

CELLULAR-COMPACT AND RELATED SPACES ON HYPERSPACES

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ABSTRACT. In this paper, we study cellular-compact and cellular-Lindelöf spaces on hyperspaces with the Pixley-Roy topology and the Vietoris topology.

1. INTRODUCTION AND PRELIMINARIES

Recently, the generalized metric properties on hyperspaces with the Pixley-Roy topology and the Vietoris topology have been studied by many authors ([2–8, 10–12, 15, 17–22]).

In 2018, cellular-Lindelöf spaces were introduced and studied by W.F. Xuan and Y.K. Song ([23]). After, V.V. Tkachuk and R.G. Wilson introduced and studied cellular-compact spaces in 2019 ([16]). They obtained some good results. In this paper, we study them on hyperspaces with the Pixley-Roy topology and the Vietoris topology. For a space X and $n \in \mathbb{N}$, we prove that

- (1) $\text{PR}[X]$ is cellular-compact (resp., cellular-Lindelöf) if and only if X is finite (resp., countable);
- (2) If $\mathbb{K}(X)$ or $\mathcal{F}(X)$ or $\mathcal{F}_n(X)$ is cellular-compact (resp., cellular-Lindelöf), then X is cellular-compact (resp., cellular-Lindelöf).

Throughout this paper, all spaces are assumed to be Hausdorff, \mathbb{N} denotes the set of all positive integers, the first infinite ordinal denoted by ω .

The set $\text{PR}[X]$ is the set of all non-empty finite subsets of a space X . For each $F \in \text{PR}[X]$ and $A \subset X$, denote

$$[F, A] = \{H \in \text{PR}[X] : F \subset H \subset A\}.$$

The *Pixley-Roy hyperspace* $\text{PR}[X]$ over a space X , defined by C. Pixley and P. Roy in [13], with the topology generated by the sets of the form $[F, V]$, where $F \in \text{PR}[X]$ and V is an open subset in X containing F . For any space X , $\text{PR}[X]$ is zero-dimensional, completely regular and hereditarily metacompact (see [1]).

For each $n \in \mathbb{N}$, let $\text{PR}_n[X] = \{F \in \text{PR}[X] : |F| \leq n\}$. Then, $\text{PR}[X] = \bigcup_{n=1}^{\infty} \text{PR}_n[X]$ and $\text{PR}_n[X] \subset \text{PR}_{n+1}[X]$ for each $n \in \mathbb{N}$.

Remark 1.1. Let X be a space and $n \in \mathbb{N}$.

2020 *Mathematics Subject Classification.* 54B20, 54D20,

Key words and phrases. Hyperspace, Pixley-Roy topology, Vietoris topology, cellular-compact, cellular-Lindelöf.

- (1) $\text{PR}_n[X]$ is a closed subspace of $\text{PR}[X]$ and in particular, $\text{PR}_1[X]$ is a closed discrete subspace of $\text{PR}[X]$ [14].
- (2) Every $\text{PR}_m[X]$ is a closed subspace of $\text{PR}_n[X]$ for each $m, n \in \mathbb{N}$, $m < n$ [11].

Given a space X , we define its *hyperspaces* as the following sets:

- (1) $CL(X) = \{A \subset X : A \text{ is closed and nonempty}\}$;
- (2) $\mathbb{K}(X) = \{A \in CL(X) : A \text{ is compact}\}$;
- (3) $\mathcal{F}_n(X) = \{A \in CL(X) : |A| \leq n\}$, where $n \in \mathbb{N}$;
- (4) $\mathcal{F}(X) = \{A \in CL(X) : A \text{ is finite}\}$.

Let P_1, \dots, P_s be subsets of a space X . Then, we denote

$$\langle P_1, \dots, P_s \rangle = \left\{ A \in CL(X) : A \subset \bigcup_{i \leq s} P_i, A \cap P_i \neq \emptyset \text{ for each } i \leq s \right\}.$$

The set $CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\mathcal{B} = \{ \langle U_1, \dots, U_k \rangle : U_1, \dots, U_k \text{ are open subsets of } X, k \in \mathbb{N} \}.$$

Note that, by definition, $\mathbb{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

- (1) $CL(X)$ is called the *hyperspace of nonempty closed subsets of X* ;
- (2) $\mathbb{K}(X)$ is called the *hyperspace of nonempty compact subsets of X* ;
- (3) $\mathcal{F}_n(X)$ is called the *n -fold symmetric product of X* ;
- (4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of X* .

