# On 3-generated axial algebras of Jordan type half

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*Abstract: Axial algebras of Jordan type $\eta$ are a special type of commutative non-associative algebras. They are generated by idempotents whose adjoint operators have the minimal polynomial dividing $(x - 1)x(x - \eta)$, where $\eta$ is a fixed value that is not equal to $0$ or $1$. These algebras have restrictive multiplication rules that generalize the Pierce decomposition for idempotents in Jordan algebras. Axial algebras of Jordan type $1/2$ we will call Jordan type half algebras. In fact, Jordan algebras generated by idempotents are an Jordan type half algebras.*

*An universal 3-generated Jordan type half algebra as an algebra with 4 parameters was construsted by I. Gorshkov and A. Staroletov. Depending on the value of the parameter, the universal algebra may contain a non-trivial form radical. In the present paper, all semisimple 3-generated Jordan type half algebras have been described.*

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## Introduction

Axial algebras of Jordan type $\eta$ were introduced by Hall, Rehren and Shpektorov [2] within the framework of the general theory of axial algebras. These algebras are commutative non-associative algebras over a field $\mathbb{F}$, generated by a set of special idempotents known as primitive axes. While Jordan algebras generated by primitive idempotents are an example of Jordan type half algebras, not all algebras of this type are Jordan algebras. The Matsuo algebras constructed from the group of 3-transpositions are an example of such algebras. It was proved in [2] (with a correction in [3]) that, for $\eta \neq 1/2$, algebras of Jordan type $\eta$ are the Matsuo algebras or their factor algebras. Therefore, the case $\eta = 1/2$ is special for algebras of Jordan type and for this $\eta$ they are called algebras of Jordan type half. The class of Matsuo algebras was introduced by Matsuo [6] and later generalized in [2].

Jordan type half algebras are not exhausted by Matsuo algebras and their factors. Moreover, the factor algebras of Matsuo algebras do not contain all Jordan algebras generated by primitive idempotents. So, for example, the 27-dimensional Albert algebra is generated by 4 primitive idempotents and hence it is a Jordan type half algebra but not a Matsuo algebra [5].

An universal 3-generated algebra $A(\alpha, \beta, \gamma, \phi)$ of Jordan type half was constructed in [1], where $\alpha, \beta, \gamma, \phi$ are parameters. There it is proved that if $(\alpha + \beta + \gamma - 2\phi - 1)(\alpha\beta\gamma - \phi^2) \neq 0$ and $\phi^2 - \alpha\beta\gamma$ is a square in $\mathbb{F}$, then $A(\alpha, \beta, \gamma, \phi)$ is isomorphic to the matrix algebra $M_3^+(\mathbb{F})$ of $3 \times 3$ matrices with Jordan multiplication. Otherwise, the algebra $A(\alpha, \beta, \gamma, \phi)$ is not simple.

A Frobenius form ( , ) on $A$ is a nonzero symmetric bilinear form, which associates with multiplication in $A$ i.e. $\forall a, b, c \in A$ we have $(ab, c) = (ac, b)$. Hall, Rehren and Shpectorov [2] showed that in Jordan type algebras there exists a unique Frobenius form with the property $(a, a) = 1$ for any primitive axis $a$.

Let $A$ be an algebra with a Frobenius form ( , ). The radical of the form ( , ) is an ideal $R(A)$ generated by elements $x$ such that $(x, a) = (a, x) = 0$ for any element $a \in A$.

The purpose of this article is to describe all 3-generated algebras of Jordan type half with trivial radical.

# 1 Preliminary results

We consider commutative (but usually non-associative) algebras over a ground field $\mathbb{F}$ with characteristic not two. For the definitions and preliminary results, we will almost always follow [2] and [4].

Let's denote as $L\langle X \rangle$ the linear span of the set $X$ over $F$, as $\langle\langle X \rangle\rangle$ the algebra generated by the set $X$, as $\{a, b, c\}$ the set which contains $a, b, c$.

**Definition 1.** *Given an element $a \in A$ and $\lambda \in \mathbb{F}$, we introduce the subspace*
$A_\lambda(a) = \{u \in A \mid au = \lambda u\}$.

Obviously, $A_\lambda(a)$ is an eigenspace of operator $ad_a : x \to ax$, associated with $\lambda \in \mathbb{F}$.

