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BOUNDED k-LINEAR FUNCTIONALS AND k-CONTINUOUS FUNCTIONS ON THE n-NORMED SPACES

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Abstract: In this paper, we studied bounded k-linear functionals ($k \in \mathbb{N}$), on the n-normed spaces. We defined k-linear functionals in new several types of boundedness and showed that those are equivalent. We then formed dual spaces with respect to the type of boundedness. Since the types of boundedness are equivalent, those spaces are identical as a set. We also defined two norms on the dual spaces and showed that both norms are equivalent. Moreover, we gave some examples of bounded k-linear functionals on an n-inner product space and calculated their norms with respect to the types of boundedness. We also gave a relation between the bounded k-linear functional and k-continuous function in n-normed spaces.

Keywords: bounded k-linear functionals, n-normed spaces, k-continuous function.

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1 Introduction

The concept of the n-normed spaces was initially introduced by S. Gähler in the 1960's. One can see it in [1, 2, 3, 4]. Various characteristis on n-normed spaces have been studied by many researchers since then, see for instance [5, 6, 7, 8, 9, 10, 11, 12]. Let n be a nonnegative integer and X be a real vector space with $\dim X \geq n$. An n-norm on X is a function $\|\cdot, \ldots, \cdot\| : X^n \ldots \mathbb{R}$ which satisfies the following conditions:

N1. $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ linearly dependent,
N2. $\|x_1, \ldots, x_n\|$ is invariant under permutation,
N3. $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,
N4. $\|x_1 + x_1', \ldots, x_n\| \leq \|x_1, \ldots, x_n\| + \|x_1', \ldots, x_n\|$.

The pair $(X, \|\cdot, \cdot\|)$ is called an n-normed space. One can observe that (N3) and (N4) imply $\|x_1, \ldots, x_n\| \geq 0$, for any $x_1, \ldots, x_n \in X$. In the case $n = 1$, this will be the definition of a normed space. [13] Moreover, in an n-normed space one may see that

$$\|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \ldots, x_n\|. \tag{1}$$

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, we can define the standard n-norm on X by

$$\|x_1, \ldots, x_n\|^S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{\frac{1}{2}}.$$

Geometrically, the value of $\|x_1, \ldots, x_n\|^S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$.

Next, a function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ that satisfies these following comditions:

I1. $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$ and it is equal to 0 if and only if $x_1, \ldots, x_n$ are linearly dependent,
I2. $\langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle = \langle x_1, x_1 | x_2, \ldots, x_n \rangle$ for any permutation $\{i_1, \ldots, i_n\}$ of $\{1, \ldots, n\}$,
I3. $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$,
I4. $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle$,
I5. $\langle x + x', y | x_2, \ldots, x_n \rangle = \langle x, y | x_2, \ldots, x_n \rangle + \langle x', y | x_2, \ldots, x_n \rangle$,

is called an n-inner product on X, and the pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an n-inner product space. In the case $n = 1$, this will be the definition of the inner product space. Moreover, for every n-inner product $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on X, we have

$$\|x_1, x_2, \ldots, x_n\| = \langle x_1, x_1 | x_2, \ldots, x_n \rangle^{\frac{1}{2}}, \tag{2}$$

with $x_1, \ldots, x_n \in X$. The n-norm defined in equation (2) is called the n-normed induced by the n-inner product on X. Furthermore, in an n-inner product space we have Cauchy-Schwarz inequality [14]

$$|\langle x, y | x_2, \ldots, x_n \rangle| \leq \|x, x_2, \ldots, x_n\| \|y, x_2, \ldots, x_n\|,$$

for any $x, y, x_2, \ldots, x_n \in X$.

Moreover, let X_i with $i = 1, \ldots, k$ be vector spaces and $f : \prod_{i=1}^{k} X_i \to \mathbb{R}$ be a functional. The functional f is called k-linear if for any $x_i \in X_i$, we have

$$f(x_1 \ldots, \alpha x_i + \beta x_i', \ldots, x_n) = \alpha f(x_1, \ldots, x_i, \ldots, x_n) + \beta f(x_1, \ldots, x_i', \ldots, x_n),$$

where $\alpha, \beta \in \mathbb{R}$. The function f is linear for each part. We will investigate the boundedness of a k-linear functional in the next section.