On the other hand, it is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in \mathbb{N}$.

Remark 1.2 ([15]). Let X be a space and let $n \in \mathbb{N}$.

- (1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.
- (2) $f_1 : X \rightarrow \mathcal{F}_1(X)$ given by $f_1(x) = \{x\}$ is a homeomorphism.
- (3) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in \mathbb{N}$, $m < n$.

Definition 1.3. Let X be a space.

- (1) X is *cellular-Lindelöf* [23], if for any disjoint family \mathcal{U} of non-empty open sets, there is a Lindelöf subspace L such that $L \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.
- (2) X is *cellular-compact* [16], if for any disjoint family \mathcal{U} of non-empty open sets, there is a compact subspace K such that $K \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

Remark 1.4. (1) Compact \Rightarrow cellular-compact \Rightarrow cellular-Lindelöf.

- (2) Compact \Rightarrow Lindelöf \Rightarrow cellular-Lindelöf.

2. MAIN RESULTS

Theorem 2.1. *Let X be a space and $n \in \mathbb{N}$, $n \geq 2$. Then, the following statements are equivalent:*

- (1) $\text{PR}[X]$ is compact;
- (2) $\text{PR}[X]$ is cellular-compact;
- (3) $\text{PR}_n[X]$ is cellular-compact;

- (4) $\text{PR}_1[X]$ is cellular-compact;
- (5) X is finite.

Proof. (5) \Rightarrow (1) \Rightarrow (2) is obvious. For each $n \in \mathbb{N}$, $n \geq 2$, since $\text{PR}_n[X]$ is a regular closed subset of $\text{PR}[X]$ and $\text{PR}_1[X]$ is a regular closed subset of $\text{PR}_n[X]$, by [16, Proposition 3.4], we claim that (2) \Rightarrow (3) \Rightarrow (4).

It follows from Remark 1.1(1) that if $\text{PR}_1[X]$ is cellular-compact, then $\text{PR}_1[X]$ is finite. Therefore, (4) \Rightarrow (5). \square

Notation 2.2. Let P_1, \dots, P_s be subsets of a space X . Then, we denote

$$\begin{aligned} \langle P_1, \dots, P_s \rangle_n &= \langle P_1, \dots, P_s \rangle \cap \mathcal{F}_n(X) \text{ for each } n \in \mathbb{N}, \\ \langle P_1, \dots, P_s \rangle_{\mathcal{F}(X)} &= \langle P_1, \dots, P_s \rangle \cap \mathcal{F}(X), \\ \langle P_1, \dots, P_s \rangle_{\mathbb{K}(X)} &= \langle P_1, \dots, P_s \rangle \cap \mathbb{K}(X). \end{aligned}$$

Theorem 2.3. Let X be a space and $n \in \mathbb{N}$. If $\mathbb{K}(X)$ or $\mathcal{F}(X)$ or $\mathcal{F}_n(X)$ is cellular-compact, then X is cellular-compact.

Proof. Assume that \mathcal{U} is a disjoint family of non-empty open sets in X . Put

$$\begin{aligned} \mathfrak{U}_{\mathbb{K}(X)} &= \{ \langle U \rangle_{\mathbb{K}(X)} : U \in \mathcal{U} \}; \\ \mathfrak{U}_{\mathcal{F}(X)} &= \{ \langle U \rangle_{\mathcal{F}(X)} : U \in \mathcal{U} \}; \\ \mathfrak{U}_n &= \{ \langle U \rangle_n : U \in \mathcal{U} \}. \end{aligned}$$

Then, it is obvious that $\mathfrak{U}_{\mathbb{K}(X)}$ (resp., $\mathfrak{U}_{\mathcal{F}(X)}$, \mathfrak{U}_n) is a disjoint family of non-empty open sets in $\mathbb{K}(X)$ (resp., $\mathcal{F}(X)$, $\mathcal{F}_n(X)$). If $\mathbb{K}(X)$ is cellular-compact, then there is a compact subspace \mathcal{K} of $\mathbb{K}(X)$ such that $\mathcal{K} \cap \langle U \rangle_{\mathbb{K}(X)} \neq \emptyset$ for each $U \in \mathcal{U}$. Thus, for each $U \in \mathcal{U}$, there exists $F \in \mathcal{K}$ such that $F \subset U$. Hence, $(\bigcup \mathcal{K}) \cap U \neq \emptyset$. Moreover, it follows from [9, Theorem 2.5.2] that $\bigcup \mathcal{K}$ is compact in X . Therefore, X is cellular-compact.

