \hat{l}^\tau](t) := \partial_t^\alpha h_i(t) + Av(\mathbf{x}_i, t) - \int_0^\tau l(t-s)h_i'(s)ds - \int_\tau^t \hat{l}^\tau(t-s)h_i'(s)ds - \tilde{f}_i(t), \tag{4.4}$$

and $\hat{l}^\tau(t) = \int_\tau^t r(s)ds$. Obviously, if we prove that there exists a solution $(v, y, r) \in Y_\tau^T$ with some $T \ge 2\tau$, then $(\tilde{u}, \tilde{q}, \tilde{k})$ defined by

$$(\tilde{u}, \tilde{q}, \tilde{k}) = \begin{cases} (u, q, k), & t \in [0, \tau], \\ (v, y, r), & t \in [\tau, 2\tau], \end{cases} \tag{4.5}$$

is a solution of the inverse problem (4.1)-(4.3) on the larger interval $[0, 2\tau]$.

We repeat a similar fixed-pointed argument to prove the existence of (v, y, r). Define an operator

$$K : \tilde{B}_{\tilde{\rho},T} \to Y_\tau^T, \qquad (\bar{v}, \bar{y}, \bar{r}) \to (v, y, r) \tag{4.6}$$

with $(\bar{v}, \bar{y}, \bar{r}) \in \tilde{B}_{\tilde{\rho},T}$, where

$$\tilde{B}_{\tilde{\rho},T} = \big\{(\bar{v}, \bar{y}, \bar{r}) \in Y_\tau^T : \ \bar{v}(\mathbf{x}, \tau) = u(\mathbf{x}, \tau), \, \bar{v}_t(\mathbf{x}, \tau) = u_t(\mathbf{x}, \tau), \, \mathbf{x} \in \Omega$$
$$\bar{v}(\mathbf{x}, t) = 0, \ (\mathbf{x}, t) \in \Sigma_\tau^T, \ \|\bar{v}\|_{X_\tau^T} + \|\bar{y}\|_{C^1[\tau,T]} + \|\bar{r}\|_{C[\tau,T]} \le \tilde{\rho}\big\}.$$

Here v is the solution to the initial and boundary value problem

$$\begin{cases} (\partial_t^\alpha v)(\mathbf{x}, t) + Av(\mathbf{x}, t) = \tilde{F}(\mathbf{x}, t), & (\mathbf{x}, t) \in Q_\tau^T, \\ v(\mathbf{x}, \tau) = u(\mathbf{x}, \tau), \quad v_t(\mathbf{x}, \tau) = u_t(\mathbf{x}, \tau), & \mathbf{x} \in \Omega, \\ v(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \Sigma_\tau^T, \end{cases} \tag{4.7}$$

where

$$\tilde{F}(\mathbf{x}, t) = \bar{y}(t)\bar{v}_t(\mathbf{x}, t) + (k * u)(\tau) + (\bar{r} * \bar{v})(\tau + t) + f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in Q_\tau^T. \tag{4.8}$$

Furthermore, y is the solution of (4.2) in terms of v and r is (4.3). Additionally, we have $u(\cdot, \tau) \in D(A^{\gamma_0 + \frac{1}{\alpha}})$ and $u_t(\cdot, \tau) \in D(A^{\gamma_0})$. Indeed, in view of (2.13), $u(\mathbf{x}, \tau)$ can be written as

$$u(\mathbf{x}, \tau) = Z_1(\tau)a(\mathbf{x}) + Z_2(\tau)b(\mathbf{x}) + \int_0^\tau A^{-1}Y(\tau - s)F(\mathbf{x}, s)ds \tag{4.9}$$

with $F(\mathbf{x},t) = q(t)u_t(\mathbf{x},t) + (k*u)(t) + f(\mathbf{x},t) \in C([0,\tau];D(A^\gamma))$ such that $\|F\|_{C([0,\tau];D(A^\gamma))} \le c_5(\rho,\tau,\lambda_1,f)$. Then, by Lemma 1.1, we have

