On 3-generated axial algebras of Jordan type 1/2

Ravil Bildanov, Ilya Gorshkov

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Abstract: Axial algebras of Jordan type η are a special type of commutative non-associative algebras. They are generated by idempotents whose adjoint operators have the minimal polynomial dividing $(x-1)x(x-\eta)$, where η is a fixed value that is not equal to 0 or 1. These algebras have restrictive multiplication rules that generalize the Peirce decomposition for idempotents in Jordan algebras.

A universal 3-generated algebra of Jordan type $\frac{1}{2}$ as an algebra with 4 parameters was constructed by I. Gorshkov and A. Staroletov. Depending on the value of the parameter, the universal algebra may contain a non-trivial form radical. In this paper, we describe all semisimple 3-generated algebras of Jordan type $\frac{1}{2}$ over a quadratically closed field.

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Introduction

Axial algebras of Jordan type η were introduced by Hall, Rehren, and Shpectorov [2] within the framework of the general theory of axial algebras. These algebras are commutative non-associative algebras over a field $\mathbb{F}$, generated by special idempotents known as primitive axes. While Jordan algebras generated by primitive idempotents are an example of Jordan type $\frac{1}{2}$ algebras , not all algebras of this type are Jordan algebras. The Matsuo algebras, constructed from the group of 3-transpositions, are an example of such algebras. It was proved in [2] (with a correction in [3]) that for $\eta \neq 1/2$, algebras of Jordan type η are the Matsuo algebras or their quotient algebras. Therefore, the case $\eta = 1/2$ is special for algebras of Jordan type, and for this η, they are called Jordan type half algebras. The class of Matsuo algebras was introduced by Matsuo [6] and later generalized in [2].

Jordan type half algebras are not exhausted by Matsuo algebras and their quotient algebras. Moreover, the quotient algebras of Matsuo algebras do not contain all Jordan algebras generated by primitive idempotents. For example, the 27-dimensional Albert algebra is generated by 4 primitive idempotents and hence it is a Jordan type half algebra but not a Matsuo algebra [5].

A universal 3-generated algebra $A(\alpha,\beta,\gamma,\psi)$ of Jordan type half was constructed in [1], where α,β,γ,ψ are parameters. It is proved there that if $(\alpha+\beta+\gamma-2\psi-1)(\alpha\beta\gamma-\psi^2) \neq 0$

and $\psi^2 - \alpha\beta\gamma$ is a square in $\mathbb{F}$, then $A(\alpha,\beta,\gamma,\psi)$ is isomorphic to the matrix algebra $M_3^+(\mathbb{F})$ of 3×3 matrices with Jordan multiplication. Otherwise, the algebra $A(\alpha,\beta,\gamma,\psi)$ is not simple.

A Frobenius form $(\ ,\ )$ on A is a nonzero symmetric bilinear form that associates with multiplication in A, i.e., $\forall a,b,c \in A$, we have $(ab,c) = (ac,b)$. Hall, Rehren, and Shpectorov [2] showed that for Jordan type algebras, there exists a unique Frobenius form with the property $(a,a) = 1$ for any primitive axis a.

Let A be an algebra with a Frobenius form $(\ ,\ )$. The radical of the form $(\ ,\ )$ is an ideal $R(A)$ generated by elements x such that $(x,a) = (a,x) = 0$ for any element $a \in A$.

The purpose of this article is to describe all 3-generated algebras of Jordan type half with trivial radical.

1 Preliminary results

We consider commutative non-associative algebras over a ground field $\mathbb{F}$ of characteristic not two. For definitions, we almost always follow [2] and [4].

By $L\langle X\rangle$ denote the linear span of the set X over $\mathbb{F}$, and by $\langle\langle X\rangle\rangle$ denote the algebra generated by the set X.

Definition 1. *Given $a \in A$ and $\lambda \in \mathbb{F}$, consider the subspace $A_\lambda(a) = \{u \in A \mid au = \lambda u\}$.*

Obviously, $A_\lambda(a)$ is an eigenspace of the operator $ad_a : x \to ax$, associated with $\lambda \in \mathbb{F}$.