Next, let $\mathcal{F}(X)$ or $\mathcal{F}_n(X)$ be cellular-compact. If \mathcal{K} is compact in $\mathcal{F}(X)$ or $\mathcal{F}_n(X)$, then \mathcal{K} is compact in $\mathbb{K}(X)$. Therefore, similar to the above proof, we claim that X is cellular-compact. \square

Theorem 2.4. Let X be a space and $n \in \mathbb{N}$, $n \geq 2$. Then, the following statements are equivalent:

- (1) $\text{PR}[X]$ is Lindelöf;
- (2) $\text{PR}[X]$ is cellular-Lindelöf;
- (3) $\text{PR}_n[X]$ is cellular-Lindelöf;
- (4) $\text{PR}_1[X]$ is cellular-Lindelöf;
- (5) X is countable.

Proof. (5) \Rightarrow (1) \Rightarrow (2) is obvious. For each $n \in \mathbb{N}$, $n \geq 2$, since $\text{PR}_n[X]$ is a regular closed subset of $\text{PR}[X]$ and $\text{PR}_1[X]$ is a regular closed subset of $\text{PR}_n[X]$, by [25, Theorem 3.5], we claim that (2) \Rightarrow (3) \Rightarrow (4).

It follows from Remark 1.1(1) and [24, Theorem 3.1] that if $\text{PR}_1[X]$ is cellular-Lindelöf, then $\text{PR}_1[X]$ is countable. Hence, (4) \Rightarrow (5). \square

Lemma 2.5. Let X be a space and $A \subset X$. If \mathcal{L} is Lindelöf in subspace $\langle A \rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$, then $\bigcup \mathcal{L}$ is Lindelöf in subspace A of X .

Proof. Let \mathcal{U} be an open cover of $\bigcup \mathcal{L}$ in subspace A of X . Then, for each $U \in \mathcal{U}$, there exists V_U open in X such that $U = V_U \cap A$. Take any $E \in \mathcal{L}$, we have that $E \subset \bigcup \mathcal{L} \subset \bigcup_{U \in \mathcal{U}} V_U$. Since E is a compact subset of X , there exists a finite subcover $\{V_{U_1}, \dots, V_{U_{n(E)}}\}$ of E such that $E \cap V_{U_i} \neq \emptyset$ for each $i \leq n(E)$. Thus, $E \in \langle V_{U_1}, \dots, V_{U_{n(E)}} \rangle_{\mathbb{K}(X)}$. Now, if we put

$$\mathfrak{U} = \left\{ \langle V_{U_1}, \dots, V_{U_{n(E)}} \rangle_{\mathbb{K}(X)} : E \in \mathcal{L} \right\},$$

then \mathfrak{U} is an open cover of \mathcal{L} in $\mathbb{K}(X)$. Hence, $\{\mathcal{H} \cap \langle A \rangle_{\mathbb{K}(X)} : \mathcal{H} \in \mathfrak{U}\}$ is an open cover of \mathcal{L} in subspace $\langle A \rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$. Since \mathcal{L} is Lindelöf in subspace $\langle A \rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$, there exists a countable subfamily \mathfrak{U}_0 of \mathfrak{U} such that $\mathcal{L} \subset \bigcup_{\mathcal{H} \in \mathfrak{U}_0} (\mathcal{H} \cap \langle A \rangle_{\mathbb{K}(X)})$. Put

$$\begin{aligned} \mathfrak{U}_0 &= \left\{ \mathcal{O}_{E_j} = \langle V_{U_{1(E_j)}}, \dots, V_{U_{n(E_j)}} \rangle_{\mathbb{K}(X)} : j \in \mathbb{N} \right\}, \\ \mathcal{V} &= \{U_{1(E_j)}, \dots, U_{n(E_j)} : j \in \mathbb{N}\}. \end{aligned}$$