$$\|u(\cdot,\tau)\|^2_{D(A^{\gamma_0+\frac{1}{\alpha}})} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0+\frac{2}{\alpha}} |(a,e_n)|^2 |E_{\alpha,1}(-\lambda_n\tau^\alpha)|^2 + \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0+\frac{2}{\alpha}} |(b,e_n)|^2 |\tau E_{\alpha,2}(-\lambda_n\tau^\alpha)|^2$$

$$+ \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0+\frac{2}{\alpha}} \left| \int_0^\tau (F(\cdot,s),e_n)(\tau-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(\tau-s)^\alpha)ds \right|^2$$

$$\le c_{11}^2 \|a\|^2_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + c_{12}^2 \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |(b,e_n)|^2 \left| \frac{(\lambda_n\tau^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n\tau^\alpha} \right|^2$$

$$+ c_{13}^2 \sum_{n=1}^{\infty} \max_{0\le s\le\tau} |(A^\gamma[F](\cdot,s),e_n)|^2 \left| \int_0^\tau (\tau-s)^{\alpha-\alpha\varepsilon-1} ds \right|^2 \lambda_n^{-2(\gamma-\gamma_0-\frac{1}{\alpha}+\varepsilon)}.$$

By $\gamma > \gamma_0 + \frac{1}{\alpha} - \varepsilon$, together with (1.4), we get

$$\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} \le c_{14} \left(\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})} + \lambda_1^{-\gamma+\gamma_0+\frac{1}{\alpha}-\varepsilon} \tau^{\alpha(1-\varepsilon)} \|F\|_{C([0,\tau];\mathcal{D}(\mathcal{L}^\gamma))} \right). \quad (4.10)$$

According to the (2.18), we have

$$u_t(x,\tau) = \sum_{n=1}^{\infty} \left\{ -\lambda_n \tau^{\alpha-1} (a,e_n) E_{\alpha,\alpha}(-\lambda_n\tau^\alpha) + (b,e_n) E_{\alpha,1}(-\lambda_n\tau^\alpha) \right\} e_n(x)$$

$$+ \sum_{n=1}^{\infty} \left\{ \int_0^\tau (F(\cdot,s),e_n)(\tau-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(\tau-s)^\alpha) ds \right\} e_n(x). \quad (4.11)$$

Then, by Lemma 1.1 and applying (1.4) again, we have

$$\|u_t(\cdot,\tau)\|^2_{\mathcal{D}(A^{\gamma_0})} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} \lambda_n^2 |(a,e_n)|^2 |\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)|^2 + \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |(b,e_n)|^2 |E_{\alpha,1}(-\lambda_n\tau^\alpha)|^2$$

$$+ \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} \left| \int_0^\tau (F(\cdot,s),e_n)(\tau-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(\tau-s)^\alpha) ds \right|^2$$

$$\le c_{15}^2 \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0+\frac{2}{\alpha}} (a,e_n)^2 \left(\frac{(\lambda_n\tau^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n\tau^\alpha} \right)^2 + c_{11}^2 \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} (b,e_n)^2$$

$$+ \sum_{n=1}^{\infty} \max_{0\le s\le\tau} |(A^\gamma[F](\cdot,s),e_n)|^2 \left| \int_0^\tau s^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n s^\alpha) ds \right|^2 \lambda_n^{-2\gamma+2\gamma_0}$$

$$\le c_{16}^2 \|a\|^2_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + c_{11}^2 \|b\|^2_{\mathcal{D}(A^{\gamma_0})} + \sum_{n=1}^{\infty} \max_{0\le s\le\tau} |(A^\gamma[F](\cdot,s),e_n)|^2 \left| \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n\tau^\alpha) \right|^2 \lambda_n^{-2\gamma+2\gamma_0}$$

$$\le c_{16}^2 \|a\|^2_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + c_{11}^2 \|b\|^2_{\mathcal{D}(A^{\gamma_0})} + \sum_{n=1}^{\infty} \max_{0\le s\le\tau} |(A^\gamma[F](\cdot,s),e_n)|^2 \left| \frac{(\lambda_n\tau^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n\tau^\alpha} \right|^2 \lambda_n^{-2\gamma+2\gamma_0+\frac{2}{\alpha}-2}, \quad (4.12)$$

where we have used

$$\int_0^t s^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n s^\alpha) ds = \int_0^t \frac{d}{ds} \left(s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha) \right) ds = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha).$$

