

MULTIVALUED GROUPS AND NEWTON
POLYHEDRONV.G. BARDAKOV  AND T.A. KOZLOVSKAYA 

Abstract: On the set of complex number $\mathbb{C}$ it is possible to define n -valued group for any positive integer n . The n -multiplication defines a symmetric polynomial $p_n = p_n(x, y, z)$ with integer coefficients. By the theorem on symmetric polynomials, one can present p_n as polynomial in elementary symmetric polynomials e_1, e_2, e_3 . V. M. Buchstaber formulated a question on description coefficients of this polynomial. Also, he formulated the next question: How to describe the Newton polyhedron of p_n ? In the present paper we find all coefficients of p_n under monomials of the form $e_1^i e_2^j$ and prove that the Newton polyhedron of p_n is a right triangle.

Keywords: multi-set, multivalued group, symmetric polynomial, Newton polyhedron.

One branch of Abstract Algebra is studying algebraic systems with multivalued operations. Solutions of the Yang-Baxter equation (2-simplex equation) and its generalization, n -simplex equations, $n \geq 3$, are examples of multivalued operations. In 1971, V. M. Buchstaber and S. P. Novikov [2] introduced

BARDAKOV, V.G., MULTIVALUED GROUPS AND NEWTON POLYHEDRON.

© 2023 BARDAKOV V.G.

This work was supported by the state contract of the Sobolev Institute of Mathematics, SB RAS (no. I.1.5, project FWNF-2022-0009).

KOZLOVSKAYA, T.A., MULTIVALUED GROUPS AND NEWTON POLYHEDRON.

© 2023 KOZLOVSKAYA T.A.

This study was supported by the Tomsk State University Development Programme (Priority-2030) and the article was prepared within the framework of the project “Mirror Laboratories” HSE University, RF.

Received January, 1, 2023, Published December, 31, 2023.

a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is an n -multi-set, the set of n points with multiplicities. This construction leads to the notion of n -valued group.

A good survey on n -valued groups and its applications can be found in [1]. In Section 5 of this paper, n -valued groups were constructed on the set of complex numbers $\mathbb{C}$ for any natural n . The n -valued multiplication is described by the polynomials $p_n = p_n(z; x, y)$ which are x, y, z -symmetric polynomials with integer coefficients. If we introduce elementary symmetric polynomials

$$e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz,$$

then $p_n = P_n(e_1, e_2, e_3)$ is a polynomial with integer coefficients in variables e_1, e_2, e_3 . In [1] two questions were formulated on the description the coefficients of P_n as well as a question on the Newton polyhedron of p_n .

In the present paper we find the coefficients for monomials of the form $e_1^i e_2^j$ in P_n . It gives particular answer to the first two questions. Also, we prove that if

$$f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$$

is a symmetric homogeneous polynomial of degree k , which contains a monomial ax_1^k for some non-zero a , then its Newton polyhedron is the $k\Delta^{n-1}$ -simplex. From this theorem follows that the Newton polyhedron of p_n is the right triangle with side which depend on n . This is the complete answer to the third question.

At the end of the paper we formulate some open questions.

1 Multivalued groups and Buchstaber's questions

1.1. Multivalued groups. Recall definitions and some facts from the theory of multivalued groups (see, for example, [1]).

Let X be a non-empty set. An n -valued multiplication on X is a map

$$\mu: X \times X \rightarrow (X)^n = \text{Sym}^n X, \quad \mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k,$$

where $(X)^n = \text{Sym}^n X$ is the n -th symmetric power of X , that is the quotient X^n / S_n of the Cartesian power X^n under the action of S_n by permutations of components. The next axioms are natural generalizations of the classical axioms of group multiplication.

Associativity. The n^2 -multi-sets:

$$[x * (y * z)_1, x * (y * z)_2, \dots, x * (y * z)_n], \quad [(x * y)_1 * z, (x * y)_2 * z, \dots, (x * y)_n * z]$$

are equal for all $x, y, z \in X$.

Unit. An element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for all $x \in X$.

Inverse. A map $inv: X \rightarrow X$ such that

$$e \in inv(x) * x \text{ and } e \in x * inv(x)$$

for all $x \in X$.

The map μ defines n -valued group structure $\mathcal{X} = (X, \mu, e, inv)$ on X if it is associative, has a unit and an inverse.

