

MULTIVALUED GROUPS AND NEWTON  
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**Abstract:** On the set of complex number  $\mathbb{C}$  it is possible to define  $n$ -valued group for any positive integer  $n$ . The  $n$ -multiplication defines a symmetric polynomial  $p_n = p_n(x, y, z)$  with integer coefficients. By the theorem on symmetric polynomials, one can present  $p_n$  as polynomial in elementary symmetric polynomials  $e_1, e_2, e_3$ . V. M. Buchstaber formulated a question on description coefficients of this polynomial. Also, he formulated the next question: How to describe the Newton polyhedron of  $p_n$ ? In the present paper we find all coefficients of  $p_n$  under monomials of the form  $e_1^i e_2^j$  and prove that the Newton polyhedron of  $p_n$  is a right triangle.

**Keywords:** multi-set, multivalued group, symmetric polynomial, Newton polyhedron.

One branch of Abstract Algebra is studying algebraic systems with multivalued operations. Solutions of the Yang-Baxter equation (2-simplex equation) and its generalization,  $n$ -simplex equations,  $n \geq 3$ , are examples of multivalued operations. In 1971, V. M. Buchstaber and S. P. Novikov [2] introduced

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a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is an  $n$ -multi-set, the set of  $n$  points with multiplicities. This construction leads to the notion of  $n$ -valued group.

A good survey on  $n$ -valued groups and its applications can be found in [1]. In Section 5 of this paper,  $n$ -valued groups were constructed on the set of complex numbers  $\mathbb{C}$  for any natural  $n$ . The  $n$ -valued multiplication is described by the polynomials  $p_n = p_n(z; x, y)$  which are  $x, y, z$ -symmetric polynomials with integer coefficients. If we introduce elementary symmetric polynomials

$$e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz,$$

then  $p_n = P_n(e_1, e_2, e_3)$  is a polynomial with integer coefficients in variables  $e_1, e_2, e_3$ . In [1] two questions were formulated on the description the coefficients of  $P_n$  as well as a question on the Newton polyhedron of  $p_n$ .

In the present paper we find the coefficients for monomials of the form  $e_1^i e_2^j$  in  $P_n$ . This gives particular answer to the first two questions. Also, we prove that if

$$f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$$

is a symmetric homogeneous polynomial of degree  $k$ , which contains a monomial  $ax_1^k$  for some non-zero  $a$ , then its Newton polyhedron is the  $k\Delta^{n-1}$ -simplex. From this theorem follows that the Newton polyhedron of  $p_n$  is the right triangle with side which depend on  $n$ . This is the complete answer to the third question.

At the end of the paper we formulate some open questions.

## 1 Multivalued groups and Buchstaber's questions

**1.1. Multivalued groups.** Recall definitions and some facts from the theory of multivalued groups (see, for example, [1]).

Let  $X$  be a non-empty set. An  $n$ -valued multiplication on  $X$  is a map

$$\mu: X \times X \rightarrow (X)^n = \text{Sym}^n X, \quad \mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k,$$

where  $(X)^n = \text{Sym}^n X$  is the  $n$ -th symmetric power of  $X$ , that is the quotient  $X^n / S_n$  of the Cartesian power  $X^n$  under the action of  $S_n$  by permutations of components. The next axioms are natural generalizations of the classical axioms of group multiplication.

*Associativity.* The  $n^2$ -multi-sets:

$$[x * (y * z)_1, x * (y * z)_2, \dots, x * (y * z)_n], \quad [(x * y)_1 * z, (x * y)_2 * z, \dots, (x * y)_n * z]$$

are equal for all  $x, y, z \in X$ .

*Unit.* An element  $e \in X$  such that

$$e * x = x * e = [x, x, \dots, x]$$

for all  $x \in X$ .

*Inverse.* A map  $inv: X \rightarrow X$  such that

$$e \in inv(x) * x \text{ and } e \in x * inv(x)$$

for all  $x \in X$ .

The map  $\mu$  defines  $n$ -valued group structure  $\mathcal{X} = (X, \mu, e, inv)$  on  $X$  if it is associative, has a unit and an inverse.