Then, \mathcal{V} is a countable subfamily of \mathcal{U} . Moreover, $\bigcup \mathcal{L} \subset \bigcup \mathcal{V}$. Indeed, let $z \in \bigcup \mathcal{L}$. Then, there exists $E \in \mathcal{L}$ such that $z \in E$. Since $E \in \mathcal{L}$, there exists $j \in \mathbb{N}$ such that

$$E \in \mathcal{O}_{E_j} \cap \langle A \rangle_{\mathbb{K}(X)} = \langle V_{U_{1(E_j)}}, \dots, V_{U_{n(E_j)}} \rangle_{\mathbb{K}(X)} \cap \langle A \rangle_{\mathbb{K}(X)}.$$

This implies that there exists $1 \leq i \leq n$ such that

$$z \in V_{U_{i(E_j)}} \cap A = U_{i(E_j)} \subset \bigcup \mathcal{V}.$$

Therefore, $\bigcup \mathcal{L}$ is Lindelöf in subspace A of X . \square

Theorem 2.6. *Let X be a space and $n \in \mathbb{N}$. If $\mathbb{K}(X)$ or $\mathcal{F}(X)$ or $\mathcal{F}_n(X)$ is cellular-Lindelöf, then X is cellular-Lindelöf.*

Proof. Suppose that \mathcal{U} is a disjoint family of non-empty open sets in X . Put

$$\begin{aligned} \mathfrak{U}_{\mathbb{K}(X)} &= \{\langle U \rangle_{\mathbb{K}(X)} : U \in \mathcal{U}\}; \\ \mathfrak{U}_{\mathcal{F}(X)} &= \{\langle U \rangle_{\mathcal{F}(X)} : U \in \mathcal{U}\}; \\ \mathfrak{U}_n &= \{\langle U \rangle_n : U \in \mathcal{U}\}. \end{aligned}$$

Then, it is obvious that $\mathfrak{U}_{\mathbb{K}(X)}$ (resp., $\mathfrak{U}_{\mathcal{F}(X)}$, \mathfrak{U}_n) is a disjoint family of non-empty open sets in $\mathbb{K}(X)$ (resp., $\mathcal{F}(X)$, $\mathcal{F}_n(X)$). If $\mathbb{K}(X)$ is cellular-Lindelöf, then there is a Lindelöf subspace \mathcal{L} of $\mathbb{K}(X)$ such that $\mathcal{L} \cap \langle U \rangle_{\mathbb{K}(X)} \neq \emptyset$ for each $U \in \mathcal{U}$. This implies that for each $U \in \mathcal{U}$, there exists $F \in \mathcal{L}$ such that $F \subset U$. Hence, $(\bigcup \mathcal{L}) \cap U \neq \emptyset$. Moreover, since $\mathbb{K}(X) = \langle X \rangle_{\mathbb{K}(X)}$, $\bigcup \mathcal{L}$ is Lindelöf in X by Lemma 2.5. Therefore, X is cellular-Lindelöf.

Next, assume that $\mathcal{F}(X)$ or $\mathcal{F}_n(X)$ is cellular-Lindelöf. If \mathcal{L} is Lindelöf in $\mathcal{F}_n(X)$ or $\mathcal{F}(X)$, then \mathcal{L} is Lindelöf in $\mathbb{K}(X)$. Therefore, similar to the above proof, we claim that X is cellular-Lindelöf. \square

Remark 2.7. Let X be a space. Then, X is cellular-compact (resp., cellular-Lindelöf) if and only if $\mathcal{F}_1(X)$ is cellular-compact (resp., cellular-Lindelöf) by Remark 1.2(2).

By Theorems 2.3, 2.6 and Remark 2.7, we have the following question.

Question 2.8. *If X is cellular-compact (resp., cellular-Lindelöf), then are $\mathbb{K}(X)$, $\mathcal{F}(X)$ and $\mathcal{F}_n(X)$ for some $n \in \mathbb{N}$ ($n \geq 2$) cellular-compact (resp., cellular-Lindelöf)?*

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