**Definition 2.** *An idempotent $a \in A$ is primitive if $\dim(A_1(a)) = 1$.*

**Definition 3.** *An algebra $A$ is an algebra of Jordan type half, if $A$ is generated by the set of primitive idempotents $X$ such that for every $x \in X$ we have a decomposition $A = A_0(x) \oplus A_1(x) \oplus A_{1/2}(x)$ with following fusion (multiplication) rules:*

$$A_0(a)A_{1/2}(a) \subseteq A_{1/2}(a), A_1(a)A_{1/2}(a) \subseteq A_{1/2}(a), A_0(a)A_1(a) \subseteq \{0\}$$

$$A_0^2(a) \subseteq A_0(a), A_1^2(a) \subseteq A_1(a), A_{1/2}^2(a) \subseteq A_0 \oplus A_1$$

.

Let us introduce some classes of simple Jordan algebras.

**Definition 4.** *Denote by $M_n^+(\mathbb{F})$ the matrix algebra $M_n(\mathbb{F})$ with Jordan product*
$A \circ B = \frac{1}{2}(AB + BA)$.

**Definition 5.** *If $\mathbb{F}$ has an involution $\theta \in Aut(\mathbb{F})$, then denote by*
$H_n^+(\mathbb{F}) = \{A \in M_n^+(\mathbb{F}) \mid \theta(A)^T = A\}$ *the Hermitian Jordan algebra.*

**Definition 6.** *We define an algebra $JForm_n(\mathbb{F})$ on $\mathbb{F} \oplus V$ over an arbitrary field $\mathbb{F}$ and vector space $V$ of dimension $n$ over $\mathbb{F}$ with bilinear form $\phi$ with the product*

$$(a \oplus \mathbf{v}) \bullet (b \oplus \mathbf{w}) = (ab + \phi(\mathbf{v}, \mathbf{w})) \oplus (a\mathbf{w} + b\mathbf{v}), \ \ where \ a, b \in \mathbb{F} \ and \ \mathbf{v}, \mathbf{w} \in V.$$

It is well known that $M_n^+(\mathbb{F})$, $H_n^+(\mathbb{F})$, $JForm_n(\mathbb{F})$ are simple Jordan algebras generated by primitive idempotents, so they are algebras of Jordan type half.

**Lemma 1.** *(Theorem 4.1. [3]) Every algebra of Jordan type $\eta$ admits a unique Frobenius form which satisfies the property $(a, a) = 1$ for all axes $a \in X$.*

**Lemma 2.** *(Proposition 2.7. [2]) The radical of Frobenius form $R(A)$ coincides with the largest ideal of $A$ containing no axes from $A$.*

**Lemma 3.** *Let $A$ be an algebra of Jordan type $\eta$. Then for all $a, b \in A$ and their images $\bar{a}, \bar{b} \in A/R(A)$ $(a, b) = (\bar{a}, \bar{b})$.*

*Proof.* Let $a = \bar{a} + r_a, b = \bar{b} + r_b$, where $\bar{a}, \bar{b} \in A/R(A), r_a, r_b \in R(A)$. Then $(a, b) = (\bar{a} + r_a, \bar{b} + r_b) = (\bar{a}, \bar{b}) + (\bar{a}, r_b) + (\bar{b}, r_a) + (r_a, r_b) = (\bar{a}, \bar{b})$. $\qquad\square$

**Lemma 4.** *(Lemma 1. [4]) Let $A$ be a finitely generated algebra of Jordan type half, $a, b$ are axes, $\alpha = (a, b)$. Then we have the following equalities:*

1. $a_0^2(b) = (1 - \alpha)a_0(b)$

2. $a_{1/2}^2(b) = \alpha a_0(b) + (\alpha - \alpha^2)a$

3. $a_0(b)a_{1/2}(b) = \frac{1}{2}(1 - \alpha)a_{1/2}(b)$

**Lemma 5.** *Let $A = \langle\langle a, b \rangle\rangle$ be a 2-generated algebra of Jordan type half. Then it is one of following:*

1. $\dim(A) = 1, (a, b) = 1, a = b$

2. $\dim(A) = 2, (a, b) = 0, A \cong \mathbb{F} \oplus \mathbb{F}$

3. $\dim(A) = 2, (a, b) = 1, \dim(R(A)) = 1$

4. $\dim(A) = 3, (a, b) = 0, \dim(R(A)) = 1, A/R(A) \cong \mathbb{F} \oplus \mathbb{F}$

5. $\dim(A) = 3, (a, b) = 1, \dim(R(A)) = 2$

6. $\dim(A) = 3, (a, b) \neq 0, 1$ *and $A$ is a Matsuo algebra.*

*Proof.* The assertion of the lemma is a simple consequence of Proposition 1 [3] $\qquad\square$