Let μ be the multiplication

$$\mu: \mathbb{C} \times \mathbb{C} \rightarrow (\mathbb{C})^n$$

that is defined by the formula

$$\mu(x, y) = x * y = [(\sqrt[n]{x} + \epsilon^r \sqrt[n]{y})^n, \quad 1 \leq r \leq n],$$

where ϵ is a primitive n -th root of unity. This multiplication endows $\mathbb{C}$ with the structure of an n -valued group with the unit $e = 0$. The inverse element is given by the map $inv(x) = (-1)^n x$.

The n -valued multiplication is described by the polynomials

$$p_n = p_n(z; x, y) = \prod_{k=1}^n (z - (inv(x) * inv(y))_k),$$

whence the product $x * y$ is defined by z -roots of the equation $p_n = 0$. The polynomials $p_n(z; x, y)$ are x, y, z -symmetric polynomials with integral coefficients, e.g.,

$$p_1 = x + y + z, \quad p_2 = (x + y + z)^2 - 4(xy + yz + zx).$$

Set

$$e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz.$$

Then

$$p_1 = e_1,$$

$$p_2 = e_1^2 - 2^2 e_2,$$

$$p_3 = e_1^3 - 3^3 e_3,$$

$$p_4 = e_1^4 - 2^3 e_1^2 e_2 + 2^4 e_2^2 - 2^7 e_1 e_3,$$

$$p_5 = e_1^5 - 5^4 e_1^2 e_3 + 5^5 e_2 e_3,$$

$$p_6 = e_1^6 - 2^2 \cdot 3 e_1^4 e_2 + 2^4 \cdot 3 e_1^2 e_2^2 - 2^6 e_2^3 - 2 \cdot 3^4 \cdot 17 e_1^3 e_3 - 2^3 \cdot 3^4 \cdot 19 e_1 e_2 e_3 + 3^3 \cdot 19^3 e_3^2,$$

$$p_7 = e_1^7 - 5 \cdot 7^4 e_1^4 e_3 + 2 \cdot 7^6 e_1^2 e_2 e_3 - 7^7 e_2^2 e_3 + 7^8 e_1 e_3^2.$$

The following questions were formulated in [1].

(1) What is the relationship between prime factors of n and prime factors of the coefficients of the polynomials p_n ?

- (2) How to distinguish the monomials that have zero coefficient?
 (3) How to describe the Newton polyhedron of p_n ?

2 Coefficients and the Newton polyhedron of p_n

Since p_n is a symmetric homogeneous polynomial of degree n , by the theorem on symmetric polynomials we can present p_n as a polynomial on the elementary symmetric polynomials e_1 , e_2 , and e_3 ,

$$p_n = \sum_{\substack{k_1 \geq k_2 \geq k_3 \geq 0 \\ k_1 + k_2 + k_3 = n}} A_{k_1, k_2, k_3} e_1^{k_1 - k_2} e_2^{k_2 - k_3} e_3^{k_3} \in \mathbb{Z}[e_1, e_2, e_3].$$

The main problem is to find the coefficients A_{k_1, k_2, k_3} .

We can write p_n in the form

$$\begin{aligned} p_n &= \prod_{k=1}^n (z - ((\text{inv}(x) * \text{inv}(y))_k)) = \prod_{k=1}^n (z - ((-1)^n x * (-1)^n y)_k) = \\ &= \prod_{k=1}^n \left(z - \left(\sqrt[n]{(-1)^n x} + \epsilon^k \sqrt[n]{(-1)^n y} \right)^n \right). \end{aligned}$$

If $y = 0$, then

$$\bar{p}_n = p_n(z; x, 0) = \prod_{k=1}^n \left(z - (\sqrt[n]{(-1)^n x})^n \right) = \prod_{k=1}^n (z - (-1)^n x) = (z - (-1)^n x)^n.$$

Denote by

$$\bar{e}_1 = e_1(z; x, 0) = x + z, \quad \bar{e}_2 = e_2(z; x, 0) = zx.$$

We see that $e_3(z; x, 0) = 0$.

The next proposition gives particular answers to the first two questions.

Proposition 1. 1) If n is odd, then all $A_{k_1, k_2, 0}$, $k_2 \neq 0$, are zero, i.e. in this case p_n does not contain monomials $e_1^i e_2^j$, $j > 0$.

