ON TWO-DIMENSIONAL LANDSBERG SPACE WITH A SPECIAL CUBIC (α, β)-METRIC

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Abstract. In 1992 [22], Matsumoto proposed the concept of (α, β)-metrics in Finsler spaces. In this study, we investigate the special cubic (α, β)-metric ([1], [2]), which is a particular class of p-power Finsler metric, and prove that a two-dimensional Finsler space with such a specific metric is a Landsberg space. We find the main scalar I of two-dimensional Finsler space with the cubic (α, β)-metric. It is demonstrated that if a Finsler space with the cubic (α, β)-metric is a Landsberg space, then it is a Berwald space.

Keywords: Finsler space, (α, β)-metric, Cubic (α, β)-metric, Cartan connection, Berwald connection, Landsberg space, Main scalar.

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1. Introduction

A Finsler space with the Cartan connection $C\Gamma$ is considered. The Finsler space is known as a Landsberg space if the covariant derivative $C_{hij|k}$ of the C-torsion tensor $C_{hij} = \dot{\partial}_h \dot{\partial}_i \dot{\partial}_j (\frac{L^2}{4})$ satisfies $C_{hij|k}(x, y)y^k = 0$. A Berwald space is defined as $C_{hij|k} = 0$. Berwald spaces are very interesting and significant since the connection is linear, and several examples of Berwald spaces have been known. However, there is no concrete example of a Landsberg space that is not a Berwald space. If a Finsler space is a Landsberg space and satisfies some additional conditions, it is merely a Berwald space ([16], [26]). In the two-dimensional case, however, a general Finsler space is a Landsberg space if and only if its main scalar $I(x, y)$ satisfies $I_{|i}y^i = 0$ [17].

A generalized form of an (α, β)-metric on an n-dimensional manifold M^n is defined as

$$F = \alpha\left(1 + \frac{\beta}{\alpha}\right)^p \tag{1}$$

is known as the class p-power (α, β)-metrics [5], where $p \neq 0$ is a real constant. If $p = 1$, equation (1) reduces to Rander's metric $(L = \alpha + \beta)$, which has important and interesting curvature properties and was recommended for the first time by Ingarden in 1957. When $p = 2$, it becomes a square metric, commonly known as Z. Shen's square metric $\left(L = \frac{(\alpha+\beta)^2}{\alpha}\right)$. If $p = -1$, it reduces to the Matsumoto type metric, which may be used for measuring the slope of a mountain and so on.

In 2000, H. S. Park and I. Y. Lee [10] studied the condition for a two-dimensional Finsler space with a special (α, β)-metric is to be a Landsberg space. Further, they clarified that if a Finsler space with a special (α, β)-metric is a Landsberg space, then it is also a Berwald space. Later, in 2003, I. Y. Lee and D. G. Jun [13] obtained the conditions for a cubic Finsler space, which is a Berwald space, and a two-dimensional cubic Finsler space, which is a Landsberg space. Further, they have proved that if a two-dimensional cubic Finsler space is a Landsberg space, then it can also be a Berwald space.

In the present paper, we consider $p = 3$ in equation (1) and get the special class of (α, β)-meric in the form of

$$L = \frac{(\alpha + \beta)^3}{\alpha^2} \tag{2}$$

and named as cubic (α, β)-metric in an n-dimensional manifold M^n and an n-dimensional Finsler space F^n equipped with this cubic (α, β)-metric is known as Finsler space with the cubic (α, β)-metric ([1], [2]). Further, we investigate the conditions under which the Finsler space F^2 with the cubic (α, β)-metric is a Landsberg space in which the main scalar I plays an important role, and we show that if F^2 with the metric above is a Landsberg space, then it is a Berwald space. We next use the metric (2) to determine the difference vector and the main scalar of F^2.