2 Results and Discussions

In this section we define bounded k-linear functionals in several ways. We also discuss the duality properties for bounded k-linear functionals. In the end, we give a relation between bounded k-linear functionals and k-continuous functions. The results are developed from [15, 16, 17]

2.1. Bounded k-Linear Functionals (of 1st index). Let $i = 1, \ldots, k$ and $(X_i, \|\cdot, \ldots, \cdot\|)$ be normed spaces. Fix a linearly independent set $Y_i = \{y_{i_1}, \ldots, y_{i_n}\} \subset X_i$. A k-linear functional $f : \prod_{i=1}^{k} X_i \to \mathbb{R}$ is said to be bounded of 1st index (with respect to Y_i), if there exist a $C > 0$ such that

$$|f(x_1, \ldots, x_k)| \leq C \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \right), \tag{3}$$

for all $x_i \in X_i, i = 1, \ldots, k$. The sum is taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$ and $j_1 \in \{1, \ldots, n\} \setminus \{j_2, \ldots, j_n\}$.

Moreover, we will see more formulas contain a sum similar to (3). To keep it simple, all the sums after this will be taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$ and $j_1 \in \{1, \ldots, n\} \setminus \{j_2, \ldots, j_n\}$, unless we specify the condition.

We define a dual space that contains all bounded linear functional (of 1st index) on $\left(\prod_{i=1}^{k} X_i\right)$. We denote it by $\left(\prod_{i=1}^{k} X_i\right)_1^*$. Next, define a function $\|\cdot\|_1 : \left(\prod_{i=1}^{k} X_i\right)_1^* \to \mathbb{R}$ defined by

$$\|f\|_1 := \inf\{C > 0 \ : \ (3) \text{ holds}\}. \tag{4}$$

One can check that this function defines a norm in $\left(\prod_{i=1}^{k} X_i\right)_1^*$. We also define an identical norm to (4). We state it in the following lemma.

Lemma 1. *The norm in (4) is identical with*

$$\|f\|_1 := \sup\left\{ |f(x_1, \ldots, x_k)| : \sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}\| \leq 1, i = 1, \ldots, k \right\} \tag{5}$$

Proof. Let $f \in \left(\prod_{i=1}^{k} X_i\right)_1^*$ and $\alpha = \inf\{C > 0 : (4) \text{ holds}\}$. For each $i = 1, \ldots, k$, let $Y_i = \{y_1, \ldots, y_{i_n}\} \subset X_i$ be linearly independent set, and $z_i =$

$\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}$, with $x_i \in X_i$. We can see that for any $\varepsilon > 0$, we have

$$\|[z_i + \varepsilon]^{-1} x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i} \leq 1.$$

As a consequence, we have

$$\left| f([z_1 + \varepsilon]^{-1} x_1, \dots, [z_n + \varepsilon]^{-1} x_k) \right| \leq \|f\|_1.$$

Then we can write

$$|f(x_1, \dots, x_k)| \leq \|f\|_1 \left(\prod_{i=1}^{k} [z_i + \varepsilon] \right),$$

for $\varepsilon \to 0$, we have

$$|f(x_1, \dots, x_k)| \leq \|f\|_1 \left(\prod_{i=1}^{k} z_i \right) = \|f\|_1 \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_2}, \dots, y_{i_{j_n}}\| \right).$$

Thus, $\alpha \leq \|f\|_1$. Conversely, if for each $x_i \in X_i, i = 1, \dots, k$ we have $|f(x_1, \dots, x_k)| \leq C \prod_{i=1}^{k} z_i$, with $z_i \leq 1$, then $|f(x_1, \dots, x_k)| \leq C$. This means $\|f\|_1 \leq C$. Thus, $\|f\|_1 \leq \alpha$. This leads us to conclude $\|f\|_1 = \alpha$. $\square$

Moreover, we call the norm in (4) and (5) inf norm-1 and sup norm-1, respectively. The above lemma shows that inf norm-1 and sup norm-1 are equivalent, precisely they are identical. Next, we give an example of bounded k-linear functionals.