**Lemma 6.** *(Corollary 1.[4]) Let $A$ be a 2-generated algebra of Jordan type half with generating axes $a, b$. Denote $\alpha = (a, b)$. Then we have*

1. $a(ab) = \frac{1}{2}(\alpha a + ab)$

| Number | Condition | $dim(A/R(A))$ | Basis of radical |
|---|---|---|---|
| 1 | $\psi = \alpha = \beta = \gamma = 1$ | 1 | $b - a, c - a, ab - a, bc - a, ac - a,$ <br> $a(bc) - a, b(ac) - a, c(ab) - a$ |
| 2 | $\psi = \alpha = \beta = 0, \gamma = 1$ | 2 | $c - a, ab, bc, ac - a,$ <br> $a(bc), b(ac), c(ab)$ |
| 3 | $\psi = \alpha = \beta = \gamma = 0$ | 3 | $ab, bc, ac, a(bc), b(ac), c(ab)$ |
| 4 | $\psi = \alpha = 0, \beta, \gamma \neq 0,$ <br> $\beta + \gamma = 1$ | 3 | $ab, \frac{1}{2}\gamma a - \frac{1}{2}\beta b - \frac{1}{2}c + bc,$ <br> $-\frac{1}{2}\gamma a + \frac{1}{2}\beta b - \frac{1}{2}c + ac,$ <br> $\frac{1}{4}\gamma a + \frac{1}{4}\beta b - \frac{1}{4}c + a(bc),$ <br> $\frac{1}{4}\gamma a + \frac{1}{4}\beta b - \frac{1}{4}c + b(ac), c(ab)$ |
| 5 | $\alpha\beta\gamma = \psi^2, \psi \neq 0, \alpha \neq 1,$ <br> $\alpha + \beta + \gamma = 2\psi + 1$ | 3 | $\alpha(\beta - 1)a + \alpha(\gamma - 1)b + \alpha(1 - \alpha)c + (2\alpha - 2\psi)ab,$ <br> $(\alpha\beta - \alpha\psi)b + (\psi - \alpha\beta)ab + (\alpha^2 - \alpha)bc,$ <br> $(\alpha\gamma - \alpha\psi)a + (\psi - \alpha\gamma)ab + (\alpha^2 - \alpha)ac,$ <br> $(\alpha\psi - \alpha^2\beta)a + (\alpha + \psi - \alpha^2 - \alpha\gamma)ab + 2\alpha(\alpha - 1)a(bc),$ <br> $\alpha(\psi - \alpha\gamma)b + (\alpha + \psi - \alpha^2 - \alpha\beta)ab + 2\alpha(\alpha - 1)b(ac),$ <br> $(\psi - \alpha\beta)a + (\psi - \alpha\gamma)b + (1 - \alpha)ab + 2(\alpha - 1)c(ab)$ |
| 6 | $\psi = \alpha = \beta = 0, \gamma \neq 0,1$ | 4 | $ab, bc, ac, a(bc), b(ac), c(ab)$ |
| 7 | $\psi^2 \neq \alpha\beta\gamma,$ <br> $\alpha + \beta + \gamma = 2\psi + 1,$ <br> $\alpha \neq 1$ | 4 | $\frac{1}{2}(\beta - 1)a + \frac{1}{2}(\beta - \alpha)b + \frac{1}{2}(1 - \alpha)c + (1 - \beta)ab + (\alpha - 1)bc,$ <br> $\frac{1}{2}(\gamma - \alpha)a + \frac{1}{2}(\gamma - 1)b + \frac{1}{2}(1 - \alpha)c + (1 - \gamma)ab + (\alpha - 1)ac,$ <br> $(2\psi - 2\alpha\beta + \beta - 1)a + (\gamma - 1)b + (1 - \alpha)c + (4 - 2\alpha - 2\gamma)ab + (4\alpha - 4)a(bc),$ <br> $(\beta - 1)a + (2\psi - 2\alpha\gamma + \gamma - 1)b + (1 - \alpha)c + (4 - 2\alpha - 2\beta)ab + (4\alpha - 4)b(ac),$ <br> $(\psi - \alpha)a + (\psi - \alpha)b + \alpha(1 - \alpha)c + (2 - \beta - \gamma)ab + (2\alpha - 2)c(ab)$ |
| 8 | $\psi = \alpha = 0, \beta, \gamma \neq 0,$ <br> $\beta + \gamma \neq 1$ | 6 | $ab, b(ac) - a(bc), cab$ |
| 9 | $\alpha\beta\gamma = \psi^2, \psi \neq 0,$ <br> $\alpha + \beta + \gamma \neq 2\psi + 1$ | 6 | $-\beta\gamma ab - \alpha\beta ac + 2\psi a(bc),$ <br> $-\beta\gamma ab - \alpha\gamma bc + 2\psi b(ac),$ <br> $-\alpha\gamma bc - \alpha\beta ac + 2\psi c(ab)$ |