2. Preliminaries

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with the cubic (α, β)-metric and $R^n = (M^n, \alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^i y^j$, $\beta = b_i(x)y^i$. The Riemannian metric α is not assumed to be positive-definite in the following, and we shall restrict our considerations to a domain of (x, y), where β does not vanish. $(;)$ denotes the covariant differentiation in the Levi-Civita connection $\gamma^i_{jk}(x)$ of R^n. Let us list the symbols here for further use:

$$L_\alpha = \frac{\partial L}{\partial \alpha},\ \ L_\beta = \frac{\partial L}{\partial \beta},\ \ L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha},\ \ L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta},$$
$$r_i = b_r r^r_i,\ \ s_i = b_r s^r_i,\ \ b^i = a^{ir} b_r,\ \ b^2 = a^{rs} b_r b_s,\ \ y_k = a_{kr} y^r,$$
$$2r_{ij} = b_{i;j} + b_{j;i},\ \ 2s_{ij} = b_{i;j} - b_{j;i},\ \ r^i_j = a^{ir} r_{rj},\ \ s^i_j = a^{ir} s_{rj}.$$

The Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$of F^n plays a significant part in this present paper. B^i_{jk} denotes the difference tensor of G^i_{jk} from γ^i_{jk}as follows [18]:

$$G^i_{jk}(x, y) = \gamma^i_{jk}(x, y) + B^i_{jk}(x, y). \tag{3}$$

With the subscript 0, transvection by y^i, we have

$$G^i_j = \gamma^i_{0j} + B^i_j, \quad 2G^i = \gamma^i_{00} + 2B^i. \tag{4}$$

and then $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$, and $\dot{\partial}_j = \dfrac{\partial}{\partial y^j}$. It is noteworthy that the Cartan connection contains the nonlinear connection G^i_j comman to $B\Gamma$. According to [18], $B^i(x, y)$ is known as the *difference vector.* If $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$, then B^i is expressed as follows:

$$B^i = \left(\frac{E^*}{\alpha}\right) y^i + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right) C^* \left\{ \left(\frac{1}{\alpha}\right) y^i - \left(\frac{\alpha}{\beta}\right) b^i \right\}, \tag{5}$$

where

$$E^{*}=\left(\frac{\beta L_{\beta}}{L}\right)C^{*};\quad C^{*}=\frac{\alpha\beta(r_{00}L_{\alpha}-2s_{0}\alpha L_{\beta})}{2(\beta^{2}L_{\alpha}+\alpha\gamma^{2}L_{\alpha\alpha})}.$$

Since $B\Gamma$ is L-metrical, $L(\alpha,\beta)$ satisfies

$$L_{|i}=\partial_{i}L-(\dot{\partial}_{r}L)G_{i}^{r}=0=L_{\alpha}\alpha_{|i}+L_{\beta}\beta_{|i},$$

where $(L_{\alpha},L_{\beta})=\left(\frac{\partial L}{\partial\alpha},\frac{\partial L}{\partial\beta}\right)$, and so

$$\alpha_{|i}=-\frac{L_{\beta}}{L_{\alpha}}\beta_{|i}. \tag{6}$$

It is observed that $\beta_{|i}=b_{s|i}y^{s}=(b_{s;i}-b_{r}B_{si}^{r})y^{s}$, which implies

$$\beta_{|i}y^{i}=r_{00}-2b_{r}B^{r}. \tag{7}$$

For the scalar b^{2} we have $b_{|i}^{2}y^{i}=(\partial_{i}b^{2})y^{i}=b_{;i}^{2}y^{i}=2b^{r}(r_{ri}+s_{ri})y^{i}$, which shows

$$b_{|i}^{2}y^{i}=2(r_{0}+s_{0}). \tag{8}$$

Next is the quadratic form

$$\gamma^{2}=b^{2}\alpha^{2}-\beta^{2}=(b^{2}a_{ij}-b_{i}b_{j})y^{i}y^{j}.$$

It plays a role in the following. It is easy to explain from the given equations

$$\gamma_{|i}^{2}y^{i}=2(r_{0}+s_{0})\alpha^{2}-2\left(\frac{L_{\beta}}{L_{\alpha}}b^{2}\alpha+\beta\right)(r_{00}-2b_{r}B^{r}). \tag{9}$$

The following Lemma has been shown as follows:

Lemma 2.1. *([11], [25]). If $\alpha^{2}\equiv 0$ (mod β), that is, $a_{ij}(x)y^{i}y^{j}$ contains $b_{i}(x)y^{i}$ as a factor, then the dimension n is equal to two and b^{2} vanishes. In this case we have $\delta=d_{i}(x)y^{i}$ satisfying $\alpha^{2}=\beta\delta$ and $d_{i}b^{i}=2$.*