Let  $\mu$  be the multiplication

$$\mu: \mathbb{C} \times \mathbb{C} \rightarrow (\mathbb{C})^n$$

that is defined by the formula

$$\mu(x, y) = x * y = [(\sqrt[n]{x} + \epsilon^r \sqrt[n]{y})^n, \quad 1 \leq r \leq n],$$

where  $\epsilon$  is a primitive  $n$ -th root of unity. This multiplication endows  $\mathbb{C}$  with the structure of an  $n$ -valued group with the unit  $e = 0$ . The inverse element is given by the map  $inv(x) = (-1)^n x$ .

The  $n$ -valued multiplication is described by the polynomials

$$p_n = p_n(z; x, y) = \prod_{k=1}^n (z - (inv(x) * inv(y))_k),$$

whence the product  $x * y$  is defined by  $z$ -roots of the equation  $p_n = 0$ . The polynomials  $p_n(z; x, y)$  are  $x, y, z$ -symmetric polynomials with integral coefficients, e.g.,

$$p_1 = x + y + z, \quad p_2 = (x + y + z)^2 - 4(xy + yz + zx).$$

Set

$$e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz.$$

Then

$$p_1 = e_1,$$

$$p_2 = e_1^2 - 2^2 e_2,$$

$$p_3 = e_1^3 - 3^3 e_3,$$

$$p_4 = e_1^4 - 2^3 e_1^2 e_2 + 2^4 e_2^2 - 2^7 e_1 e_3,$$

$$p_5 = e_1^5 - 5^4 e_1^2 e_3 + 5^5 e_2 e_3,$$

$$p_6 = e_1^6 - 2^2 \cdot 3 e_1^4 e_2 + 2^4 \cdot 3 e_1^2 e_2^2 - 2^6 e_2^3 - 2 \cdot 3^4 \cdot 17 e_1^3 e_3 - 2^3 \cdot 3^4 \cdot 19 e_1 e_2 e_3 + 3^3 \cdot 19^3 e_3^2,$$

$$p_7 = e_1^7 - 5 \cdot 7^4 e_1^4 e_3 + 2 \cdot 7^6 e_1^2 e_2 e_3 - 7^7 e_2^2 e_3 + 7^8 e_1 e_3^2.$$

The following questions were formulated in [1].

(1) What is the relationship between prime factors of  $n$  and prime factors of the coefficients of the polynomials  $p_n$ ?

- (2) How to distinguish the monomials that have zero coefficient?  
 (3) How to describe the Newton polyhedron of  $p_n$ ?

## 2 Coefficients and the Newton polyhedron of $p_n$

Since  $p_n$  is a symmetric homogeneous polynomial of degree  $n$ , by the theorem on symmetric polynomials we can present  $p_n$  as a polynomial on the elementary symmetric polynomials  $e_1$ ,  $e_2$ , and  $e_3$ ,

$$p_n = \sum_{\substack{k_1 \geq k_2 \geq k_3 \geq 0 \\ k_1 + k_2 + k_3 = n}} A_{k_1, k_2, k_3} e_1^{k_1 - k_2} e_2^{k_2 - k_3} e_3^{k_3} \in \mathbb{Z}[e_1, e_2, e_3].$$

The main problem is to find the coefficients  $A_{k_1, k_2, k_3}$ .

We can write  $p_n$  in the form

$$\begin{aligned} p_n &= \prod_{k=1}^n (z - ((inv(x) * inv(y))_k)) = \prod_{k=1}^n (z - ((-1)^n x * (-1)^n y)_k) = \\ &= \prod_{k=1}^n \left( z - \left( \sqrt[n]{(-1)^n x} + \epsilon^k \sqrt[n]{(-1)^n y} \right)^n \right). \end{aligned}$$

If  $y = 0$ , then

$$\bar{p}_n = p_n(z; x, 0) = \prod_{k=1}^n \left( z - (\sqrt[n]{(-1)^n x})^n \right) = \prod_{k=1}^n (z - (-1)^n x) = (z - (-1)^n x)^n.$$

Denote by

$$\bar{e}_1 = e_1(z; x, 0) = x + z, \quad \bar{e}_2 = e_2(z; x, 0) = zx.$$

We see that  $e_3(z; x, 0) = 0$ .

The next proposition gives particular answers to the first two questions.