2) If $n = 2k$ is even, then the coefficient $A_{2k-i, i, 0}$ at $e_1^{2(k-i)} e_2^i$, is equal to

$$A_{2k-i, i, 0} = (-4)^i C_k^i = (-4)^i \frac{k!}{i!(k-i)!}, \quad i = 1, 2, \dots, k.$$

Proof. 1) If n is odd, then

$$\bar{p}_n = (z + x)^n = \bar{e}_1^n.$$

It means that in p_n all coefficients $A_{k_1, k_2, 0}$, where $k_1 \geq k_2 > 0$ and $k_1 + k_2 = n$ are zero.

2) If $n = 2k$ is even, then

$$\bar{p}_n = (z - x)^n = (\bar{e}_1^2 - 4\bar{e}_2)^k = \sum_{i=0}^k (-4)^i C_k^i (\bar{e}_1^2)^{k-i} (\bar{e}_2)^i.$$

Hence, we have found the following coefficients in p_n .

$$A_{2k-i,i,0} = (-4)^i C_k^i, \quad i = 1, 2, \dots, k.$$

□

Example. From this proposition follows that for even n hold

$$\bar{p}_2 = \bar{e}_1^2 - 2^2 \bar{e}_2,$$

$$\bar{p}_4 = \bar{e}_1^4 - 2^3 \bar{e}_1^2 \bar{e}_2 + 2^4 \bar{e}_2^2,$$

$$\bar{p}_6 = \bar{e}_1^6 - 2^2 \cdot 3 \bar{e}_1^4 \bar{e}_2 + 2^4 \cdot 3 \bar{e}_1^2 \bar{e}_2^2 - 2^6 \bar{e}_2^3,$$

$$\bar{p}_8 = \bar{e}_1^8 - 2^4 \bar{e}_1^6 \bar{e}_2 + 2^5 \cdot 3 \bar{e}_1^4 \bar{e}_2^2 - 2^8 \bar{e}_1^2 \bar{e}_2^3 + 2^8 \bar{e}_2^4.$$

It is easy to see that for even n all coefficients of $\bar{p}_n$ except the coefficient at $\bar{e}_1^n$ are even. This is not true for polynomials p_n , as example p_6 shows. We can formulate

Conjecture. 1) If $n = p^m$ is a power of a prime p , then all coefficients, except the coefficient at e_1^n are divided into p . 2) If n is even, then all coefficients A_{k_1, k_2, k_3} are non-zero.

2.1. Newton polyhedron. In this subsection we give a complete answer to the third question in [1]. Recall the necessary definition. Let

$$f = f(x_1, x_2, \dots, x_n) = \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \in \mathbb{Z}[x_1, x_2, \dots, x_n]$$

be a polynomial with integer coefficients. Denote by I_f the set of multi indexes $(i_1, \dots, i_n)$ such that $a_{i_1 \dots i_n} \neq 0$. The convex hull

$$N_f = \text{Conv}(I_f) \subset \mathbb{R}^n$$

is said to be a *Newton polyhedron* of f .

To find Newton polyhedra for polynomials p_n , consider them for small n ,

$$p_1 = x + y + z,$$

$$p_2 = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,$$

$$p_3 = (z + x + y)^3 - 27xyz,$$

$$p_4 = ((x + y + z)^2 - 4(xy + yz + zx))^2 - 2^7(x + y + z)xyz = p_2^2 - 2^7 p_1 xyz.$$

Denote by $N_i \subset \mathbb{R}^3$ the Newton polyhedron for p_i . Then

- N_1 is the right triangle $A_1 B_1 C_1$ with the vertices $A_1 = (1, 0, 0)$, $B_1 = (0, 1, 0)$, $C_1 = (0, 0, 1)$;
- N_2 is the right triangle $A_2 B_2 C_2$ with the vertices $A_2 = (2, 0, 0)$, $B_2 = (0, 2, 0)$, $C_2 = (0, 0, 2)$;
- N_3 is the right triangle $A_3 B_3 C_3$ with the vertices $A_3 = (3, 0, 0)$, $B_3 = (0, 3, 0)$, $C_3 = (0, 0, 3)$;

– N_4 is the right triangle $A_4B_4C_4$ with the vertices $A_4 = (4, 0, 0)$, $B_4 = (0, 4, 0)$, $C_4 = (0, 0, 4)$.

To describe N_k for $k > 2$ we introduce the next definition.