For $i = 1, \dots, k$, let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_{X_i})$ be n-inner product spaces and fix a linearly independent set $Y_i = \{y_{i_1}, \dots, y_{i_n}\} \subset X_i$. We give a functional $f : \prod_{i=1}^{k} X_i \to \mathbb{R}$ is defined by

$$f(x_1, \dots, x_k) = \prod_{i=1}^{k} \left(\sum \langle x, y_{i_{j_1}} | y_{i_{j_2}}, \dots, y_{i_{j_n}} \rangle_{X_i} \right). \tag{6}$$

It is easy to see that f is a k-linear functional. We give a following fact of f.

Fact 1. *The k-linear functional defined on (6) is bounded (of 1st index) with*

$$\|f\|_1 = \prod_{i=1}^{k} \|y_{i_1}, \dots, y_{i_n}\|_{X_i}.$$

Proof. By using Triangle inequality and Cauch-Schwarz inequality, we have

$$\begin{aligned} |f(x_1, \dots, x_k)| &\leq \prod_{i=1}^{k} \left(\sum |\langle x, y_{i_{j_1}} | y_{i_{j_2}}, \dots, y_{i_{j_n}} \rangle_{X_i}| \right) \\ &\leq \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i} \|y_{i_{j_1}}, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i} \right) \\ &= \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i} \|y_{i_1}, y_{i_2}, \dots, y_{i_n}\|_{X_i} \right). \end{aligned}$$

Thus, we have

$$|f(x_1,\ldots,x_k)| \le \left(\prod_{i=1}^{k} \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\|\right)\left(\prod_{i=1}^{k}\left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{x_i}\right)\right),$$

which means f is bounded (of 1st index). By the definition of inf norm-1, we also have

$$\|f\|_1 \le \prod_{i=1}^{k} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}.$$

To obtain the equality, choose $x_i = \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1} y_{i_1}$ for each $i = 1,, \ldots, k$. We have

$$\begin{aligned}
\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\| &= \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1} \sum \|y_{i_1}, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \\
&= \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i} \\
&= 1.
\end{aligned}$$

One can see that for the last sum, all terms will be 0 except for $\|y_{i_1}, \ldots, y_{i_n}\|_{X_i}$. Next,

$$\begin{aligned}
|f(x_1,\ldots,x_k)| &= \left| f\left(\|y_{i_1}, \ldots, y_{i_n}\|_{X_1}^{-1} y_{i_1}, \ldots, \|y_{n_1}, \ldots, y_{n_n}\|_{X_n}^{-1}\right)\right| \\
&= \left(\prod_{i=1}^{k} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1}\right)\left(\prod_{i=1}^{k}\left(\sum \langle y_{i_1}, y_{i_{j_1}} | y_{i_{j_2}}, \ldots, y_{i_{j_n}}\rangle_{X_i}\right)\right) \\
&= \left(\prod_{i=1}^{k} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1}\right)\left(\prod_{i=1}^{k} \langle y_{i_1}, y_{i_1} | y_{i_2}, \ldots, y_{i_n}\rangle_{X_i}\right) \\
&= \left(\prod_{i=1}^{k} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1}\right)\left(\prod_{i=1}^{k} \|y_{i_1}, y_{i_1} | y_{i_2}, \ldots, y_{i_n}\|_{X_i}^{2}\right) \\
&= \prod_{i=1}^{k} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}.
\end{aligned}$$

Therefore, with respect to sup norm-1 we have $\|f\|_1 = \prod_{i=1}^{k} \|y_{i_n}, \ldots, y_{i_n}\|_{X_i}$. □