Definition 2. *An idempotent $a \in A$ is said to be primitive if $\dim(A_1(a)) = 1$.*

Definition 3. *An algebra A is an algebra of Jordan type half if A is generated by the set of primitive idempotents X such that for every $x \in X$, we have a decomposition $A = A_0(x) \oplus A_1(x) \oplus A_{1/2}(x)$ with the following fusion (multiplication) rules:*

$$A_0(a)A_{1/2}(a) \subseteq A_{1/2}(a), A_1(a)A_{1/2}(a) \subseteq A_{1/2}(a), A_0(a)A_1(a) \subseteq \{0\}$$

$$A_0^2(a) \subseteq A_0(a), A_1^2(a) \subseteq A_1(a), A_{1/2}^2(a) \subseteq A_0 \oplus A_1.$$

Such idempotents are called axes. By an n-generated algebra we mean an algebra generated by n primitive axes.

Let us introduce some classes of simple Jordan algebras.

Definition 4. *Denote by $M_n^+(\mathbb{F})$ the matrix algebra $M_n(\mathbb{F})$ with Jordan product $A \circ B = \frac{1}{2}(AB + BA)$.*

Definition 5. *If j is an involution of $M_n(\mathbb{F})$, then define the Hermitian Jordan algebra $H(M_n(\mathbb{F}), j)$ as $\{A \in M_n^+(\mathbb{F}) \mid j(A) = A\}$.*

Definition 6. *Define the Jordan form algebra $JForm_n(\mathbb{F})$ on $\mathbb{F} \oplus V$ over an arbitrary field $\mathbb{F}$ and vector space V of dimension n over $\mathbb{F}$ with bilinear form ϕ, with the product*

$$(a \oplus \mathbf{v}) \bullet (b \oplus \mathbf{w}) = (ab + \phi(\mathbf{v},\mathbf{w})) \oplus (a\mathbf{w} + b\mathbf{v}), \text{ where } a,b \in \mathbb{F} \text{ and } \mathbf{v},\mathbf{w} \in V.$$

It is well known that $M_n^+(\mathbb{F})$, $H_n^+(\mathbb{F})$, $JForm_n(\mathbb{F})$ for $n \geq 2$ are simple Jordan algebras generated by primitive idempotents, so they are algebras of Jordan type half.

Lemma 1. *[3, Theorem 4.1.] Every algebra of Jordan type η admits a unique Frobenius form which satisfies the property $(a, a) = 1$ for all axes $a \in X$.*

Lemma 2. *[2, Proposition 2.7.] The radical of Frobenius form $R(A)$ coincides with the largest ideal of A containing no axes from A.*

Lemma 3. *Let A be an algebra of Jordan type η. Then for all $a, b \in A$ and their images $\overline{a}, \overline{b} \in A/R(A)$, $(a, b) = (\overline{a}, \overline{b})$.*

Proof. Let $a = \overline{a} + r_a, b = \overline{b} + r_b$, where $\overline{a}, \overline{b} \in A/R(A), r_a, r_b \in R(A)$. Then $(a, b) = (\overline{a} + r_a, \overline{b} + r_b) = (\overline{a}, \overline{b}) + (\overline{a}, r_b) + (\overline{b}, r_a) + (r_a, r_b) = (\overline{a}, \overline{b})$. □

Lemma 4. *[4, Lemma 1.] Let A be a finitely generated algebra of Jordan type half, a, b are axes, $\alpha = (a, b)$. Then we have the following equalities:*

1. $a_0^2(b) = (1 - \alpha)a_0(b)$;
2. $a_{1/2}^2(b) = \alpha a_0(b) + (\alpha - \alpha^2)a$;
3. $a_0(b)a_{1/2}(b) = \frac{1}{2}(1 - \alpha)a_{1/2}(b)$.

Lemma 5. *Let $A = \langle\langle a, b\rangle\rangle$ be a 2-generated algebra of Jordan type half. Then one of the following holds:*

1. $\dim(A) = 1$, $(a, b) = 1$, $a = b$;
2. $\dim(A) = 2$, $(a, b) = 0$, $A \cong \mathbb{F} \oplus \mathbb{F}$;
3. $\dim(A) = 2$, $(a, b) = 1$, $\dim(R(A)) = 1$;
4. $\dim(A) = 3$, $(a, b) = 0$, $\dim(R(A)) = 1$, $A/R(A) \cong \mathbb{F} \oplus \mathbb{F}$;
5. $\dim(A) = 3$, $(a, b) = 1$, $\dim(R(A)) = 2$;
6. $\dim(A) = 3$, $(a, b) \neq 0, 1$, *and A is a Matsuo algebra. In particular, it is a simple Jordan algebra isomorphic to* $JForm_2(\mathbb{F})$.