Table 1: Bases of the radical

2. $(ab)b = \frac{1}{2}(\alpha b + ab)$

3. $(ab)(ab) = \frac{\alpha}{4}(a + b + 2ab)$

**Lemma 7.** *(Main theorem [1]) There is a 3-generated 9-dimensional algebra $A(\alpha, \beta, \gamma, \phi)$ such that each 3-generated algebra of Jordan type half is a factor algebra of this algebra for suitable values of parameters.*

Let $A = \langle\langle a, b, c\rangle\rangle, \dim(A) = 9, \alpha = (a, b), \beta = (b, c), \gamma = (a, c), \phi = (ab, c)$. We put in Table 1 all possible conditions for $\alpha, \beta, \gamma, \phi$ for $A(\alpha, \beta, \gamma, \phi)$ to be non-simple. We also put in this table from [1] basis of the radical $R(A)$ and dimension of the factor algebra $A/R(A)$ in each case.

## 2 Factors

In this section we will describe 3-generated algebra of Jordan types half with trivial radical. We will prove the following theorem.

**Theorem 1.** *Let $A$ be a 3-generated algebra of Jordan type half with trivial radical over a ground field $\mathbb{F}$ with characteristic not equal to two or three. Then $A$ is isomorphic to one of the following algebras:*

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1. $\mathbb{F}^n, n \in \{1,2,3\}$

2. $JForm_3(\mathbb{F})$

3. $\mathbb{F} \oplus JForm_3(\mathbb{F})$

4. $M_2^+(\mathbb{F})$

5. $H_3^+(\mathbb{F})$

6. $M_3^+(\mathbb{F})$

It follows from Lemma 7 that we must describe the factors of the algebra $A(\alpha, \beta, \gamma, \psi)$ by the radical. We will use the description of the algebra $A(\alpha, \beta, \gamma, \psi)$ in Theorem 2 from [1].

Follow by [1], denote $\alpha = (a, b), \beta = (b, c), \gamma = (a, c), \phi = (ab, c)$.

In [1], the dimensions and bases of the radicals of the algebra $A(\alpha, \beta, \gamma, \psi)$ are described (see Table 6). Table 6 of [1] we write in present text, see Table 1. Denote by $A_i$ the universal 9-dimensional algebra $A(\alpha_i, \beta_i, \gamma_i, \psi_i)$ with parameters and numeration from Table 1, by $R_i$ the radical of this algebra and by $S_i$ the factor algebra by radical.

We begin from two trivial propositions for 1-dimensional and 2-dimensional algebras. It is obvious that such algebras aren't generated by three linear independent axes.

**Proposition 1.** *Let $A$ be a 1-dimensional algebra of Jordan type half with trivial radical. Then $A \cong S_1$.*

*Proof.* It is easy to see that $S_1 \cong \mathbb{F}$. We have $A$ is 1-dimensional, so $\dim L\langle a, b, c\rangle = 1$ and $a = b = c$. Hence $A \cong \mathbb{F} \cong S_1$. $\square$

**Proposition 2.** *Let $A$ be a 2-dimensional 3-generated algebra of Jordan type half with trivial radical. Then $A \cong \mathbb{F} \oplus \mathbb{F} \cong S_2$.*

*Proof.* By Lemma 5, there is only one 2-dimensional algebra of Jordan type half with trivial radical, so $A \cong \mathbb{F} \oplus \mathbb{F} \cong S_2$. $\square$

**Proposition 3.** *Let $A$ be 3-dimensional 3-generated algebra of Jordan type half with trivial radical. Then $A$ is isomorphic to one of the algebras $S_3, S_5$.*

*Proof.* Assume that the algebra $A$ generated by $a$ and $b$. From Lemma 5 it follows that there is only one 3-dimensional 2-generated Jordan type half algebra with trivial radical. In this case, as the axis $c$ we can choose any axis of the algebra $A$. Put $c = a^{\tau_b} = a - 4ab + 4\alpha b$. We have $\beta = \alpha, \gamma = (1-2\alpha)^2$ and $\psi = \alpha(2\alpha-1)$. Therefore $\alpha\beta\gamma = \psi^2, \psi \neq 0, \alpha \neq 1, \alpha+\beta+\gamma = 2\psi+1$. In particular in this case $A \simeq S_5$.