Note that for $j_1 \ne 1$ and $\{j_2, \ldots, j_n\} = \{1, \ldots, n\} \setminus \{j_1\}$, we have

$$|\langle y_{i_1}, y_{i_{j_1}} | y_{i_{j_2}}, \ldots, y_{i_{j_n}}\rangle_{X_i}| \le \|y_{i_1}, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \|y_{i_{j_1}}, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} = 0,$$

because one of $y_{i_{j_2}}, \ldots, y_{i_{j_n}}$ will be equal to y_{i_1}. This implies $\left(\prod_{i=1}^{k} \langle y_{i_1}, y_{i_1} | y_{i_2}, \ldots, y_{i_n}\rangle_{X_i}\right) = \left(\prod_{i=1}^{k} \|y_{i_1}, y_{i_1} | y_{i_2}, \ldots, y_{i_n}\|_{X_i}^{2}\right)$.

2.2. Bounded k-Linear Functionals (of p-th index). Let $i = 1, \dots, k$ and $(X_i, \|\cdot, \dots, \cdot\|)$ be n-normed spaces. Fix linearly independent set $Y_i = \{y_{i_1}, \dots, y_{i_n}\} \subset X_i$. A k-linear functional $f : \prod_{i=1}^k X_i \to \mathbb{R}$ is said to be bounded of p-th index (with respect to Y_i), if there exist a $C > 0$ such that

$$|f(x_1, \dots, x_n)| \le C \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}}, \tag{7}$$

with $p \ge 1$ and for all $x_i \in X_i$. For $p = \infty$, the inequality (7) will be

$$|f(x_1, \dots, x_n)| \le C \prod_{i=1}^{k} \left(\max\{\|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}\} \right) \tag{8}$$

The max is taken over $\{j_2, \dots, j_n\} \subset \{1, \dots, n\}$.

Let $\left(\prod_{i=1}^k X_i\right)_p^*$ be a dual space contains all bounded k-linear functionals of p-th index on $\prod_{i=1}^k X_i$. For each $f \in \left(\prod_{i=1}^k X_i\right)_p^*$ we define

$$\|f\|_p := \{C > 0 \,:\, (8)\,\text{holds}\}. \tag{9}$$

Equation (9) defines a norm on $\left(\prod_{i=1}^k X_i\right)_p^*$, we call it inf norm-$p$. Moreover, we define an identical norm of the in norm-p in the following lemma. We call the norm on the following lemma the sup norm-p.

Lemma 2. *The norm defined in (9) is identical with*

$$\|f\|_p := \sup \left\{ |f(x_1, \dots, x_n)| : \left(\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}} \le 1 \right\} \tag{10}$$

Proof. Let

$$\|f\|_p := \sup \left\{ |f(x_1, \dots, x_n)| : \left(\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}} \le 1 \right\},$$

and

$$\|f\|_p := \{C > 0 \,:\, (8)\,\text{holds}\}.$$

For $i = 1, \dots, k$, let $z_i = \left(\sum \|x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}}$, with $x_i \in X_i$ and a linearly independent set $Y_i = \{y_{i_1}, \dots, y_{i_n}\} \subset X_i$. For any $\varepsilon > 0$, it i easy to see that

$$\left(\sum \|[z_i + \varepsilon]^{-1} x_i, y_{i_{j_2}}, \dots, y_{i_{j_n}}\|_{X_i}^p \right) \le 1,$$

for each $i = 1, \dots, k$. Moreover, we have

$$|f([z_1 + \varepsilon]^{-1} x_1, \dots, [z_k + \varepsilon]^{-1} x_k)| \le \|f\|_p,$$

which means

$$|f(x_1, \dots, x_k)| \le \|f\|_p \prod_{i=1}^{k} (z_i + \varepsilon).$$

For $\varepsilon \to 0$, we have

$$\begin{aligned} |f(x_1,\ldots,x_k)| &\le \|f\|_p \, z_i \\ &= \|f\|_p \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, $\alpha_p \le \|f\|_p$.