Proof. The assertion of the lemma is a simple consequence of [3, Proposition 1]. □

Lemma 6. *[4, Corollary 1] Let A be a 2-generated algebra of Jordan type half with generating axes a and b. Denote $\alpha = (a, b)$. Then we have*

1. $a(ab) = \frac{1}{2}(\alpha a + ab)$;
2. $(ab)b = \frac{1}{2}(\alpha b + ab)$;
3. $(ab)(ab) = \frac{\alpha}{4}(a + b + 2ab)$.

Lemma 7. *[1, Main theorem] There exists a 3-generated 9-dimensional algebra $A(\alpha, \beta, \gamma, \psi)$ such that each 3-generated algebra of Jordan type half is a quotient algebra of this algebra for suitable values of parameters.*

Let $A = \langle\langle a, b, c\rangle\rangle$, $\dim(A) = 9$, $\alpha = (a, b)$, $\beta = (b, c)$, $\gamma = (a, c)$, $\psi = (ab, c)$. In the table 1 below (that is similar to [1, Table 6] up to renumbering rows), we present all possible relations for $\alpha, \beta, \gamma, \psi$ for $A(\alpha, \beta, \gamma, \psi)$ to not be simple.

Number	Relations	$\dim(A/R(A))$	Basis of the radical
1	$\psi = \alpha = \beta = \gamma = 1$	1	$b - a, c - a, ab - a, bc - a, ac - a,$ $a(bc) - a, b(ac) - a, c(ab) - a$
2	$\psi = \alpha = \beta = 0, \gamma = 1$	2	$c - a, ab, bc, ac - a,$ $a(bc), b(ac), c(ab)$
3	$\psi = \alpha = \beta = \gamma = 0$	3	$ab, bc, ac, a(bc), b(ac), c(ab)$
4	$\psi = \alpha = 0, \beta, \gamma \neq 0,$ $\beta + \gamma = 1$	3	$ab, \frac{1}{2}\gamma a - \frac{1}{2}\beta b - \frac{1}{2}c + bc,$ $-\frac{1}{2}\gamma a + \frac{1}{2}\beta b - \frac{1}{2}c + ac,$ $\frac{1}{4}\gamma a + \frac{1}{4}\beta b - \frac{1}{4}c + a(bc),$ $\frac{1}{4}\gamma a + \frac{1}{4}\beta b - \frac{1}{4}c + b(ac), c(ab)$
5	$\alpha\beta\gamma = \psi^2, \psi \neq 0, \alpha \neq 1,$ $\alpha + \beta + \gamma = 2\psi + 1$	3	$\alpha(\beta - 1)a + \alpha(\gamma - 1)b + \alpha(1 - \alpha)c + (2\alpha - 2\psi)ab,$ $(\alpha\beta - \alpha\psi)b + (\psi - \alpha\beta)ab + (\alpha^2 - \alpha)bc,$ $(\alpha\gamma - \alpha\psi)a + (\psi - \alpha\gamma)ab + (\alpha^2 - \alpha)ac,$ $(\alpha\psi - \alpha^2\beta)a + (\alpha + \psi - \alpha^2 - \alpha\gamma)ab + 2\alpha(\alpha - 1)a(bc),$ $\alpha(\psi - \alpha\gamma)b + (\alpha + \psi - \alpha^2 - \alpha\beta)ab + 2\alpha(\alpha - 1)b(ac),$ $(\psi - \alpha\beta)a + (\psi - \alpha\gamma)b + (1 - \alpha)ab + 2(\alpha - 1)c(ab)$
6	$\psi = \alpha = \beta = 0, \gamma \neq 0, 1$	4	$ab, bc, ac, a(bc), b(ac), c(ab)$
7	$\psi^2 \neq \alpha\beta\gamma,$ $\alpha + \beta + \gamma = 2\psi + 1,$ $\alpha \neq 1$	4	$\frac{1}{2}(\beta - 1)a + \frac{1}{2}(\beta - \alpha)b + \frac{1}{2}(1 - \alpha)c + (1 - \beta)ab + (\alpha - 1)bc,$ $\frac{1}{2}(\gamma - \alpha)a + \frac{1}{2}(\gamma - 1)b + \frac{1}{2}(1 - \alpha)c + (1 - \gamma)ab + (\alpha - 1)ac,$ $(2\psi - 2\alpha\beta + \beta - 1)a + (\gamma - 1)b + (1 - \alpha)c + (4 - 2\alpha - 2\gamma)ab + (4\alpha - 4)a(bc),$ $(\beta - 1)a + (2\psi - 2\alpha\gamma + \gamma - 1)b + (1 - \alpha)c + (4 - 2\alpha - 2\beta)ab + (4\alpha - 4)b(ac),$ $(\psi - \alpha)a + (\psi - \alpha)b + \alpha(1 - \alpha)c + (2 - \beta - \gamma)ab + (2\alpha - 2)c(ab)$
8	$\psi = \alpha = 0, \beta, \gamma \neq 0,$ $\beta + \gamma \neq 1$	6	$ab, b(ac) - a(bc), c(ab)$
9	$\alpha\beta\gamma = \psi^2, \psi \neq 0,$ $\alpha + \beta + \gamma \neq 2\psi + 1$	6	$-\beta\gamma ab - \alpha\beta ac + 2\psi a(bc),$ $-\beta\gamma ab - \alpha\gamma bc + 2\psi b(ac),$ $-\alpha\gamma bc - \alpha\beta ac + 2\psi c(ab)$