Assume that the algebra $A$ is not generated by 2 axes. Therefore, based on the dimension of $A$, we conclude that $A$ is the linear span of the axes $a$, $b$, and $c$.

Assume that $ab \notin L\langle a, b\rangle$. Hence $\dim\langle\langle a, b\rangle\rangle = 3$. Therefore $c \in L\langle a, b, ab\rangle = A$; the contradiction. By the same way, we can show that if $ac \in L\langle a, c\rangle$ and $bc \in L\langle b, c\rangle$. In particular, we have $\dim(\langle\langle a, b\rangle\rangle) = \dim(\langle\langle a, c\rangle\rangle) = \dim(\langle\langle c, b\rangle\rangle) = 2$. From Lemma 5 it

follows that $\{(a,b),(a,c),(b,c)\} \subseteq \{0,1\}$. Moreover, if $(a,b) = 0$, then $\langle\langle a,b\rangle\rangle \simeq \mathbb{F} \oplus \mathbb{F}$. Therefore, if $(a,b) = (a,c) = (b,c) = 0$ then $A \simeq \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $\psi = 0$. In this case the Gram matrix of the algebra $A$ is the identity matrix and hence the radical of $A$ is trivial. In We conclude that in this case $A \simeq S_3$.

Assume that $(a,c) \neq 0$. We have $(a,c) = 1$. In this case, $R(\langle\langle a,c\rangle\rangle)$ is not trivial and contains the element $a - c$. Assume that $(a,b) = (b,c) = 0$. In this case we have $(a - c,b) = 0$. Consequently $a - c \in R(A)$; the contradiction. Therefore, without loss of generality, we can assume that $(b,c) = 1$. If $(a,b) = 1$ then $(a - c,b) = 0$ and consequently $a - c \in R(A)$; the contradiction. Therefore $(a,b) = 0$. From the description of two generated algebras we have $ab = 0$, $a = c + a_h$, $b = c + b_h$ where $a_h, b_h \in A_{1/2}(c)$. Therefore $0 = ab = (c + a_h)(c + b_h) = c + 1/2(a_h + b_h) + a_h b_h$, where $c + a_h b_h \in A_{0+1}(c)$ and $a_h + b_h \in A_{1/2}(c)$. Therefore $a_h + b_h = 0$. In particular $b = a^{\tau_c}$ and $\dim(A) = 2$. $\qquad\square$

**Proposition 4.** *Algebras $S_4$ and $S_5$ are isomorphic.*

*Proof.* It follows from Lemma 5 that the subalgebra of the algebra $S_4$ generated by the axes $a$ and $b$ has dimension 3. Hence $S_5$ is generated by 2 axes and is isomorphic to $S_3$. $\qquad\square$

**Proposition 5.** *Let $A$ be 4-dimensional 3-generated Jordan type half algebra with trivial radical. Then one of the assertions hold:*

1. *$A \simeq S_6 \simeq \mathbb{F} \oplus JForm_3(\mathbb{F})$*

2. *$A \simeq S_7 \simeq M_2^+(\mathbb{F})$*

*Proof.* The algebra $M_2^+(\mathbb{F})$ is a simple Jordan algebra. Algebra $\mathbb{F} \oplus JForm_3(\mathbb{F})$ includes non-trivial ideals. Therefore $M_2^+(\mathbb{F}) \not\simeq \mathbb{F} \oplus JForm_3(\mathbb{F})$. Hence, to prove this proposition, it suffices to show that $S_6 \simeq \mathbb{F} \oplus JForm_3(\mathbb{F})$ and $S_7 \simeq M_2^+(\mathbb{F})$. $\qquad\square$

**Lemma 8.** *$S_6$ is isomorphic to $\mathbb{F} \oplus JForm_3(\mathbb{F})$.*

*Proof.* Let $\langle\langle a,b,c\rangle\rangle \simeq S_6$. We have $(a,c) \notin \{0,1\}$. Therefore $\langle\langle a,c\rangle\rangle$ is isomorphic to $JForm_3(\mathbb{F})$. From Table 1 it follows that radical of $A(0,0,\gamma,0)$ contains $ab$ and $bc$. Therefore $ab = bc = 0$ and $S_6 \simeq \langle\langle a,c\rangle\rangle \oplus \langle\langle b\rangle\rangle \simeq \mathbb{F} \oplus JForm_3(\mathbb{F})$.