**Proposition 1.** 1) If  $n$  is odd, then all  $A_{k_1, k_2, 0}$ ,  $k_2 \neq 0$ , are zero, i.e. in this case  $p_n$  does not contain monomials  $e_1^i e_2^j$ ,  $j > 0$ .

2) If  $n = 2k$  is even, then the coefficient  $A_{2k-i, i, 0}$  at  $e_1^{2(k-i)} e_2^i$ , is equal to

$$A_{2k-i, i, 0} = (-4)^i C_k^i = (-4)^i \frac{k!}{i!(k-i)!}, \quad i = 1, 2, \dots, k.$$

*Proof.* 1) If  $n$  is odd, then

$$\bar{p}_n = (z + x)^n = \bar{e}_1^n.$$

It means that in  $p_n$  all coefficients  $A_{k_1, k_2, 0}$ , where  $k_1 \geq k_2 > 0$  and  $k_1 + k_2 = n$  are zero.

2) If  $n = 2k$  is even, then

$$\bar{p}_n = (z - x)^n = (\bar{e}_1^2 - 4\bar{e}_2)^k = \sum_{i=0}^k (-4)^i C_k^i (\bar{e}_1^2)^{k-i} (\bar{e}_2)^i.$$

Hence, we have found the following coefficients in  $p_n$ .

$$A_{2k-i,i,0} = (-4)^i C_k^i, \quad i = 1, 2, \dots, k.$$

□

**Example.** From this proposition follows that for even  $n$  hold

$$\bar{p}_2 = \bar{e}_1^2 - 2^2 \bar{e}_2,$$

$$\bar{p}_4 = \bar{e}_1^4 - 2^3 \bar{e}_1^2 \bar{e}_2 + 2^4 \bar{e}_2^2,$$

$$\bar{p}_6 = \bar{e}_1^6 - 2^2 \cdot 3 \bar{e}_1^4 \bar{e}_2 + 2^4 \cdot 3 \bar{e}_1^2 \bar{e}_2^2 - 2^6 \bar{e}_2^3,$$

$$\bar{p}_8 = \bar{e}_1^8 - 2^4 \bar{e}_1^6 \bar{e}_2 + 2^5 \cdot 3 \bar{e}_1^4 \bar{e}_2^2 - 2^8 \bar{e}_1^2 \bar{e}_2^3 + 2^8 \bar{e}_2^4.$$

It is easy to see that for even  $n$  all coefficients of  $\bar{p}_n$  except the coefficient at  $\bar{e}_1^n$  are even. This is not true for polynomials  $p_n$ , as example  $p_6$  shows. We can formulate

**Conjecture.** 1) If  $n = p^m$  is a power of a prime  $p$ , then all coefficients, except the coefficient at  $e_1^n$  are divided into  $p$ . 2) If  $n$  is even, then all coefficients  $A_{k_1, k_2, k_3}$  are non-zero.

**2.1. Newton polyhedron.** In this subsection we give a complete answer to the third question in [1]. Recall the necessary definition. Let

$$f = f(x_1, x_2, \dots, x_n) = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in \mathbb{Z}[x_1, x_2, \dots, x_n]$$

be a polynomial with integer coefficients. Denote by  $I_f$  the set of multi indexes  $(i_1, \dots, i_n)$  such that  $a_{i_1 \dots i_n} \neq 0$ . The convex hull

$$N_f = \text{Conv}(I_f) \subset \mathbb{R}^n$$

is said to be a *Newton polyhedron* of  $f$ .

To find Newton polyhedra for polynomials  $p_n$ , consider them for small  $n$ ,  
 $p_1 = x + y + z,$

$$p_2 = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,$$

$$p_3 = (z + x + y)^3 - 27xyz,$$

$$p_4 = ((x + y + z)^2 - 4(xy + yz + zx))^2 - 2^7(x + y + z)xyz = p_2^2 - 2^7 p_1 xyz.$$

Denote by  $N_i \subset \mathbb{R}^3$  the Newton polyhedron for  $p_i$ . Then

- $N_1$  is the right triangle  $A_1 B_1 C_1$  with the vertices  $A_1 = (1, 0, 0)$ ,  $B_1 = (0, 1, 0)$ ,  $C_1 = (0, 0, 1)$ ;
- $N_2$  is the right triangle  $A_2 B_2 C_2$  with the vertices  $A_2 = (2, 0, 0)$ ,  $B_2 = (0, 2, 0)$ ,  $C_2 = (0, 0, 2)$ ;
- $N_3$  is the right triangle  $A_3 B_3 C_3$  with the vertices  $A_3 = (3, 0, 0)$ ,  $B_3 = (0, 3, 0)$ ,  $C_3 = (0, 0, 3)$ ;

–  $N_4$  is the right triangle  $A_4B_4C_4$  with the vertices  $A_4 = (4, 0, 0)$ ,  $B_4 = (0, 4, 0)$ ,  $C_4 = (0, 0, 4)$ .