By $\gamma > \gamma_0 + \frac{1}{\alpha} - \varepsilon$ for $\frac{1}{\alpha} < \varepsilon < 1$, we have $2\gamma - 2\gamma_0 - \frac{2}{\alpha} + 2 > 0$.

Thus,

$$\|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})} \le c_{17} \Big(\|a\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|b\|_{\mathcal{D}(A^{\gamma_0})} + \|F\|_{C([0,\tau];\mathcal{D}(A^\gamma))} \Big). \quad (4.13)$$

Moreover, by (3.12) we have

$$\|v\|_{X_\tau^T} \le c_{18}\lambda_1^{-(\gamma_0-\gamma)}\left(\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})}\right)$$

$$+c_{19}\left((T-\tau)^{\alpha-1} + (T-\tau)^{\alpha(1-\varepsilon)}\right)\left(\|\bar{y}(t)\bar{v}_t\|_{C([\tau,T],\mathcal{D}(A^\gamma))} + \|(k*u)\|_{C([0,\tau],\mathcal{D}(A^\gamma))}\right.$$

$$\left.+\|(\bar{r}*\bar{u})\|_{C([\tau,T],\mathcal{D}(A^\gamma))} + \|f\|_{C([\tau,T],\mathcal{D}(A^\gamma))}\right) \le c_{20}\lambda_1^{-(\gamma_0-\gamma)}(\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})})$$

$$+c_{21}\left((T-\tau)^{\alpha-1} + (T-\tau)^{\alpha(1-\varepsilon)}\right)\left(\tilde{\rho} + \lambda_1^{-\frac{1}{\alpha}}\rho\tau + \lambda_1^{-\frac{1}{\alpha}}\tilde{\rho}(T-\tau) + c_1\right). \tag{4.14}$$

On the other hand, by (4.2), and using (3.33), we have the following estimate for y

$$\|y\|_{C^1[\tau,T]} \le |h_2(0)|\|p^{-1}\|_{C^1[\tau,T]}\|\mathcal{N}_1[v,\hat{l}^\tau]\|_{C^1[\tau,T]} + |h_1(0)|\|p^{-1}\|_{C^1[\tau,T]}\|\mathcal{N}_2[v,\hat{l}^\tau]\|_{C^1[\tau,T]}$$

$$\le \sum_{i=1}^{2}\Bigg[C(h_i)\Big(1 + \|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})} + C(h_i,\tilde{f}_i)$$

$$+C(h_i)\left((T-\tau)^{\alpha/2} + (T-\tau)^{\alpha-1}\right)\|\bar{y}\|_{C[\tau,T]}\|\bar{v}_t\|_{C([\tau,T];\mathcal{D}(A^\gamma))} + C(h_i)\left((T-\tau)^{\alpha/2+1} + (T-\tau)^{\alpha}\right)$$

$$\times\|\bar{r}\|_{C[\tau,T]}\|\bar{v}\|_{C([\tau,T];\mathcal{D}(A^{\gamma+\frac{1}{\alpha}}))} + C(h_i,f)\left((T-\tau)^{\alpha/2} + (T-\tau)^{\alpha-1}\right) + \|\hat{l}^\tau * h_i'\|_{C^1[\tau,T]}$$

$$+C(h_i)C^*\Bigg] \le c_{22}\Big(\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})}\Big) + c_{23} + c_{24}\left((T-\tau)^{\alpha/2} + (T-\tau)^{\alpha-1}\right)$$

$$+c_{25}\left((T-\tau)^{\alpha/2+1} + (T-\tau)^{\alpha}\right) + c_{26}\left(T-\tau + (T-\tau)^{3/2}\right). \tag{4.15}$$

