SOME FRACTIONAL CORRECTED DUAL EULER-SIMPSON TYPE INEQUALITIES FOR DIFFERENTIABLE s-CONVEX FUNCTIONS

S. SALAH[1] AND B. MEFTAH[2]

ABSTRACT. In this paper, some corrected dual Euler-Simpson type inequalities via Riemann-Liouville fractional integrals and s-convexity of the module of the first derivatives are established. The obtained results are based on a new integral identiy. some application to Numerical integration as well as analytique inequalities are given.

1. Introduction

Integral inequalities are extensively recognized as a fundamental mathematical tool, playing a critical role in numerous fields such as real, complex, and numerical analysis, number theory, differential and integral equations, probability theory, among others. Over the past decades, many mathematicians have delved into the investigation of error estimations for Newton-Cotes formulas for a diverse range of function classes, including but not limited to convex functions, bounded functions, and other such classes see $[1, 2, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22]$ and references therein.

We recall that a function $f : I \to \mathbb{R}$ is said to be convex, if for all $x, y \in I$ and all $t \in [0, 1]$ (see [18]), we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

The concept of convex functions has been also generalized in diverse manners. One of them is the so-called s-convex function or Breckner convex function defined as follows:

A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$ (see [3]).

Nowadays, the fractional calculus has become an attractive field of research, quenseqently several articles have been appeared, in particular those using the Riemann-Liouville operator whose definition we recall

---

Key words and phrases. Corrected dual Euler-Simpson inequality, s-convex functions, Hölder inequality, Riemann-Liouville integral operator.

Mathematics subject classification (2010): 26D10, 26D15, 26A51.

Definition 1 ([10]). *Let $\mathcal{H} \in L^1[i,j]$ with $i \geq 0$ and $\alpha > 0$. The following operators*

$$I_{i^+}^{\alpha}\mathcal{H}(v) = \frac{1}{\Gamma(\alpha)}\int_i^v (v-u)^{\alpha-1}\mathcal{H}(u)du, \qquad v > i,$$

$$I_{j^-}^{\alpha}\mathcal{H}(v) = \frac{1}{\Gamma(\alpha)}\int_v^j (u-v)^{\alpha-1}\mathcal{H}(t)du, \qquad j > v$$

denotes The Riemann-Liouville fractional integrals, where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t}t^{\alpha-1}dt$ is the gamma function and $I_{a^+}^0\mathcal{H}(x) = I_{b^-}^0\mathcal{H}(x) = \mathcal{H}(x)$.

The corrected dual Euler-Simpson formulae (see [7]) is as follows

$$\int_a^b \mathcal{H}(t)dt \simeq \frac{b-a}{15}\left(8\mathcal{H}\left(\frac{3a+b}{4}\right) - \mathcal{H}\left(\frac{a+b}{2}\right) + 8\mathcal{H}\left(\frac{a+3b}{4}\right)\right).$$

In this investigation we first establish a new fractional identity. Based on this identity we derive some corrected dual Euler-Simpson type inequalities for s-convex differentiable functions via Riemann-Liouville fractional integrals. some particularly cases are discussed. Application to Numerical integration as well as analytique inequalities are given.

2. Main results

Befor giving our results we start this section by considering the following special functions (see [10]).

For all reals and nonpositive integers x, y the beta function is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

For all positive numbers $c > b > 0$ and $|z| < 1$ the hypergeometric function is defined by

$${}_2F_1(a,b,c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

where $B(.,.)$ is the beta function.

Lemma 1. *Let $\mathcal{H} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $u, v \in I^{\circ}$ with $u < v$, and $\mathcal{H}' \in L^1[u, v]$, then the following equality holds*

$$\begin{aligned}
&\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right) - \mathcal{H}\left(\tfrac{u+v}{2}\right) + 8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right) - \tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H} \\
= \; & \tfrac{v-u}{16}\left(\int_0^1 w^{\alpha}\mathcal{H}'\left((1-w)u + t\tfrac{3u+v}{4}\right)dw\right. \\
& - \int_0^1 \left((1-w)^{\alpha} + \tfrac{2}{15}\right)\mathcal{H}'\left((1-w)\tfrac{3u+v}{4} + w\tfrac{u+v}{2}\right)dw \\
& + \int_0^1 \left(w^{\alpha} + \tfrac{2}{15}\right)\mathcal{H}'\left((1-w)\tfrac{u+v}{2} + w\tfrac{u+3v}{4}\right)dw \\
& \left. - \int_0^1 (1-w)^{\alpha}\mathcal{H}'\left((1-w)\tfrac{u+3v}{4} + wv\right)dw\right),
\end{aligned}$$