$\qquad\square$

**Lemma 9.** *$S_7$ is isomorphic to $M_2^+(\mathbb{F})$.*

*Proof.* Let

$$A = \begin{pmatrix} 1 & 0 \\ \lambda_a & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 0 & \lambda_b \end{pmatrix} C = \begin{pmatrix} -2\lambda_b\lambda_c & 2\lambda_c(1 + 2\lambda_b\lambda_c) \\ -\lambda_b & 1 + 2\lambda_b\lambda_c \end{pmatrix}$$

Where $\lambda_a, \lambda_b, \lambda_c \in \mathbb{F} \setminus \{0\}$. Consider the following map $f : S_7 \to M_2^+(\mathbb{F}), f(a) = A, f(b) = B, f(c) = C$. Using calculations, we show that multiplication table for $f(\langle\langle a,b,c\rangle\rangle)$ [1] coincides with multiplication table for $S_7$. Hence, $f$ is an isomorphism.

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[1]Calculations for multiplication table in $S_7$ can be found in `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/S8multiplicationtable.nb`, see paragraph Tables.

All calculations are straightforward and can be done by hand [2].

$\square$

**Proposition 6.** *Let $A$ be 6-dimensional 3-generated algebra of Jordan type half with trivial radical. Then $A \simeq S_8 \simeq S_9 \simeq H_3^+(\mathbb{F})$.*

For proof the isomorphism between $S_8$ and $S_9$, we will proof the isomorphism of both algebras to $H_3^+(\mathbb{F})$.

**Lemma 10.** $S_8 \cong H_3^+(\mathbb{F})$

*Proof.* Consider the following matrices in $H_3^+(\mathbb{F})$ and map $f : S_8 \to H_3^+(\mathbb{F})$, $f(a) = A$, $f(b) = B$, $f(c) = C$, $\lambda_a, \lambda_b, \lambda_c \in \mathbb{F} \setminus \{0\}$ are the invariant by $\theta$ parameters which are defined later from conditions to $\alpha, \beta, \gamma$ and $\psi$.

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad
B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1+\sqrt{1-4\lambda_b^2}}{2} & \lambda_b \\ 0 & \lambda_b & \frac{1-\sqrt{1-4\lambda_b^2}}{2} \end{pmatrix} \quad
C = \begin{pmatrix} \frac{1+\sqrt{1-4\lambda_c^2}}{2} & 0 & \lambda_c \\ 0 & 0 & 0 \\ \lambda_c & 0 & \frac{1-\sqrt{1-4\lambda_c^2}}{2} \end{pmatrix}
$$

Below we show that the mapping $f$ is an isomorphism between the algebras $S_8$ and $H_3^+(\mathbb{F})$. All calculations are straightforward and can be done by hand [3].

It is easy to see that $A^2 = A, B^2 = B, C^2 = C$. Symmetric involution matrices $e_{12}, e_{13}, e_{23}$ from $H_3^+(\mathbb{F})$ are in $\langle\langle A, B, C \rangle\rangle$, hence $\langle\langle A, B, C \rangle\rangle = H_3^+(\mathbb{F})$.

A map $(\ ,\ ) : H_3^+(\mathbb{F})^2 \to \mathbb{F}$ such that $(X, Y) = tr(XY) = tr(X \circ Y)$, where $X, Y \in H_3^+(\mathbb{F})$, is a symmetric bilinear form on $H_3^+(\mathbb{F})$. This form associates with the product $\circ$. Clearly, we have $tr(A \circ A) = tr(B \circ B) = tr(C \circ C) = 1$.

Furthermore, we see that $tr(A \circ B) = 0, tr(B \circ C) = \frac{1}{4}(1 - \sqrt{1-4\lambda_b^2})(1 - \sqrt{1-4\lambda_c^2}) = \beta, tr(A \circ C) = \frac{1+\sqrt{1-4\lambda_c^2}}{2} = \gamma$ and $tr(A \circ (B \circ C)) = tr(B \circ (A \circ C)) = tr(C \circ (A \circ B)) = 0$. So $\lambda_b = \pm\sqrt{\dfrac{-\beta(1 + \beta + \gamma)}{(\gamma - 1)^2}}, \lambda_c = \pm\sqrt{-\gamma(\gamma - 1)}$ and we can take any solution of this equation.

Take basis $a, b, c, b \cdot c, a \cdot c, a \cdot (b \cdot c)$ for $S_8$. Multiplication table for $f(\langle\langle a, b, c \rangle\rangle)$ [4] coincides with multiplication table for $S_8$. We have $A, B, C, B \cdot C, A \cdot C, A \cdot (B \cdot C)$ is a basis of the algebra $H_+^3(\mathbb{F})$ and hence the kernel of $f$ is trivial. Thus $f$ is an isomorphism of the algebras $S_8$ and $H_3^+(\mathbb{F})$.