The last term becomes from

$$\|\hat{l}^\tau * h_i'\|_{C^1[\tau,T]} \le |h_i'(0)|\|\hat{l}^\tau\|_{C[\tau,T]} + \|\hat{l}^\tau\|_{C[\tau,T]}\|h_i''\|_{L^1(\tau,T)} + (T-\tau)^{1/2}\|\hat{l}^\tau\|_{C[\tau,T]}\|h_i'\|_{L^2(\tau,T)},$$

here we notice that, by the Sobolev embedding theorem, we have $\|h_i\|_{W^{2,1}(\tau,T)} \le c\|\partial_t^\alpha h_i\|_{C^1[0,T]}$ (see Remark 1.1). Similarly, we have

$$\|r\|_{C[\tau,T]} \le \tilde{c}_{22}\Big(\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})}\Big) + \tilde{c}_{23} + \tilde{c}_{24}\left((T-\tau)^{\alpha/2} + (T-\tau)^{\alpha-1}\right)$$

$$+\tilde{c}_{25}\left((T-\tau)^{\alpha/2+1} + (T-\tau)^{\alpha}\right) + \tilde{c}_{26}\left(T-\tau + (T-\tau)^{3/2}\right). \tag{4.16}$$

We set $T-\tau \le 1$. Combining the estimates (4.14)-(4.16), as a result we have

$$\|(v,y,r)\|_{Y_\tau^T} \le c_{27}\Big(\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})}\Big) + c_{28}\left((T-\tau)^{\alpha-1} + (T-\tau)^{\alpha(1-\varepsilon)}\right)(1+\tau)$$

$$+c_{29}\left((T-\tau)^{\alpha} + (T-\tau)^{\alpha(1-\varepsilon)+1}\right) + c_{30}\left((T-\tau)^{\alpha/2} + (T-\tau)^{\alpha-1}\right)$$

$$+c_{31}\left((T-\tau)^{\alpha/2+1} + (T-\tau)^{\alpha}\right) + c_{32}\left(T-\tau + (T-\tau)^{3/2}\right) + c_{33}. \tag{4.17}$$

Moreover, using (4.10) and (4.13), by similar calculations to (3.22), we have

$$\|K(v_1,y_1,r_1) - K(v_2,y_2,r_2)\|_{Y_\tau^T} \le c_{34}\Big[\left((T-\tau)^{\alpha-1} + (T-\tau)^{\alpha(1-\varepsilon)}\right)(1+\tau)$$

$$+(T-\tau)^{\alpha} + (T-\tau)^{\alpha(1-\varepsilon)+1} + (T-\tau)^{\alpha/2} + (T-\tau)^{\alpha-1}$$

$$+T-\tau + (T-\tau)^{3/2}\Big]\|v_1-v_2, y_1-y_2, r_1-r_2\|_{Y_\tau^T}. \tag{4.18}$$

We choose $\tilde{\rho}$ such that $\tilde{\rho} \ge \rho$ and

$$c_{27}\Big(\|u(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0+\frac{1}{\alpha}})} + \|u_t(\cdot,\tau)\|_{\mathcal{D}(A^{\gamma_0})}\Big) + c_{33} \le \frac{\tilde{\rho}}{2}.$$

It is easy to see that if we choose $\tilde{\rho}$ larger, then we could get larger $T-\tau$ to satisfy

$$c_{28}\left((T-\tau)^{\alpha-1}+(T-\tau)^{\alpha(1-\varepsilon)}\right)(1+\tau)$$

$$+c_{29}\left((T-\tau)^{\alpha}+(T-\tau)^{\alpha(1-\varepsilon)+1}\right)+c_{30}\left((T-\tau)^{\alpha/2}+(T-\tau)^{\alpha-1}\right)$$

$$+c_{31}\left((T-\tau)^{\alpha/2+1}+(T-\tau)^{\alpha}\right)+c_{32}\left(T-\tau+(T-\tau)^{3/2}\right)\leq\frac{\tilde{\rho}}{2}. \tag{4.19}$$