Lemma 2.2. *([11], [19]). We consider the two-dimensional case.*

1. *If $b^{2}\neq 0$, then there exist a sign $\varepsilon=\pm 1$ and $\delta=d_{i}(x)y^{i}$ such that $\alpha^{2}=\frac{\beta^{2}}{b^{2}}+\varepsilon\delta^{2}$ and $d_{i}b^{i}=0$.*
2. *If $b^{2}=0$, then there exists $\delta=d_{i}(x)y^{i}$ such that $\alpha^{2}=\beta\delta$ and $d_{i}b^{i}=2$.*

If there are two functions $f(x)$ and $g(x)$ that satisfy $f\alpha^{2}+g\beta^{2}=0$, then $f=g=0$ is obvious, since $f\neq 0$ implies the contradiction $\alpha^{2}=\left(\frac{-g}{f}\right)\beta^{2}$.

Throughout the paper, we shall say "homogeneous polynomial (s) in y^{i} of degree r" as $hp(r)$ for brevity. Thus γ_{00}^{i} are $hp(2)$.

The main scalar I of two-dimensional Finsler space F^{2} with the (α,β)-metric, according to [23], is given by

$$\epsilon I^{2}=\left(\frac{L}{\alpha}\right)^{4}\left[\frac{\gamma^{2}(T_{2})^{2}}{4T^{3}}\right], \tag{10}$$

where ϵ is the signature of the space, $\gamma^2 = b^2\alpha^2 - \beta^2$,

$$\begin{cases} T = P(P + P_0 b^2 + P_{-1}\beta) + \{P_0 P_{-2} - (P_{-1})^2\}\gamma^2; \\ T_2 = \dfrac{\partial T}{\partial \beta}. \end{cases} \tag{11}$$

and

$$\begin{cases} P = \dfrac{LL_1}{\alpha}, \\ P_0 = LL_{22} + (L_2)^2, \\ P_{-1} = \dfrac{1}{\alpha}(LL_{12} + L_1 L_2), \\ P_{-2} = \dfrac{L}{\alpha^2}(L_{11} - \dfrac{L_1}{\alpha}) + \dfrac{L_1^2}{\alpha^2}. \end{cases} \tag{12}$$

3. Main scalar I of a special cubic (α, β)-metric

In the following section, we obtained the equation for the main scalar of two-dimensional Finsler spaces using a specialcubic (α, β)-metric.
Let us consider the special cubic metric given by (1).
Now, the partial derivatives of (1) with respect to α and β are as follows:

$$L_\alpha = \frac{(\alpha+\beta)^2(\alpha - 2\beta)}{\alpha^3}, \quad L_\beta = \frac{3(\alpha+\beta)^2}{\alpha^2}, \quad L_{\alpha\alpha} = \frac{6\beta^2(\alpha+\beta)}{\alpha^4}, \tag{13}$$
$$L_{\beta\beta} = \frac{6(\alpha+\beta)}{\alpha^2}, \quad L_{\alpha\beta} = L_{\beta\alpha} = \frac{-6\beta(\alpha+\beta)}{\alpha^3}.$$

Putting (13) into (12), we get

$$P = \frac{(\alpha+\beta)^5(\alpha - 2\beta)}{\alpha^6}, \quad P_0 = \frac{15(\alpha+\beta)^4}{\alpha^4}, \tag{14}$$
$$P_{-1} = \frac{3(\alpha+\beta)^4(\alpha - 4\beta)}{\alpha^6}, \quad P_{-2} = \frac{-3\beta(\alpha+\beta)^4(\alpha - 4\beta)}{\alpha^8}.$$

Again Putting (14) into (11), we get

$$T = \frac{(\alpha+\beta)^{10}}{\alpha^{12}}\Omega. \tag{15}$$

where

$$\Omega = \alpha^2(1 + 6b^2) - \alpha\beta - 8\beta^2.$$

and

$$T_2 = \frac{(\alpha+\beta)^9}{\alpha^{12}}\{10\Omega - (\alpha+\beta)(\alpha + 16\beta)\} \tag{16}$$