Conversely, if we have $|f(x_1,\ldots,x_k)| \le C \prod_{i=1}^k z_i$ with $z_i \le 1$, then $|f(x1,\ldots,x_k)| \le C$. This implies $\|f\|_p \le C$. Thus, $\|f\|_p \le \alpha_p$. Therefore, we obtain $\|f\|_p = \alpha_p$, which means the sup norm-p and the inf norm-p are identical. □

Now, we give another fact about the k-linear functional defined in (6) with respect to the boundedness (of p-th index)

Fact 2. *The k-linear functional defined in (6) is bounded (of p-th index) with*

$$\|f\|_p = n^{\frac{k}{q}} \prod_{i=1}^k \|y_{i_1}, \ldots, y_{i_n}\|_{X_i},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using triangle inequality, and Hölder inequality, we have

$$\begin{aligned} |f(x_1,\ldots,x_n)| &\le \prod_{i=1}^k \left(\sum |\langle x, y_{i_{j_1}} | y_{i_{j_2}}, \ldots, y_{i_{j_n}} \rangle_{X_i}| \right) \\ &\le \prod_{i=1}^k \left(\sum \|x, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \|y_{i_{j_1}}, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \right) \\ &\le \prod_{i=1}^k \left(\sum \|x, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\|_{X_i} \right) \\ &\le \prod_{i=1}^k \left[\left(\sum \|x_i, y_{i_{j_2}}, \ldots . y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}} \left(\sum \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\|_{X_i}^p \right)^{\frac{1}{q}} \right] \\ &\le \prod_{i=1}^k n^{\frac{1}{q}} \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| \left(\sum \|x_i, y_{i_{j_2}}, \ldots . y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}} \\ &= n^{\frac{k}{q}} \prod_{i=1}^k \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| \left(\sum \|x_i, y_{i_{j_2}}, \ldots . y_{i_{j_n}}\|_{X_i}^p \right)^{\frac{1}{p}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which means f is bounded of p-th index with

$$\|f\|_p \le n^{\frac{k}{q}} \prod_{i=1}^k \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\|_{X_i}^p.$$

To get the equality, choose $x_i = n^{\frac{1}{p}} \|y_{i_1}, \ldots, y_{i_n}\| \ (y_{i_1} + \cdots + y_{i_n})$. Since,

$$\begin{aligned} \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^p &= \left(n^{-\frac{1}{p}} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1}\right)^p \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^p \\ &= \frac{1}{n}, \end{aligned}$$

we have

$$\left(\sum \|x_i, y_{i_{j_2}}, \ldots . y_{i_{j_n}}\|_{X_i}^p\right) = 1.$$

For the chosen x_i, we obtain

$$\begin{aligned} |f(x_1, \ldots, x_k)| &= \prod_{i=1}^k n^{-\frac{1}{p}} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1} \left(\sum |\langle y_{i_1} + \cdots + y_{i_n}, y_{i_{j_1}} | y_{i_{j_2}}, \ldots, y_{i_{j_n}} \rangle_{X_i}|\right) \\ &= \prod_{i=1}^k n^{-\frac{1}{p}} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1} (n \langle y_{i_{j_1}}, y_{i_{j_1}} | y_{i_{j_2}}, \ldots, y_{i_{j_n}} \rangle_{X_i}) \\ &= \prod_{i=1}^k n^{\frac{1}{q}} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^{-1} \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}^2 \\ &= n^{\frac{k}{q}} \prod_{i=1}^k \|y_{i_1}, \ldots, y_{i_n}\|_{X_i}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. □

Note that, if $p = 1$, then $q = \infty$, which means

$$\|f\|_1 = \prod_{i=1}^k \|y_{i_1}, \ldots, y_{i_n}\|_{X_i},$$

exactly the same with the result we had earlier.

Furthermore, Fact (1) and (2) indicate that there is a relationship between $(X_i)_1^*$ and $(X_i)_p^*$ for any $p \geq 1$. We found that the dual spaces $(X_i)_1^*$ and $(X_i)_p^*$ are identical. It can be seen in the following theorem.