To describe  $N_k$  for  $k > 2$  we introduce the next definition.

**Definition 1.** *Let  $k$  be a positive integer. The standard  $n$ -simplex of size  $k$  is the subset of  $\mathbb{R}^{n+1}$  given by*

$$k\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = k \text{ and } t_i \geq 0 \text{ for } i = 0, 1, \dots, n \right\}.$$

For simplicity we shall call the standard  $n$ -simplex of size  $k$  by  $k\Delta^n$ -simplex.

For  $k = 1$  we get the definition of the standard  $n$ -simplex (or unit simplex).

The  $k\Delta^n$ -simplex has  $n + 1$  vertices,

$$E_0 = (k, 0, 0, \dots, 0, 0), E_1 = (0, k, 0, \dots, 0, 0), \dots, E_n = (0, 0, 0, \dots, 0, k).$$

Now we are ready to prove the main result of the present subsection.

**Theorem 1.** *Let  $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$  be a symmetric homogeneous polynomial of degree  $k$ , which contains a monomial  $ax_1^k$  for some non-zero  $a$ . Then its Newton polyhedron  $N_f$  is the  $k\Delta^{n-1}$ -simplex.*

*Proof.* Since  $ax_1^k$  is a monomial of  $f$  and  $f$  is symmetric, it contains monomials  $ax_i^k$  for all  $i = 1, 2, \dots, n$ . Hence,  $N_f$  contains the vertices

$$E_0 = (k, 0, 0, \dots, 0, 0), E_1 = (0, k, 0, \dots, 0, 0), \dots, E_{n-1} = (0, 0, 0, \dots, 0, k) \in \mathbb{R}^n$$

and hence it contains  $k\Delta^{n-1}$ -simplex. Let us show that any other vertex of  $N_f$ , which corresponds a monomial in  $f$  lies in this simplex. Indeed, any such monomial has the form

$$bx_1^{k_1}x_2^{k_2} \dots x_n^{k_n}, \quad b \in \mathbb{R}, \quad b \neq 0.$$

Since

$$k_1 + k_2 + \dots + k_n = k, \quad k_i \geq 0 \text{ for } i = 1, 2, \dots, n,$$

the corresponding vertex lies in  $k\Delta^{n-1}$ -simplex.  $\square$

We seen that the polynomial  $p_k$  is homogeneous and has the form  $p_k = e_1^k + \dots$ . Hence, the answer to the third question of V. M. Buchstaber follows from Theorem 1.

**Corollary.** *The Newton polyhedron that corresponds to the polynomial  $p_k(x, y, z)$ ,  $k \geq 1$ , is the  $k\Delta^2$ -simplex that is a right triangle with sides of length  $\sqrt{2}k$ .*

### 3 Some open questions

The following questions seem interesting:

- (1) Let  $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$  be a symmetric polynomial,  $N_f$  is its Newton polyhedron. Let us present  $f$  as a polynomial in elementary symmetric polynomials,  $f = F[e_1, e_2, \dots, e_n] \in \mathbb{Z}[e_1, e_2, \dots, e_n]$  and construct its Newton polyhedron  $N_F$ . What is the relation between  $N_f$  and  $N_F$ ?
- (2) Let  $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$  be a symmetric polynomial of degree  $k$ , which does not contain  $x_1^k$ . What can we say about its Newton polyhedron  $N_f$ ?
- (3) Since there is a homomorphism of one multivalued group to another multivalued group and the kernel of this homomorphism is defined, we can talk about extensions of multivalued groups. Construct a theory of extensions of multivalued groups.
- (4) Is it possible to define (co)homology for multivalued groups?

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