Definition 1. *Let k be a positive integer. The standard n -simplex of size k is the subset of $\mathbb{R}^{n+1}$ given by*

$$k\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = k \text{ and } t_i \geq 0 \text{ for } i = 0, 1, \dots, n \right\}.$$

For simplicity we shall call the standard n -simplex of size k by $k\Delta^n$ -simplex.

For $k = 1$ we get the definition of the standard n -simplex (or unit simplex). The $k\Delta^n$ -simplex has $n + 1$ vertices,

$$E_0 = (k, 0, 0, \dots, 0, 0), E_1 = (0, k, 0, \dots, 0, 0), \dots, E_n = (0, 0, 0, \dots, 0, k).$$

Now we are ready to prove the main result of the present subsection.

Theorem 1. *Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric homogeneous polynomial of degree k , which contains a monomial ax_1^k for some non-zero a . Then its Newton polyhedron N_f is the $k\Delta^{n-1}$ -simplex.*

Proof. Since ax_1^k is a monomial of f and f is symmetric, it contains monomials ax_i^k for all $i = 1, 2, \dots, n$. Hence, N_f contains the vertices

$$E_0 = (k, 0, 0, \dots, 0, 0), E_1 = (0, k, 0, \dots, 0, 0), \dots, E_{n-1} = (0, 0, 0, \dots, 0, k) \in \mathbb{R}^n$$

and hence it contains $k\Delta^{n-1}$ -simplex. Let us show that any other vertex of N_f , which corresponds a monomial in f lies in this simplex. Indeed, any such monomial has the form

$$bx_1^{k_1}x_2^{k_2}\dots x_n^{k_n}, \quad b \in \mathbb{R}, \quad b \neq 0.$$

Since

$$k_1 + k_2 + \dots + k_n = k, \quad k_i \geq 0 \text{ for } i = 1, 2, \dots, n,$$

the corresponding vertex lies in $k\Delta^{n-1}$ -simplex. $\square$

We seen that the polynomial p_k is homogeneous and has the form $p_k = e_1^k + \dots$. Hence, the answer to the third question of V. M. Buchstaber follows from Theorem 1.

Corollary. *The Newton polyhedron that corresponds to the polynomial $p_k(x, y, z)$, $k \geq 1$, is the $k\Delta^2$ -simplex that is a right triangle with sides of length $\sqrt{2}k$.*

3 Some open questions

The following questions seem interesting:

- (1) Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric polynomial, N_f is its Newton polyhedron. Let us present f as a polynomial in elementary symmetric polynomial, $f = F[e_1, e_2, \dots, e_n] \in \mathbb{Z}[e_1, e_2, \dots, e_n]$ and construct its Newton polyhedron N_F . What is the relation between N_f and N_F ?
- (2) Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric polynomial of degree k , which does not contain x_1^k . What can we say about its Newton polyhedron N_f ?
- (3) Since there is a homomorphism of one multivalued group to another multivalued group and the kernel of this homomorphism is defined, we can talk about extensions of multivalued groups. Construct a theory of extensions of multivalued groups.
- (4) Is it possible to define (co)homology for multivalued groups?

Acknowledgments. The authors thank V. M. Buchstaber and D. V. Talalaev for interesting discussions and useful suggestions.

References

- [1] Buchstaber, V. M. *n-valued groups: theory and applications*, Moscow Math. J., 6, no. 1 (2006), 57–84.
- [2] V. M. Buchstaber, S. P. Novikov, *Formal groups, power systems and Adams operators*, Russian Mat. Sb. (N.S.), **84 (126)**, 1971, 81–118.

VALERIY GEORGIEVICH BARDAKOV
 SOBOLEV INSTITUTE OF MATHEMATICS,
 PR. KOPTYUGA, 4,
 630090, NOVOSIBIRSK, RUSSIA
 NOVOSIBIRSK STATE AGRARIAN UNIVERSITY
 DOBROLYUBOVA STREET, 160 NOVOSIBIRSK 630039, RUSSIA
Email address: bardakov@math.nsc.ru

TATYANA ANATOLEVNA KOZLOVSKAYA
 REGIONAL SCIENTIFIC AND EDUCATIONAL MATHEMATICAL CENTER OF TOMSK STATE
 UNIVERSITY,
 36 LENIN AVE.,
 634050, TOMSK, RUSSIA
Email address: t.kozlovskaya@math.tsu.ru