Now, the main scalar of two-dimensional space is

$$\epsilon I^2 = \frac{9\gamma^2 W^2}{4\Omega^3}. \tag{17}$$

Where

$$W = (3 + 20b^2)\alpha^2 - 9\alpha\beta - 32\beta^2.$$

Thus, we have the following theorem:

Theorem 3.1. *The main scalar of two dimensional Finsler space F^2 with the special cubic (α, β)-metric is given by (17).*

4. Landsberg space of dimension two with a special cubic (α, β)-metric

Let $F^n = (M^n, L(\alpha, \beta))$ denote an n-dimensional Finsler space with the Cubic (α, β)-metric denoted by (2). The difference vector B^i of F^n is given by using the approach described in ([4], [18]).

$$2B^i = \frac{6\alpha^2 A}{(\alpha - 2\beta)\Omega}\left[b^i + \frac{(\alpha - 4\beta)}{2\alpha^2}y^i\right] + \frac{6\alpha^2}{(\alpha - 2\beta)}s^i_0, \tag{18}$$

Where

$$A = (\alpha - 2\beta)r_{00} - 6\alpha^2 s_0,$$
$$\Omega = \alpha^2(1 + 6b^2) - \alpha\beta - 8\beta^2.$$

It is obvious that $\alpha \neq 0, \Omega \neq 0$ and $(\alpha - 2\beta) \neq 0$, since α is irrational in (y^i). It follows from (18) that

$$r_{00} - 2b_r B^r = \frac{(\alpha - 2\beta)A}{\Omega}. \tag{19}$$

Now we consider the condition for a two-dimensional Finsler space F^2 with the cubic (α, β)-metric given by (2) to be a Landsberg space. In the two-dimensional case, it is widely known that a general Finsler space is a Landsberg space if and only if its main scalar $I(x, y)$ satisfies $I_{|i}y^i = 0$ [11].

With regard to ([23], [24]), the main scalar I of a two-dimensional Finsler space F^2 with a cubic (α, β)-metric occurs as follows:

$$\epsilon I^2 = \frac{9\gamma^2 W^2}{4\Omega^3}, \tag{20}$$

Where

$$W = (3 + 20b^2)\alpha^2 - 9\alpha\beta - 32\beta^2.$$

Now, the covariant differentiation of (20) yields

$$\begin{aligned} 4\Omega^4\epsilon I^2_{|i} = 9W\Big[&\Omega W\gamma^2_{|i} + \gamma^2\{2\Omega(6\alpha + 40b^2\alpha - 9\beta) - 3W(2\alpha + 12b^2\alpha - \beta)\}\alpha_{|i} \\ &+\gamma^2\{3W(\alpha + 16\beta) - 2\Omega(9\alpha + 64\beta)\}\beta_{|i} + 2\gamma^2\alpha^2(20\Omega - 9W)b^2_{|i}\Big]. \end{aligned} \tag{21}$$

Transvecting (21) by y^i, we get

$$4\Omega^4\epsilon I^2_{|i}y^i = 9W\Big[H\gamma^2_{|i}y^i + I\alpha_{|i}y^i + J\beta_{|i}y^i + Kb^2_{|i}y^i\Big], \tag{22}$$

Where

$$\begin{aligned} H &= 3\alpha^4 + 38b^2\alpha^4 + 120b^4\alpha^4 - 12\alpha^3\beta - 74b^2\alpha^3\beta - 47\alpha^2\beta^2 - 352b^2\alpha^2\beta^2 + 104\alpha\beta^3 + 256\beta^4, \\ I &= -6b^2\alpha^5 - 76b^4\alpha^5 - 240b^6\alpha^5 + 33b^2\alpha^4\beta + 196b^4\alpha^4\beta + \alpha^3\beta^2 + 163b^2\alpha^3\beta^2 + 752b^4\alpha^3\beta^2 \\ &\quad - 33\alpha^2\beta^3 - 148b^2\alpha^2\beta^3 - 87\alpha\beta^4 - 512b^2\alpha\beta^4 - 48\beta^5, \\ J &= -9b^2\alpha^5 - 48b^4\alpha^5 + 7b^2\alpha^4\beta + 192b^4\alpha^4\beta + 9\alpha^3\beta^2 - 208b^2\alpha^3\beta^2 - 7\alpha^2\beta^3 - 704b^2\alpha^2\beta^3 \\ &\quad + 256\alpha\beta^4 + 512\beta^5, \\ K &= -14b^2\alpha^6 - 120b^4\alpha^6 + 122b^2\alpha^5\beta + 14\alpha^4\beta^2 + 376b^2\alpha^4\beta^2 - 122\alpha^3\beta^3 - 256\alpha^2\beta^4. \end{aligned}$$