$\square$

**Lemma 11.** $S_9 \cong H_3^+(\mathbb{F})$

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[2] One can find our calculations here: `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/M2+(S7).nb`

[3] One can find our calculations here: `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/H3+(S8).nb`

[4] Calculations for multiplication table in $S_8$ can be found in `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/S8multiplicationtable.nb`, see paragraph Tables.

*Proof.* Let $f : S_9 \to H_3^+(\mathbb{F})$, $f(a) = A, f(b) = B, f(c) = C$, and $\lambda_a, \lambda_b, \lambda_c \in \mathbb{F}$ are the invariant by $\theta$ parameters which are defined later from conditions to $\alpha, \beta, \gamma$ and $\psi$.

$$A = \begin{pmatrix} \frac{1+\sqrt{1-4\lambda_b^2}}{2} & \lambda_a & 0 \\ \lambda_a & \frac{1+\sqrt{1-4\lambda_b^2}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1+\sqrt{1-4\lambda_b^2}}{2} & \lambda_b \\ 0 & \lambda_b & \frac{1-\sqrt{1-4\lambda_b^2}}{2} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1+\sqrt{1-4\lambda_c^2}}{2} & 0 & \lambda_c \\ 0 & 0 & 0 \\ \lambda_c & 0 & \frac{1-\sqrt{1-4\lambda_c^2}}{2} \end{pmatrix}$$

Below we show that the mapping $f$ is an isomorphism between the algebras $S_8$ and $H_3^+(\mathbb{F})$. All calculations are straightforward and can be done by hand[5]. It is easy to see that $A^2 = A, B^2 = B, C^2 = C$. Symmetric involution matrices $e_{12}, e_{13}, e_{23}$ from $H_3^+(\mathbb{F})$ are in $\langle\langle A, B, C\rangle\rangle$, hence $\langle\langle A, B, C\rangle\rangle = H_3^+(\mathbb{F})$. A map $(\ ,\ ) : H_3^+(\mathbb{F})^2 \to \mathbb{F}$ such that $(X, Y) = tr(XY) = tr(X \circ Y)$, where $X, Y \in H_3^+(\mathbb{F})$, is a symmetric bilinear form on $H_3^+(\mathbb{F})$. It is easy to see that this form associates with the product $\circ$. Clearly, we have $tr(A \circ A) = tr(B \circ B) = tr(C \circ C) = 1$.

Furthermore, we see that $tr(A \circ B) = \frac{1}{4}(1 - \sqrt{1-4\lambda_a^2})(1 - \sqrt{1-4\lambda_b^2}) = \alpha, tr(B \circ C) = \frac{1}{4}(1 - \sqrt{1-4\lambda_b^2})(1 - \sqrt{1-4\lambda_c^2}) = \beta, tr(A \circ C) = \frac{1}{4}(1 - \sqrt{1-4\lambda_a^2})(1 - \sqrt{1-4\lambda_c^2}) = \gamma$ and $tr(A \circ (B \circ C)) = tr(B \circ (A \circ C)) = tr(C \circ (A \circ B)) = \lambda_a\lambda_b\lambda_c = \psi$. So, we have equations to $\lambda_a, \lambda_b, \lambda_c$. We can obtain roots of these expressions by Wolfram Mathematica, but the roots are too long to show it here.

We can easily check in Wolfram Mathematica that $\psi^2 = \alpha\beta\gamma$ and $2\psi A \circ (B \circ C) - \beta\gamma(A \circ B) - \alpha\beta(A \circ C) = 2\psi B \circ (A \circ C) - \beta\gamma(A \circ B) - \alpha\gamma(B \circ C) = 2\psi C \circ (A \circ B) - \alpha\gamma(B \circ C) - \alpha\beta(A \circ C) = 0$.

Take basis $A, B, C, A \cdot B, B \cdot C, A \cdot C$ for $S_8$. Multiplication table for $f(\langle\langle a, b, c\rangle\rangle)$ [6] coincides with multiplication table for $S_9$.