Table 1: Bases of the radical

2 Quotient algebras

In this section we describe 3-generated algebras of Jordan type half over a quadratically closed field F with trivial radical and prove the following theorem.

Theorem 1. *Let A be a 3-generated algebra of Jordan type half with trivial radical over a ground quadratically closed field $\mathbb{F}$ with characteristic not equal to two or three. Then A is isomorphic to one of the following algebras:*

1. $\mathbb{F}^n, n \in \{1, 2, 3\}$;
2. $JForm_2(\mathbb{F})$;

3. $\mathbb{F} \oplus JForm_2(\mathbb{F})$*;*

4. $M_2^+(\mathbb{F})$*;*

5. $H(M_3(\mathbb{F}), j)$ *with* $j(X) = X^T$*;*

6. $M_3^+(\mathbb{F})$*.*

It follows from Lemma 7 that we need to describe the quotient algebras of the algebra $A(\alpha, \beta, \gamma, \psi)$ by its radical. We use the description of the algebra $A(\alpha, \beta, \gamma, \psi)$ from [1, Theorem 2].

Following [1], denote $\alpha = (a, b), \beta = (b, c), \gamma = (a, c), \psi = (ab, c)$.

In [1, Table 6], one can find the dimensions and bases of the radicals of the algebra $A(\alpha, \beta, \gamma, \psi)$. Denote by A_i the universal 9-dimensional algebra $A(\alpha_i, \beta_i, \gamma_i, \psi_i)$ with parameters and numeration from Table 1, R_i the radical of this algebra and by S_i the quotient algebra A_i/R_i.

We begin with two trivial propositions for 1-dimensional and 2-dimensional algebras, which are not generated by three linearly independent axes.

Proposition 1. *If A is a 1-dimensional algebra of Jordan type half with trivial radical, then $A \cong S_1$.*

Proof. It is easy to see that $S_1 \cong \mathbb{F}$. We have that A is 1-dimensional, so $\dim L\langle a, b, c\rangle = 1$ and $a = b = c$. Hence $A \cong \mathbb{F} \cong S_1$. □

Proposition 2. *If A is a 2-dimensional 3-generated algebra of Jordan type half with trivial radical, then $A \cong \mathbb{F} \oplus \mathbb{F} \cong S_2$.*

Proof. By Lemma 5, there is only one 2-dimensional algebra of Jordan type half with trivial radical, so $A \cong \mathbb{F} \oplus \mathbb{F} \cong S_2$. □

Proposition 3. *If A is a 3-dimensional 3-generated algebra of Jordan type half with trivial radical, then A is isomorphic to either S_3 or S_5.*

Proof. Assume that A is generated by axes a and b. From Lemma 5, it follows that there is only one 3-dimensional 2-generated Jordan type half algebra with trivial radical. In this case, we can choose any other axis of the algebra A as the axis c. Put $c = a^{\tau_b} = a - 4ab + 4\alpha b$. We have $\beta = \alpha$, $\gamma = (1-2\alpha)^2$ and $\psi = \alpha(2\alpha - 1)$. Therefore $\alpha\beta\gamma = \psi^2$, $\psi \neq 0, \alpha \neq 1$, $\alpha + \beta + \gamma = 2\psi + 1$. So in this case $A \simeq S_5$.

Assume that A is not generated by 2 axes. Therefore, based on the dimension of A, we conclude that A is the linear span of the axes a, b, and c.