As a consequence, the two-dimensional Finsler space F^2 with the cubic (α,β)-metric is a Landsberg space, if and only if

$$9W\Big[H\gamma^2_{|i}y^i + I\alpha_{|i}y^i + J\beta_{|i}y^i + Kb^2_{|i}y^i\Big] = 0,$$

Which implies

$$H\gamma^2_{|i}y^i + I\alpha_{|i}y^i + J\beta_{|i}y^i + Kb^2_{|i}y^i = 0, \tag{23}$$

Since $W \neq 0$, because $W = 0$ means $b^2 = 0$, this is a contradiction.

The equation is rewritten as follows using (6),(7),(8), and (9):

$$\{2(\alpha-2\beta)\}\,(\alpha^2H+K)(r_0+s_0) + \big[-2(3b^2\alpha^2+\alpha\beta-2\beta^2)H$$
$$-3\alpha I + J(\alpha-2\beta)\big](r_{00}-2b_rB^r) = 0.$$

Substituting the values of H, I, J, K, and (19) into the above equation yields

$$\begin{aligned}
&\alpha^2\big[6\alpha^5+84b^2\alpha^5+288b^4\alpha^5-30\alpha^4\beta-96b^2\alpha^4\beta+576b^4\alpha^4\beta-90\alpha^3\beta^2-828b^2\alpha^3\beta^2 \qquad (24)\\
&+288b^4\alpha^3\beta^2+222\alpha^2\beta^3-1032b^2\alpha^2\beta^3+564\alpha\beta^4-384b^2\alpha\beta^4+288\beta^5\big](r_0+s_0)\\
&+\big[-9b^2\alpha^6-48b^4\alpha^6-6\alpha^5\beta-60b^2\alpha^5\beta+39\alpha^4\beta^2+27b^2\alpha^4\beta^2-144b^4\alpha^4\beta^2\\
&+66\alpha^3\beta^3-486b^2\alpha^3\beta^3-192b^4\alpha^3\beta^3-456b^2\alpha^2\beta^4-105\alpha^2\beta^4+576b^4\alpha^2\beta^4\\
&-222\alpha\beta^5-96\beta^6\big]r_{00}+\alpha^2\big[54b^2\alpha^5+288b^4\alpha^5+36\alpha^4\beta+468b^2\alpha^4\beta\\
&+576b^4\alpha^4\beta-162\alpha^3\beta^2+774b^2\alpha^3\beta^2+288b^4\alpha^3\beta^2-720\alpha^2\beta^3\\
&-1368b^2\alpha^2\beta^3+1728b^4\alpha^2\beta^3-810\alpha\beta^4-288\beta^5\big]s_0=0.
\end{aligned}$$

Separating (24) in the rational and irrational terms of (y^i), we have

$$\{\alpha^2\beta P_1(r_0+s_0)+P_2r_{00}+\alpha^2\beta P_3s_0\}+\alpha\{\alpha^2Q_1(r_0+s_0)+\beta Q_2r_{00}+\alpha^2P_3s_0\}=0, \tag{25}$$

Where

$$\begin{aligned}
P_1 &= -30\alpha^4-96b^2\alpha^4+576b^4\alpha^4+222\alpha^2\beta^2-1032b^2\alpha^2\beta^2+288\beta^4,\\
P_2 &= -9b^2\alpha^6-48b^4\alpha^6+39\alpha^4\beta^2+27b^2\alpha^4\beta^2-144b^4\alpha^4\beta^2-456b^2\alpha^2\beta^4-105\alpha^2\beta^4\\
&\quad +576b^4\alpha^2\beta^4-96\beta^6,\\
P_3 &= 36\alpha^4+468b^2\alpha^4+576b^4\alpha^4-720\alpha^2\beta^2-1368b^2\alpha^2\beta^2+1728b^4\alpha^2\beta^2-288\beta^4,\\
Q_1 &= 6\alpha^4+84b^2\alpha^4+288b^4\alpha^4-90\alpha^2\beta^2-828b^2\alpha^2\beta^2+288b^4\alpha^2\beta^2+564\beta^4-384b^2\beta^4,\\
Q_2 &= -6\alpha^4-60b^2\alpha^4+66\alpha^2\beta^2-486b^2\alpha^2\beta^2-192b^4\alpha^2\beta^2-222\beta^4,\\
Q_3 &= 54b^2\alpha^4+288b^4\alpha^4-162\alpha^2\beta^2+774b^2\alpha^2\beta^2+288b^4\alpha^2\beta^2-810\beta^4.
\end{aligned}$$