Theorem 1. *Let $i = 1, \ldots, k$ and $(X_i, \|\cdot, \ldots, \cdot\|)$ be normed spaces. A k-linear functional f is bounded of 1st index if and only if it is bounded of p-th index.*

Proof. Let f be a k-linear functional that bounded of p-th index, $p \geq 1$. If for each $x_i \in X_i$ satisfies $\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \leq 1$, then each term of the sum is less or equal 1, or we can write $\|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \leq 1$ for each $i = 1, \ldots, k$. This implies

$$\|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^p \leq \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \leq 1.$$

Therefore,

$$\prod_{i=1}^k \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^p \leq \prod_{i=1}^k \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} \leq 1.$$

Since $|f(x_1,\ldots,x_k)| \leq \|f\|_p$, f is bounded of 1st index with

$$\|f\|_1 \leq \|f\|_p,$$

for any $p \leq 1$.

Conversely, let f be bounded of 1st index. If for each $x_i \in X_i$ with $i = 1,\ldots,k$ satisfies

$$\left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^p\right)^{\frac{1}{p}} \leq 1.$$

By using Hölder inequality we have

$$\begin{aligned}\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i} &\leq \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^p\right)^{\frac{1}{p}} \left(\sum 1^q\right)^{\frac{1}{q}} \\ &\leq n^{\frac{1}{q}}.\end{aligned}$$

We obtain

$$\left|f\left(\frac{x_1}{n^{\frac{1}{q}}}, \ldots, \frac{x_k}{n^{\frac{1}{q}}}\right)\right| \leq \|f\|_1,$$

or

$$|f(x_1,\ldots,x_k)| \leq n^{\frac{k}{q}}\|f\|_1,$$

which means f is bounded of p-th index with

$$\|f\|_p \leq n^{\frac{k}{q}}\|f\|_1.$$

Then this convincces us that any k-linear functional f is bounded of 1st index if and only if it is bounded of p-th index □

Remark 1. *The above theorems shows that $\|\cdot\|_1$ and $\|\cdot\|_p$ are equivalent, with*

$$\|f\|_1 \leq \|f\|_p \leq n^{\frac{k}{q}}\|f\|_1,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, for any $p \geq 1$ the dual spaces $(\prod_{i=1}^k X_i)_1^*$ and $(\prod_{i=1}^k X_i)_p^*$ are identical as a set.

Next is a corollary of the above theorem.

Corollary 1. *Let $i = 1,\ldots,k$, and $(X, \|\cdot,\ldots,\cdot\|)$ be normed spaces. For any $p_1, p_2 \geq 1$, the dual spaces $(\prod_{i=1}^k X_i)_{p_1}^*$ and $(\prod_{i=1}^k X_i)_{p_2}^*$ are identical a set or*

$$\left(\prod_{i=1}^k X_i\right)_{p_1}^* = \left(\prod_{i=1}^k X_i\right)_{p_2}^*.$$

Moreover, the norms $\|f\|_{p_1}$ on $(\prod_{i=1}^k X_i)_{p_1}^*$ and $\|f\|_{p_2}$ on $(\prod_{i=1}^k X_i)_{p_2}^*$ are equivalent with

$$\|f\|_{p_1} \leq n^{\frac{k}{q_1}}\|f\|_{p_2} \leq n^{\frac{k}{q_1}+\frac{k}{q_2}}\|f\|_{p_1},$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and $\frac{1}{p_2} + \frac{1}{q_2} = 1$.

Proof. The proof is simple. One can proof this corollary using Theorem (1). The norm $\|\cdot\|_1$ can be used as a bridge. □

2.3. Duality Properties for $\mathbf{p = 2}$.. We will give another example of a bounded k-linear functional on the n-inner product space. Let $i = 1, \ldots, k$ and $(X_i, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be n-inner product spaces. Fix $Y_i = \{y_{i_1}, \ldots, y_{i_n}\}$ be a linearly independent set on each X_i. We define a functional

$$f_w := \left(\prod_{i=1}^{k} X_i \right) \to \mathbb{R},$$

defined by

$$f_w(x_1, \ldots, x_n) = \prod_{i=1}^{k} \left(\sum \langle x_i, w_i | y_{i_{j_2}}, \ldots, y_{i_{j_n}} \rangle_{X_i} \right), \tag{11}$$

with a fixed $w_i \in X_i$. One can check easily that f_z is a k-linear functional. Note that (6) is a special case of (11) by taking

$$w_i = y_{i_1} + \cdots + y_{i_n}.$$

Here we give a fact about f_w.