Furthermore noticing that (4.19) and (3.15) have the same structure, we can choose $T-\tau=\tau$ to satisfy (4.19), which yields $\|K(v,y,r)\|_{Y_\tau^T}\leq\tilde{\rho}$, i.e. $K(\tilde{B}_{\tilde{\rho},T})\subset\tilde{B}_{\tilde{\rho},T}$. Additionally,

$$\|K(v_1,y_1,r_1)-K(v_2,y_2,r_2)\|_{Y_\tau^T}\leq\frac{1}{2}\|(v_1-v_2,y_1-y_2,r_1-r_2)\|_{Y_\tau^T} \tag{4.20}$$

Hence we prove that K is a contraction operator on $\tilde{B}_{\tilde{r},T}$ for $T=2\tau$.

Repeating the extension process limited times, we could obtain a solution $(u,q,k)\in Y_0^T$ of the inverse problem (2.32)-(2.34) for any T. Lemma 2.6 shows that the inverse problem (2.32)-(2.34) is equivalent to our inverse problem. Consequently, the inverse problem (1.1)-(1.4) also admits a unique solution (u,q,k) in the space $X_0^T\times C^1[0,T]\times C[0,T]$ for any T.

5 Example

In this section, as an illustration, we give an example of the inverse problem (1.1)-(1.4) when $d=2$. In this case, we assume that $A\equiv-\Delta:=-\partial_x^2-\partial_y^2$. Let $\Omega=(0,1)\times(0,1)$ be open rectangular. Then in the domain $Q_0^T:=\{(x,y,t):(x,y)\in\Omega,\ 0<t<T\}$ we have the following problem:

$$\partial_t^\alpha u(x,y,t)-\Delta u(x,y,t)=q(t)u_t(x,y,t)+\int_0^t k(t-s)u(x,y,s)ds$$

$$-\Delta a(x,y)-t\Delta b(x,y)+2\left(1-e^{-t}+(15+\pi^2)t\right)b(x,y)-(1-e^{-t})a(x,y),\quad (x,y,t)\in Q_0^T, \tag{5.1}$$

with initial

$$\begin{cases}u(x,y,0)=a(x,y):=\sin 2\pi x\sin 2\pi y, & (x,y)\in\Omega,\\ u_t(x,y,0)=b(x,y):=(10-32x^2)y\sin\pi x\sin\pi y, & (x,y)\in\Omega,\end{cases} \tag{5.2}$$

and boundary conditions

$$u(0,y,t)=u(1,y,t)=0,\quad u(x,0,t)=u(x,1,t)=0,\quad t\in(0,T), \tag{5.3}$$

In the inverse problem, it is required to find the functions $q(t)$ $k(t)$, if there are additional information regarding the solution of the direct problem (1)–(3):

$$u(\frac{1}{4},\frac{1}{4},t)=1+t,\quad u(\frac{1}{2},\frac{1}{2},t)=t,\qquad 0\leq t\leq T. \tag{5.4}$$

It is not difficult to check that all given data satisfy conditions (C1)-(C4). Then, by Lemma 2.6 the solution of the inverse problem (1.1)-(1.3) is of the form

$$u(x,t)=\sin 2\pi x\sin 2\pi y+t(10-32x^2)y\sin\pi x\sin\pi y,$$

$$k(t)=e^{-t},\quad q(t)=e^{-t}-1-(31+2\pi^2)t. \tag{5.5}$$

Of course, the solution of the inverse problem (5.1)-(5.4) also satisfies the conditions of Theorem 1.1.

Conclusion. The weak solubility of a nonlinear inverse boundary value problem for a d-dimensional fractional diffusion-wave equation with natural initial conditions was studied in the work. First, the existence and uniqueness of the direct problem were investigated. The considered problem was reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem was shown. Then, the local existence and uniqueness theorem for the auxiliary problem is proved using the Fourier method and contraction mappings principle. Further, based on the equivalency of these problems, the global existence and uniqueness theorem for the weak solution of the original inverse coefficient problem was established for any value of time.

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study. There are no conflicts of interest.

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