Assume that $ab \notin L\langle a, b\rangle$. Hence $\dim\langle\langle a, b\rangle\rangle = 3$. Therefore $c \in L\langle a, b, ab\rangle = A$, which is a contradiction. Similarly, we can show that if $ac \in L\langle a, c\rangle$ and $bc \in L\langle b, c\rangle$. In particular, we have $\dim(\langle\langle a, b\rangle\rangle) = \dim(\langle\langle a, c\rangle\rangle) = \dim(\langle\langle c, b\rangle\rangle) = 2$. From Lemma 5 it follows that $\{(a, b), (a, c), (b, c)\} \subseteq \{0, 1\}$. Moreover, if $(a, b) = 0$, then $\langle\langle a, b\rangle\rangle \simeq \mathbb{F} \oplus \mathbb{F}$. Therefore, if $(a, b) = (a, c) = (b, c) = 0$ then $A \simeq \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $\psi = 0$. In this case the Gram matrix of the algebra A is the identity matrix and hence the radical of A is trivial. We conclude that in this case $A \simeq S_3$.

Assume that $(a, c) \neq 0$. We have $(a, c) = 1$. In this case, $R(\langle\langle a, c\rangle\rangle)$ is not trivial and contains the element $a - c$. Assume that $(a, b) = (b, c) = 0$. In this case we have $(a - c, b) = 0$. Consequently $a - c \in R(A)$, which is a contradiction. Therefore, without loss of generality, we can assume that $(b, c) = 1$. If $(a, b) = 1$ then $(a - c, b) = 0$ and consequently $a - c \in R(A)$, which is a contradiction. Therefore $(a, b) = 0$. From the description of 2-generated algebras of Jordan type half we have $ab = 0$, $a = c + a_h$, $b = c + b_h$ where $a_h, b_h \in A_{1/2}(c)$. Therefore $0 = ab = (c + a_h)(c + b_h) = c + 1/2(a_h + b_h) + a_h b_h$, where $c + a_h b_h \in A_{0+1}(c)$ and $a_h + b_h \in A_{1/2}(c)$. Therefore $a_h + b_h = 0$. In particular, $b = a^{\tau_c}$ and $\dim(A) = 2$. □

Proposition 4. *Algebras S_4 and S_5 are isomorphic.*

Proof. We will first show that $S_4 = \langle\langle a, c\rangle\rangle$. Put $S = \langle\langle a, c\rangle\rangle$. We have that $S = S_0(a) + S_1(a) + S_{1/2}(a)$ and $c = c_0(a) + \gamma a + c_{1/2}(a)$, where $c_0(a) \in S_0(a)$, $c_{1/2}(a) \in S_{1/2}(a)$. If $c_{1/2} = 0$ then we can see that c is not the primitive idempotent. Thus set $c_{1/2} \neq 0$. Assume that $c_0 = 0$. Therefore, $\gamma a + c_{1/2} = c = c^2 = \gamma^2 a + \gamma c_{1/2} + c_{1/2}^2$. Hence $\gamma = 1$ and from definition of S_4 it follows that $\beta = 0$. In this case $(a - c, b) = (a, b) - (c, b) = 0$. It follows that $a - c \in R(S_4)$, which is a contradiction. Therefore, $\dim(S) = 3$. Thus, $S_4 = S$. Hence S_4 is generated by 2 axes and is isomorphic to S_5. □

It is known that A is isomorphic to $JForm_2(\mathbf{F})$ in this case.

Proposition 5. *If A is a 4-dimensional 3-generated Jordan type half algebra with trivial radical, then one of the following assertions holds:*

1. $A \simeq S_6 \simeq \mathbb{F} \oplus JForm_2(\mathbb{F})$;
2. $A \simeq S_7 \simeq M_2^+(\mathbb{F})$.

Proof. The algebra $M_2^+(\mathbb{F})$ is a simple Jordan algebra. Algebra $\mathbb{F} \oplus JForm_2(\mathbb{F})$ contains non-trivial ideals. Therefore $M_2^+(\mathbb{F}) \not\simeq \mathbb{F} \oplus JForm_2(\mathbb{F})$. Hence, to prove this proposition, it suffices to show that $S_6 \simeq \mathbb{F} \oplus JForm_2(\mathbb{F})$ and $S_7 \simeq M_2^+(\mathbb{F})$. □

Lemma 8. *S_6 is isomorphic to $\mathbb{F} \oplus JForm_2(\mathbb{F})$.*

Proof. Let $\langle\langle a, b, c\rangle\rangle \simeq S_6$. We have $(a, c) \notin \{0, 1\}$. Therefore $\langle\langle a, c\rangle\rangle$ is isomorphic to $JForm_2(\mathbb{F})$. From Table 1, it follows that the radical of $A(0, 0, \gamma, 0)$ contains ab and bc. Therefore $ab = bc = 0$ and $S_6 \simeq \langle\langle a, c\rangle\rangle \oplus \langle\langle b\rangle\rangle \simeq \mathbb{F} \oplus JForm_2(\mathbb{F})$.

