On Weakly Tripotent and Locally Invo-Regular Rings

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ABSTRACT

In this article some important observations have been reported on recent works related to weakly tripotent rings and locally invo-regular rings. We exhibit that if A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ does not necessarily hold for each $u \in J(A)$. The same observation has been reported for locally invo-regular rings also. These findings improve some recent results appeared in Rendiconti Sem. Mat. Univ. Pol. Torino (2021) and Azerbaijan Journal of Mathematics (2021).

Key-words: tripotent ring, weakly tripotent ring, locally invo-regular rings .

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1. Introduction

In this paper A is a unital and associative ring and $J(A)$ and $U(A)$ stand for the Jacobson radical of A and the set of units in A respectively. We recall that a ring A is said to be a weakly tripotent ring if $u^3 = u$ or $(1-u)^3 = 1-u$ for each $u \in A$ [1-2] and a ring A is said to be a locally invo-regular ring if $u = uvu$ or $1-u = (1-u)v(1-u)$ for each $u \in A$ and some $v \in A$ with $v^2 = 1$ [3].

It may be worth mentioning that weakly tripotent rings, locally ino-regular rings and associated notions have extensively appeared in mathematical literature [1-10]. Motivated by some of our recent works [11-12], here we take an opportunity to describe as well as rectify underlying mathematical discrepancies associated with some results appeared in [2-3].

In [2, Corollary 1] it was noted that if A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $u^2 = 2u = 0$ holds for each $u \in J(A)$. Similarly it was noted in [3, Corollary 1] that if A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $u^2 = 2u = 0$ holds for each $u \in J(A)$.

However we observe that if A is a weakly tripotent ring and it does not have non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$. Similarly we note that if A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Moreover we observe that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents such that $u^2 = 2u = 0$ for each $u \in J(A)$ then $u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid. We exhibit that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$. Thus we rectify the underlying mathematical discrepancies appeared in [2-3].

2. Some Observations

Theorem 2.1: Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.

Proof. Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A. By [1, Corollary 10], we have $u^2 = 1$ for each $u \in U(A)$ and $u^2 = 2u$ for each $u \in J(A)$. It may be noted that if $u \in J(A)$ then $1+u \in U(A)$. Similarly $1-u \in U(A)$. We note that $1+u \in U(A)$ gives that $(1+u)^2 = 1 \Rightarrow u^2 = -2u$ and $1-u \in U(A)$ gives that $(1-u)^2 = 1 \Rightarrow u^2 = 2u$. Hence $u^2 = -2u$ and $u^2 = 2u$ together give that $u^3 = 4u = 0$ for each $u \in J(RA)$.

Theorem 2.2: Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.

Proof. The proof of this Theorem follows from the proof of Proposition 2.1 and the fact that each weakly tripotent ring is a locally invo-regular ring [3].

Proposition 2.3: Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Proof. Let $A = Z_4$ and $G = \{1, g : g^2 = 1\}$. Clearly G is an abelian group under multiplication. Now we shall construct the group ring AG. It may be noted that if $a_i \in A, g_i \in G$ then $u \in AG$ is expressible as $(a_1 g_1 + a_2 g_2 + ... + a_n g_n) \in AG$ [13]. Thus the group ring AG has the following sixteen elements.

0 , 1, 2 , 3, g , $2g$, $3g$, $1+g$, $2+g$, $3+g$, $1+2g$, $2+2g$, $3+2g$, $1+3g$, $2+3g$, $3+3g$.

One may easily note that each element $u \in AG$ satisfies $u^3 = u$ or $(1-u)^3 = 1-u$. Hence AG is a weakly tripotent ring. We note that 0 and 1 are idempotent elements of R and R does not have any other idempotent element. Also 2 is nilpotent in R. We have

$U(A) = \{1, 3, \mathrm{g}, 2+\mathrm{g}, 1+2\mathrm{g}, 3+2\mathrm{g}, 2+3\mathrm{g}\}$ and
$J(A) = \{0, 2, 2\mathrm{g}, 3+\mathrm{g}, 2+2\mathrm{g}, 1+3\mathrm{g}, 3+3\mathrm{g}\}$.

Clearly $3+3g \in J(A)$, but $(3+3g)^2 = 2(3+3g) \neq 0$. Hence the proof is complete.

Proposition 2.4: Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Proof. We prove it as follows. Let us consider the ring A given above (we refer the proof of Proposition 2.3). After some computation one finds that $u = uvu$ or $1-u = (1-u)v(1-v)$ holds for each $u \in A$ and some $v \in A$ with $v^2 = 1$. Therefore A is a locally invo-regular ring.

We have already noted that 2 is nilpotent in A and A is has no non-trivial idempotent elements. Further $1+u \in J(A)$ such that $(1+u)^2 = 2(1+u) \neq 0$. Hence the proof is complete.

Proposition 2.5: Let A is a weakly tripotent ring having no non-trivial idempotents then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid.

Proof. Let A is a weakly tripotent ring such that it has no non-trivial idempotents. Let $u^2 = 2u = 0$ for each $u \in J(A)$. This gives that $u^3 = 2u^2 = 0$. This in turn implies that $u^3 = 4u = 0$ for each $u \in J(A)$ The converse is not valid. Let us consider the ring A given in the proof of Proposition 2.3. Clearly $1+u \in J(R)$ such that $(1+u)^3 = 4(1+u) = 0$ but $(1+u)^2 = 2(1+u) \neq 0$.

Proposition 2.6: Let A is a locally invo-regular ring having no non-trivial idempotents then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid.

Proof. The proof is same as the above.

3. Discussion and Conclusion.

We have observed that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$. It may be emphasized that the statement 2 is nilpotent in A implies that $3 \in U(A)$. Similarly we exhibit that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents, then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid. Further we have observed that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$. However it was noted in [1] that if A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^2 = 2u = 0$ for each $u \in J(A)$. Moreover the same result was carry forwarded in [3] as well while considering the locally invo-regular rings. We have taken an opportunity to describe as well as rectify the underlying mathematical discrepancies appeared in existing mathematical literature [2-3]

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