where

$$\mathcal{S}_{\alpha}f = I^{\alpha}_{\left(\frac{3u+v}{4}\right)^-}\mathcal{H}(u) + I^{\alpha}_{\left(\frac{3u+v}{4}\right)^+}\mathcal{H}\left(\tfrac{u+v}{2}\right) + I^{\alpha}_{\left(\frac{u+3v}{4}\right)^-}\mathcal{H}\left(\tfrac{u+v}{2}\right) + I^{\alpha}_{\left(\frac{u+3v}{4}\right)^+}\mathcal{H}(v). \tag{2.1}$$

Proof. Let

$$I = I_1 - I_2 + I_3 - I_4, \tag{2.2}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 w^{\alpha}\mathcal{H}'\left((1-w)u + w\tfrac{3u+v}{4}\right)dw \\
I_2 &= \int_0^1 \left((1-w)^{\alpha} + \tfrac{2}{15}\right)\mathcal{H}'\left((1-w)\tfrac{3u+v}{4} + w\tfrac{u+v}{2}\right)dw \\
I_3 &= \int_0^1 \left(w^{\alpha} + \tfrac{2}{15}\right)\mathcal{H}'\left((1-w)\tfrac{u+v}{2} + w\tfrac{u+3v}{4}\right)dw
\end{aligned}$$

and

$$I_4 = \int_0^1 (1-w)^{\alpha}\mathcal{H}'\left((1-w)\tfrac{u+3v}{4} + wv\right)dw.$$

Integrating by parts I_1, we get

$$\begin{aligned}
I_1 &= \tfrac{4}{v-u}w^{\alpha}\mathcal{H}\left((1-w)u + w\tfrac{3u+v}{4}\right)\Big|_{w=0}^{w=1} - \tfrac{4}{v-u}\int_0^1 \mathcal{H}\left((1-w)u + w\tfrac{3u+v}{4}\right)dw \\
&= \tfrac{4}{v-u}\mathcal{H}\left(\tfrac{3u+v}{4}\right) - \tfrac{4\alpha}{v-u}\int_0^1 w^{\alpha-1}\mathcal{H}\left((1-w)u + w\tfrac{3u+v}{4}\right)dw \\
&= \tfrac{4}{v-u}\mathcal{H}\left(\tfrac{3u+v}{4}\right) - \tfrac{4^{\alpha+1}\alpha}{(v-u)^{\alpha+1}}\int_u^{\frac{3u+v}{4}} (z-u)^{\alpha}\mathcal{H}(z)\,dz \\
&= \tfrac{4}{v-u}\mathcal{H}\left(\tfrac{3u+v}{4}\right) - \tfrac{4^{\alpha+1}\Gamma(\alpha+1)}{(v-u)^{\alpha+1}}I^{\alpha}_{\left(\frac{3u+v}{4}\right)^-}\mathcal{H}(u).
\end{aligned} \tag{2.3}$$