□

Lemma 9. *S_7 is isomorphic to $M_2^+(\mathbb{F})$.*

Proof. Let

$$A=\begin{pmatrix}1 & \lambda_a\\ 0 & 0\end{pmatrix} B=\begin{pmatrix}1 & 0\\ \lambda_b & 0\end{pmatrix}, C=\begin{pmatrix}1-\lambda_c & 1\\ \lambda_c(1-\lambda_c) & \lambda_c\end{pmatrix}$$

where $\lambda_a, \lambda_b, \lambda_c \in \mathbb{F}\setminus\{0\}$. Consider the following map $f : S_7 \to M_2^+(\mathbb{F}), f(a) = A, f(b) = B, f(c) = C$. It is easy to see that $\dim L\langle A, B, C, A\circ B\rangle = 4$, so $\langle\langle A, B, C\rangle\rangle = M_2^+(\mathbb{F})$.

A map $(\cdot\,,\cdot) : M_2^+(\mathbb{F})^2 \to \mathbb{F}$ such that $(X, Y) = tr(XY) = tr(X\circ Y)$, where $X, Y \in M_2^+(\mathbb{F})$, is a symmetric bilinear form on $M_2^+(\mathbb{F})$. This form associates with the product $\circ$. Clearly, we have $tr(A\circ A) = tr(B\circ B) = tr(C\circ C) = 1$.
Furthermore, we see that $tr(A\circ B) = 1+\lambda_a\lambda_b = \alpha, tr(B\circ C) = 1-\lambda_c+\lambda_b\lambda_c-\lambda_b\lambda_c^2 = \beta, tr(A\circ C) = 1+\lambda_a-\lambda_c = \gamma$ and $tr(A\circ(B\circ C)) = tr(B\circ(A\circ C)) = tr(C\circ(A\circ B)) = \psi = \frac{1}{2}(1-\alpha-\beta-\gamma)$. So we can take

$$\lambda_a = \frac{1}{\alpha}(\psi+\alpha\gamma\pm\sqrt{\psi^2-\alpha\beta\gamma}), \lambda_b = \frac{1}{\gamma(\gamma-1)}(\psi+\alpha\gamma\mp\sqrt{\psi^2-\alpha\beta\gamma}),$$

$$\lambda_c = \pm\frac{1}{\alpha}(\psi+\alpha+\sqrt{\psi^2-\alpha\beta\gamma}),$$

as the solution of these equations.

Using computer calculations, we show that multiplication table for $f(\langle\langle a, b, c\rangle\rangle)$ [1] coincides with multiplication table for S_7. Hence, f is an isomorphism.

We also use computer calculations to check that $R(f(\langle\langle a, b, c\rangle\rangle)) = \{0\}$ and relations between $\alpha, \beta, \gamma, \psi$ hold. [2]. □

Proposition 6. *If A is a 6-dimensional 3-generated algebra of Jordan type half with trivial radical, then $A \simeq S_8 \simeq S_9 \simeq H(M_3(\mathbb{F}), j)$, where $j(X) = X^T$.*

Lemma 10. $S_8 \cong H(M_3(\mathbb{F}), j)$

Proof. Consider the following matrices in $H(M_3(\mathbb{F}), j)$ and map $f : S_8 \to H(M_3(\mathbb{F}), j)$, $f(a) = A, f(b) = B, f(c) = C$, $\lambda_a, \lambda_b, \lambda_c \in \mathbb{F}\setminus\{0\}$ are the invariant by θ parameters which are defined later from conditions to α, β, γ and ψ.

$$A=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} B=\begin{pmatrix}0&0&0\\0&\frac{1+\sqrt{1-4\lambda_b^2}}{2}&\lambda_b\\0&\lambda_b&\frac{1-\sqrt{1-4\lambda_b^2}}{2}\end{pmatrix} C=\begin{pmatrix}\frac{1+\sqrt{1-4\lambda_c^2}}{2}&0&\lambda_c\\0&0&0\\\lambda_c&0&\frac{1-\sqrt{1-4\lambda_c^2}}{2}\end{pmatrix}$$

Below we show that the mapping f is an isomorphism between the algebras S_8 and $H(M_3(\mathbb{F}), j)$.