Fact 3. *The k-linear functional f defined on (11) is bounded of 2nd index with*

$$\|f_w\|_2 = \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{x_i}^2 \right)^{\frac{1}{2}}.$$

Proof. Let $i = 1, \ldots, k$, fix an element $w_i \in X_i$. For any $x_i \in X_i$, using triangle inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |f_z(x_1, \ldots, x_k)| &\leq \prod_{i=1}^{k} \left(\sum |\langle x_i, w_i | y_{i_{j_2}}, \ldots, y_{i_{j_n}} \rangle_{X_i}| \right) \\ &\leq \prod_{i=1}^{k} \left[\left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^2 \right)^{\frac{1}{2}} \left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^2 \right)^{\frac{1}{2}} \right] \\ &= \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^2 \right)^{\frac{1}{2}} \prod_{i=1}^{k} \left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which means f_w is bounded of 2nd index and

$$f_z \leq \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^2 \right)^{\frac{1}{2}}.$$

The equality is obtained by choosing

$$x_i = \left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|_{X_i}^2 \right)^{\frac{1}{2}} w_i.$$

Next, we can see that

$$\begin{aligned}\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} &= \sum \left\| \left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right)^{\frac{1}{2}} w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}} \right\|^2_{X_i} \\ &= \left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right)^{-1} \sum \left\| w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}} \right\|^2_{X_i} \\ &= 1.\end{aligned}$$

Moreover, we have

$$\begin{aligned}f_w(x_1, \ldots, x_k) &= \prod_{i=1}^{k} \left[\left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right)^{-\frac{1}{2}} f_w(w_1, \ldots, w_n) \right] \\ &= \prod_{i=1}^{k} \left[\left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right)^{-\frac{1}{2}} \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right) \right] \\ &= \prod_{i=1}^{k} \left(\sum \|w_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right)^{\frac{1}{2}}.\end{aligned}$$

This convinces us that $\|f_w\|_2 = \prod_{i=1}^{k} \left(\sum \|x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\|^2_{X_i} \right)^{\frac{1}{2}}$ □

Furthermore, it is easy for us to determine the norm of the functional k-linear f_w with respect to norm $\|\cdot\|_p$. On the next section we will observe continuous functions on n normed spaces and its relation with k-linear functionals

2.4. Continuous Function on the n-Normed Spaces. Let $(X, \|\cdot, \ldots, \cdot\|_X)$, $(Z, \|\cdot, \ldots, \cdot\|_Z)$ be two n-normed spaces, $Y = \{y_1, \ldots, y_n\}$, $W = \{w_1, \ldots, w_n\}$ be two linearly independent sets in X, Z respectively. A function $f : X \to Z$ is said to be continuous on $x \in X$, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $v \in X$ that satisfies $\sum \|v - x, y_{i_2}, \ldots, y_{j_n}\| < \delta$, then we have $\sum \|f(v) - f(x), w_{i_2}, \ldots, w_{j_n}\| < \varepsilon$. We said f continuous on X if f continuous on each $x \in X$. Both sums is taken over $\{j_2, \ldots, j_n\} \subset \{1, \ldots, n\}$

Moreover, one can check that the function $\|\cdot\|_X : X \to \mathbb{R}$ defined by

$$\|x\|_X = \sum \|x, y_{j_2}, \ldots, y_{j_n}\|_X,$$

and $\|\cdot\|_Z : Z \to \mathbb{R}$ defined by

$$\|z\|_Z = \sum \|z, w_{j_2}, \ldots, w_{j_n}\|,$$

define norms on X and Z respectively. In the above definition, we observe the continuity use a specific norm. We use a similar way to observed the k-continuity of a function on the n-normed space.