The equation (25) generates two equations, which are as follows:

$$\alpha^2\beta P_1(r_0+s_0)+P_2r_{00}+\alpha^2\beta P_3S_0=0, \tag{26}$$

$$\alpha^2Q_1(r_0+s_0)+\beta Q_2r_{00}+\alpha^2Q_3S_0=0. \tag{27}$$

From (26), we have

$$-96\beta^6r_{00}\equiv 0(mod\ \alpha^2). \tag{28}$$

Then there exists a function $f(x)$ such that $r_{00}=\alpha^2f(x)$. Thus, we have

$$r_{ij}=\alpha_{ij}f(x). \tag{29}$$

Transvection by $b^i y^j$ leads to

$$r_0 = \beta f(x); \quad r_j = b_j f(x). \tag{30}$$

Eliminating $(r_0 + s_0)$ from (26) and (27), then, from (29), we have

$$f(x)(P_2Q_1 - \beta^2 Q_2 P_1) + \beta(P_3Q_1 - Q_3P_1)s_0 = 0. \tag{31}$$

From $\alpha^2 \neq 0(mod\,\beta)$ it follows that there exists a function $g(x)$ satisfying $s_0 = g(x)\beta$. hence (31) reduced to

$$f(x)(P_2Q_1 - \beta^2 Q_2 P_1) + \beta^2 g(x)(P_3Q_1 - Q_3P_1) = 0, \tag{32}$$

Since the term $576\{(64b^2-94)(f(x)+3g(x))+3(37f(x)+135g(x))\}\beta^{10}$ of (32) seemingly does not contain α^2, we must have $hp(8)U_8$ such that $\beta^{10} = \alpha^2 U_8$. But it is a contradiction because of $\alpha^2 \neq 0(mod\,\beta)$, which means

$$f(x)(P_2Q_1 - \beta^2 Q_2 P_1) + \beta^2 g(x)(P_3Q_1 - Q_3P_1)$$

does not contain α^2 as a factor.

Thus, from (32) we have $g(x) = 0$. which leads to $s_0 = 0$ and $s_i = 0$. Hence

$$f(x)(P_2Q_1 - \beta^2 Q_2 P_1) + \beta^2 g(x)(P_3Q_1 - Q_3P_1) = 0. \tag{33}$$

This implies that $f(x) = g(x) = 0$. This results in $s_0 = 0$ and $s_i = 0$. We get $r_{ij} = 0$ from (29).
Summing up, we get $r_{ij} = 0$ and $s_i = 0$, which is

$$b_{i;j} + b_{j;i} = 0, \quad b^r b_{r;i} = 0. \tag{34}$$

As a result, $b_i(x)$ is the so-called killing vector field with a fixed length.
The condition (34) is equivalent to $b_{i;j} = 0$, according to [19].
As a consequence, we have the following result:

Theorem 4.1. *Let F^2 be a two-dimensional Finsler space with the special cubic (α,β)-metric (2) satisfying $b^2 \neq 0$. If F^2 is a Landsberg space, then it is a Berwald space.*

5. Conclusion

The purpose of this paper is to determine a Landsberg space in a two-dimensional Finsler space F^2 by means of a special cubic (α,β)-metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ ([1], [2]) that satisfies some conditions. First, we found that the main scalar I of F^2 which is given by (17), plays an important role. Later, with the metric (2), we get the difference vector. Further, we found a condition for a Finsler space to be a Berwald space with a special (α,β)-metric (2). Finally, we show that if the Finsler space F^2 with the metric (2) is a Landsberg space, then it becomes a Berwald space.

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