It is easy to see that $A^2 = A, B^2 = B, C^2 = C$. We check that $f(\langle\langle a, b, c\rangle\rangle) = L\langle A, B, C, A\circ C, B\circ C, A\circ(B\circ C)\rangle$. Thus, $\dim L\langle A, B, C, A\circ C, B\circ C, A\circ(B\circ C)\rangle = 6$. Hence $\langle\langle A, B, C\rangle\rangle$ and $H(M_3(\mathbb{F}), j)$ are isomorphic as vector spaces.

[1] Computer calculations for multiplication table in S_7 can be found in `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/S7multiplicationtable.nb`, see paragraph Tables.

[2] One can find our computer calculations here: `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/M2+(S7).nb`

A map $(\cdot\ ,\cdot): H(M_3(\mathbb{F}),j)^2 \to \mathbb{F}$ such that $(X,Y) = tr(XY) = tr(X \circ Y)$, where $X,Y \in H(M_3(\mathbb{F}),j)$ is a symmetric bilinear form on $H(M_3(\mathbb{F}),j)$. This form associates with the product $\circ$. Clearly, we have $tr(A\circ A) = tr(B\circ B) = tr(C\circ C) = 1$.
Furthermore, we see that $tr(A\circ B) = 0, tr(B\circ C) = \frac{1}{4}(1-\sqrt{1-4\lambda_b^2})(1-\sqrt{1-4\lambda_c^2}) = \beta, tr(A\circ C) = \frac{1+\sqrt{1-4\lambda_c^2}}{2} = \gamma$ and $tr(A\circ(B\circ C)) = tr(B\circ(A\circ C)) = tr(C\circ(A\circ B)) = 0$. So, we have conditions to λ_b, λ_c.

Take the basis $a, b, c, b\cdot c, a\cdot c, a\cdot(b\cdot c)$ for S_8. Multiplication table for $f(\langle\langle a,b,c\rangle\rangle)$ [3] coincides with multiplication table for S_8.

We also use computer calculations to check that $R(f(\langle\langle a,b,c\rangle\rangle)) = \{0\}$ and relations between α,β,γ,ψ hold. [4].

We have that $A, B, C, B\circ C, A\circ C, A\circ(B\circ C)$ is a basis of the algebra $H(M_3(\mathbb{F}),j)$ and hence the kernel of f is trivial. Thus f is an isomorphism of the algebras S_8 and $H(M_3(\mathbb{F}),j)$. $\square$

Lemma 11. *Algebras S_8 and S_9 are isomorphic.*

Proof. Assume that $\alpha = \beta = \gamma = 1$, but then $\psi = 1$ and we obtain a contradiction. Let $\alpha \neq 1$, take $d = x_a(b) = \dfrac{2ab - \alpha a - b}{\alpha - 1}$. It is known from [4], that d is a primitive idempotent in S_9 with $ad = 0$ and so d is an axis because S_9 is a Jordan algebra.

Assume that $(c,d) \neq 0$. It can be proved via computer calculations that $a, c, d, ac, cd, a(cd)$ is an additive basis of $B = \langle\langle a,c,d\rangle\rangle$. In particular $B = S_9$. Define a homomorphism f from S_9 to S_8: $f(a) = \bar{a}, f(d) = \bar{b}, f(c) = \bar{c}$. We have $(\bar{a},\bar{b}) = 0$,$(\bar{a},\bar{b}\bar{c}) = 0$,$(\bar{a},\bar{c}) = \bar{\beta} \neq 0$, $(\bar{b},\bar{c}) = \bar{\gamma} \neq 0$. Relation $\bar{\beta} + \bar{\gamma} = \dfrac{2\psi - \beta - \alpha\gamma}{\alpha - 1} + \gamma \neq 1$ is equivalent to $\alpha + \beta + \gamma \neq 2\psi + 1$, so f is an isomorphism between S_8 and S_9 iff $(c,d) \neq 0$.

Now assume that $(c,d) = 0$. We will prove that $\gamma \neq 1$ in this case. We have that

$$0 = (c, (\alpha-1)d) = (c, 2ab - \alpha a - b, c) =$$

$$= (2ab - b, c) - \alpha = (2ab, c) - (b,c) - \alpha = 2\psi - \beta - \alpha$$

If $\gamma = 1$, then $2\psi + 1 = \beta + \alpha + \gamma$, a contradiction. Therefore, $\gamma \neq 1$.

Put $d' = \dfrac{2ac - \gamma a - c}{\gamma - 1}$. Note that $(b,d') \neq 0$. Indeed, if $(b,d') = 0$, we have

$$0 = (b, 2ac - \gamma a - c) = (b, 2ac) - (b, \gamma a) - (b,c) = 2\psi - \gamma\alpha - \beta$$

Using these equalities, we obtain $\alpha = \gamma\alpha$ and so $\gamma = 1$ or $\alpha = 0$, a contradiction.