Let $i = 1, \ldots, k$, and $(X_i, \|\cdot, \ldots, \cdot\|_{X_i}), (Z, \|\cdot, \ldots, \cdot\|_Z)$ be normed spaces. Fix a linearly independent set $Y_i = \{y_{i_1}, \ldots, y_{i_n}\}$ on each X_i and $W = \{w_1, \ldots, w_n\}$ in Z. A function $f : \prod_{i=1}^{k} X_i \to Z$ is said to be k-continuous on $x = (x_1, \ldots, x_k) \in \prod_{i=1}^{k} X_i$ if for any $\varepsilon > 0$, there is a $\delta > 0$, such that for any $v = (v_1, \ldots, v_n) \in \prod_{i=1}^{k} X_i$ that satisfies $\sum \|v_i - x_i, y_{i_{j_2}}, \ldots, y_{i_{j_n}}\| < \delta$

then we have $\sum \|f(v_1,\ldots,v_k) - f(x_1,\ldots,x_k), w_{i_2},\ldots,w_{i_n}\| < \varepsilon$. The last sum is taken over $\{i_2,\ldots,i_n\} \subset \{1,\ldots,n\}$. A function f is k-continuous on $\prod_{i=1}^k X_i$ if it is k-continuous on each $x = x_1,\ldots,x_k \in \prod_{i=1}^k X_i$. From this definition we have the following theorem

Theorem 2. *If $f : \prod_{i=1}^n X_i \to \mathbb{R}$ is a bounded k-linear functional, then f is k-continuous.*

Proof. Let $i = 1,\ldots,k$, and $(X_i, \|\cdot,\ldots,\cdot\|_{X_i})$ be normed spaces. Without losing of generality let $f : \prod_{i=1}^k \to \mathbb{R}$ be a k-linear functional that bounded of 1st index, there is a $C > 0$ such that

$$|f(x_1,\ldots,x_k)| \leq C \prod_{i=1}^k \left(\sum \|x_i, y_{i_{j_2}},\ldots,y_{i_{j_n}}\|_{X_i}\right).$$

For any $\varepsilon > 0$ and for any $x = (x_1,\ldots,x_k), v = (v_1,\ldots,v_k) \in \prod_{i=1}^k X$ that satisfies

$$\prod_{i=1}^k \left(\sum \|x_i - v_i, y_{i_{j_2}},\ldots,y_{i_{j_n}}\|_{X_i}\right) < \frac{\epsilon}{C}$$

we have:

$$\begin{aligned} |f(x_1,\ldots,x_k) - f(v_1,\ldots,v_k)| &= |f(x_1 - v_1,\ldots,x_k - v_k)| \\ &\leq C \prod_{i=1}^k \left(\sum \|x_i - v_i, y_{i_{j_2}},\ldots,y_{i_{j_n}}\|_{X_i}\right) \\ &< \varepsilon. \end{aligned}$$

This means f is k-continuous in $\prod_{i=1}^k X_i$. □

Since the boundedness for all indexes are equivalent, on the above proof we use only bounded 1st index. Moreover, we define a function $\|\cdot\|_\star : \prod_{i=1}^k X_i \to \mathbb{R}$, defined by

$$\|x\|_\star = \sum_{i=1}^k \left(\sum \|x, y_{i_{j_2}},\ldots,y_{i_{j_n}}\|_{X_i}\right),$$

defines a norm on $\prod_{i=1}^k X_i$. The last sum is taken over $\{j_2,\ldots,j_n\} \subset \{1,\ldots,n\}$. Using the norm derived from n-norms of each X_i, we can observe the space $\prod_{i=1}^k X_i$.

3 Conclusion

We already investigated bounded k-linear functionals and k-continuous function in n-normed spaces. We formed some dual spaces with respect to the new several types of the boundedness. As a result we found that these doal spaces are identical as sets. Moreover, we defined

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