$\square$

# 3 Acknowlengements

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[5]One can find our calculations here: `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/H3+(S9).nb`

[6]Calculations for multiplication table in $S_9$ can be found in `https://github.com/RavilBildanov/3gen-axial-algebras/blob/main/S9multiplicationtable.nb`

# 4 Tables

Here are multiplication tables for $S_7$, $S_8$, $S_9$.

| $*$ | $a$ | $b$ | $c$ | $ab$ |
|---|---|---|---|---|
| $a$ | $a$ | $*$ | $*$ | $*$ |
| $b$ | $ab$ | $b$ | $*$ | $*$ |
| $c$ | $\frac{1}{2}((\gamma-\alpha)a+(\gamma-1)b$ $+(1-\alpha)c+2(-\gamma+1)ab)$ | $\frac{1}{2}((\beta-1)a+(\beta-\alpha)b$ $+(1-\alpha)c+2(-\beta+1)ab)$ | $c$ | $*$ |
| $ab$ | $\frac{1}{2}(a\alpha+ab)$ | $\frac{1}{2}(b\alpha+ab)$ | $(\psi-\alpha)a+(\psi-\alpha)b$ $+(\alpha-\alpha^2)c+(2-\beta-\gamma)ab$ | $\frac{1}{4}\alpha(a+b+2ab)$ |

Table 2: Multiplication table for $S_7$

| $*$ | $a$ | $b$ | $c$ | $ab$ | $bc$ | $ac$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $b$ | $ab$ | $b$ | $*$ | $*$ | $*$ | $*$ |
| $c$ | $ac$ | $bc$ | $c$ | $*$ | $*$ | $*$ |
| $ab$ | $\frac{1}{2}(\alpha a+ab)$ | $\frac{1}{2}(\alpha b+ab)$ | $-(\frac{\psi}{2\beta}bc+\frac{\psi}{2\gamma}ac)$ | $\frac{1}{4}\alpha(a+b+2ab)$ | $*$ | $*$ |
| $bc$ | $-(\frac{\psi}{2\alpha}ab+\frac{\psi}{2\gamma}ac)$ | $\frac{1}{2}(\beta b+bc)$ | $\frac{1}{2}(\beta c+bc)$ | $\frac{\psi}{4}a+\dfrac{2\beta^2-\psi}{8\beta}ab-\dfrac{\alpha\beta}{8\psi}ac$ $+\dfrac{\psi^2+2\alpha\beta\psi-\alpha\beta^2}{8\beta\psi}bc$ | $\frac{1}{4}\beta(b+c+2bc)$ | $*$ |
| $ac$ | $\frac{1}{2}(\gamma a+ac)$ | $-\frac{\psi}{2\alpha}ab-\frac{\psi}{2\beta}bc$ | $\frac{1}{2}(\gamma c+ac)$ | $\frac{\psi}{4}b+\dfrac{2\beta^2-\psi}{8\beta}ab-\dfrac{\alpha\beta}{8\psi}ac$ $+\dfrac{\alpha\beta^2+\psi^2+2\alpha\beta\psi}{8\beta\psi}bc$ | $\frac{\psi}{4}c+\dfrac{\alpha\psi-2\beta\psi}{8\alpha\beta}ab$ $+\dfrac{\alpha^2\beta^2+2\psi^3-\alpha\psi^2}{8\alpha\beta\psi}bc$ $+\dfrac{2\beta\psi-\alpha\beta}{8\psi}ac$ | $\frac{1}{4}\gamma(a+c+2ac)$ |

Table 3: Multiplication table for $S_8$

| $*$ | $a$ | $b$ | $c$ | $bc$ | $ac$ | $a(bc)$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $b$ | $ab$ | $b$ | $*$ | $*$ | $*$ | $*$ |
| $c$ | $ac$ | $bc$ | $c$ | $*$ | $*$ | $*$ |
| $bc$ | $a(bc)$ | $\frac{1}{2}(b\beta+bc)$ | $\frac{1}{2}(c\beta+bc)$ | $\frac{1}{4}(\beta(b+c+2bc))$ | $*$ | $*$ |
| $ac$ | $\frac{1}{2}(a\gamma+ac)$ | $a(bc)$ | $\frac{1}{2}(c\gamma+ac)$ | $\frac{1}{4}(\gamma bc+\beta ac+2a(bc))$ | $\frac{1}{4}\gamma(a+c+2ac)$ | $*$ |
| $a(bc)$ | $0$ | $\frac{1}{4}(\beta ac+2a(bc))$ | $\frac{1}{4}(\gamma bc+\beta ac)$ | $\frac{\beta\gamma}{8}b+\frac{\beta}{8}ac+\frac{\beta}{4}a(bc)$ | $\frac{\beta\gamma}{8}a+\frac{\gamma}{8}bc+\frac{\gamma}{4}a(bc)$ | $\frac{\beta\gamma}{16}a+\frac{\beta\gamma}{16}b$ |

Table 4: Multiplication table for $S_9$

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