We then take a, b, d' as the new generating set of S_9 and, using the same computer calculations, prove that $a, b, x, ab, ax, b(ax)$ form an additive basis of $B' = \langle\langle a,b,d'\rangle\rangle$. Define a homomorphism f' from S_9 to S_8: $f(a) = \bar{a}', f(d') = \bar{b}', f(b) = \bar{c}'$. We have $(\bar{a}',\bar{b}') =$

[3]Computer calculations for the multiplication table in S_8 can be found in `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/S8multiplicationtable.nb`, see paragraph Tables.

[4]Computer calculations for this proof can be found in `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/H3+(S8).nb`

$0,(\bar{a}',\bar{b}'\bar{c}') = 0,(\bar{a}',\bar{c}') = \bar{\beta}' \neq 0$,$(\bar{b}',\bar{c}') = \bar{\gamma}' \neq 0$. Relation $\bar{\beta}' + \bar{\gamma}' = \dfrac{2\psi - \beta - \alpha\gamma}{\gamma - 1} + \alpha \neq 1$ is equivalent to $\alpha + \beta + \gamma \neq 2\psi + 1$, so f' is an isomorphism between S_8 and S_9 iff $(b, d') \neq 0$.

We can also check that the multiplication table for $a, c, d, ac, cd, a(cd)$ coincides with the multiplication table for the standard basis $a, b, c, bc, ac, a(bc)$ of S_8. This means that S_9 contains a 6-dimensional subalgebra isomorphic to S_8.[5].

$\square$

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4 Tables

Here are the multiplication tables for S_7, S_8.

$*$	a	b	c	ab
a	a	$*$	$*$	$*$
b	ab	b	$*$	$*$
c	$\frac{1}{2(\alpha-1)}((\gamma-\alpha)a + (\gamma-1)b$ $+(1-\alpha)c + 2(-\gamma+1)ab)$	$\frac{1}{2(\alpha-1)}((\beta-1)a + (\beta-\alpha)b$ $+(1-\alpha)c + 2(-\beta+1)ab)$	c	$*$
ab	$\frac{1}{2}(a\alpha + ab)$	$\frac{1}{2}(b\alpha + ab)$	$\frac{1}{2(\alpha-1)}((\psi-\alpha)a+$ $+(\psi-\alpha)b + (\alpha-\alpha^2)c+$ $+(2-\beta-\gamma)ab)$	$\frac{1}{4}\alpha(a+$ $+b+2ab)$

Table 2: Multiplication table for S_7

$*$	a	b	c	bc	ac	$a(bc)$
a	a	$*$	$*$	$*$	$*$	$*$
b	0	b	$*$	$*$	$*$	$*$
c	ac	bc	c	$*$	$*$	$*$
bc	$a(bc)$	$\frac{1}{2}(b\beta + bc)$	$\frac{1}{2}(c\beta + bc)$	$\frac{\beta}{4}((b+c+2bc))$	$*$	$*$
ac	$\frac{1}{2}(a\gamma + ac)$	$a(bc)$	$\frac{1}{2}(c\gamma + ac)$	$\frac{\gamma}{4}bc + \frac{\beta}{4}ac + \frac{1}{2}a(bc))$	$\frac{1}{4}\gamma(a+c+2ac)$	$*$
$a(bc)$	0	$\frac{1}{4}(\beta ac + 2a(bc))$	$\frac{1}{4}(\gamma bc + \beta ac)$	$\frac{\beta\gamma}{8}b + \frac{\beta}{8}ac + \frac{\beta}{4}a(bc)$	$\frac{\beta\gamma}{8}a + \frac{\gamma}{8}bc + \frac{\gamma}{4}a(bc)$	$\frac{\beta\gamma}{16}a+$ $+\frac{\beta\gamma}{16}b$

Table 3: Multiplication table for S_8

[5]One can see our computer calculations here: `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/3genaxialalgebra.nb`, see section "Isomorphism between S_8 and S_9".

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Ravil Bildanov, Sobolev Institute of Mathematics, Novosibirsk, Russia;
Novosibirsk State University, Novosibirsk, Russia;
E-mail address: ravilbildanov@gmail.com

Ilya Gorshkov, Sobolev Institute of Mathematics, Novosibirsk, Russia;
Novosibirsk State Technical University, Novosibirsk, Russia;
E-mail address: ilygor8@gmail.com