Similarly, we get

$$
\begin{aligned}
I_2 &= \tfrac{4}{v-u}\left((1-w)^{\alpha}+\tfrac{2}{15}\right)\mathcal{H}\left((1-w)\tfrac{3u+v}{4}+w\tfrac{u+v}{2}\right)\Big|_{w=0}^{w=1} \\
&\quad +\tfrac{4\alpha}{v-u}\int_0^1(1-w)^{\alpha-1}\mathcal{H}\left((1-w)\tfrac{3u+v}{4}+w\tfrac{u+v}{2}\right)dw \\
&= \tfrac{8}{15(v-u)}\mathcal{H}\left(\tfrac{u+v}{2}\right)-\tfrac{68}{15(v-u)}\mathcal{H}\left(\tfrac{3u+v}{4}\right) \\
&\quad +\tfrac{4\alpha}{v-u}\int_0^1(1-w)^{\alpha-1}\mathcal{H}\left((1-w)\tfrac{3u+v}{4}+w\tfrac{u+v}{2}\right)dw \\
&= \tfrac{8}{15(v-u)}\mathcal{H}\left(\tfrac{u+v}{2}\right)-\tfrac{68}{15(v-u)}\mathcal{H}\left(\tfrac{3u+v}{4}\right)+\tfrac{4^{\alpha+1}\alpha}{(v-u)^{\alpha+1}}\int_{\frac{3u+v}{4}}^{\frac{u+v}{2}}\left(\tfrac{u+v}{2}-z\right)^{\alpha-1}\mathcal{H}(z)\,dz \\
&= \tfrac{8}{15(v-u)}\mathcal{H}\left(\tfrac{u+v}{2}\right)-\tfrac{68}{15(v-u)}\mathcal{H}\left(\tfrac{3u+v}{4}\right)+\tfrac{4^{\alpha+1}\Gamma(\alpha+1)}{(v-u)^{\alpha+1}}I^{\alpha}_{\left(\frac{3u+v}{4}\right)^{+}}\mathcal{H}\left(\tfrac{u+v}{2}\right), \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \tfrac{4}{v-u}\left(w^{\alpha}+\tfrac{2}{15}\right)\mathcal{H}\left((1-w)\tfrac{u+v}{2}+w\tfrac{u+3b}{4}\right)\Big|_{w=0}^{w=1} \\
&\quad -\tfrac{4\alpha}{v-u}\int_0^1 w^{\alpha-1}\mathcal{H}\left((1-w)\tfrac{u+v}{2}+w\tfrac{u+3b}{4}\right)dw \\
&= \tfrac{68}{15(v-u)}\mathcal{H}\left(\tfrac{u+3b}{4}\right)-\tfrac{8}{15(v-u)}\mathcal{H}\left(\tfrac{u+v}{2}\right) \\
&\quad -\tfrac{4\alpha}{v-u}\int_0^1 w^{\alpha-1}\mathcal{H}\left((1-w)\tfrac{u+v}{2}+w\tfrac{u+3b}{4}\right)dw \\
&= \tfrac{68}{15(v-u)}\mathcal{H}\left(\tfrac{u+3b}{4}\right)-\tfrac{8}{15(v-u)}\mathcal{H}\left(\tfrac{u+v}{2}\right)-\tfrac{4^{\alpha+1}\alpha}{(v-u)^{\alpha+1}}\int_{\frac{u+v}{2}}^{\frac{u+3b}{4}}\left(z-\tfrac{u+v}{2}\right)^{\alpha-1}\mathcal{H}(z)\,dz \\
&= \tfrac{68}{15(v-u)}\mathcal{H}\left(\tfrac{u+3b}{4}\right)-\tfrac{8}{15(v-u)}\mathcal{H}\left(\tfrac{u+v}{2}\right)-\tfrac{4^{\alpha+1}\Gamma(\alpha+1)}{(v-u)^{\alpha+1}}I^{\alpha}_{\left(\frac{u+3b}{4}\right)^{-}}\mathcal{H}\left(\tfrac{u+v}{2}\right) \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \tfrac{4}{v-u}(1-w)^{\alpha}\mathcal{H}\left((1-w)\tfrac{u+3b}{4}+wv\right)\Big|_{w=0}^{w=1} \\
&\quad +\tfrac{4\alpha}{v-u}\int_0^1(1-w)^{\alpha-1}\mathcal{H}\left((1-w)\tfrac{u+3b}{4}+wv\right)dw \\
&= -\tfrac{4}{v-u}\mathcal{H}\left(\tfrac{u+3b}{4}\right)+\tfrac{4\alpha}{v-u}\int_0^1(1-w)^{\alpha-1}\mathcal{H}\left((1-w)\tfrac{u+3b}{4}+wv\right)dt \\
&= -\tfrac{4}{v-u}\mathcal{H}\left(\tfrac{u+3v}{4}\right)+\tfrac{4^{\alpha+1}\alpha}{(v-u)^{\alpha+1}}\int_{\frac{u+3b}{4}}^{v}(v-z)^{\alpha-1}f(z)\,dz \\
&= -\tfrac{4}{v-u}\mathcal{H}\left(\tfrac{u+3v}{4}\right)+\tfrac{4^{\alpha+1}\Gamma(\alpha+1)}{(v-u)^{\alpha+1}}I^{\alpha}_{\left(\frac{u+3v}{4}\right)^{+}}\mathcal{H}(v). \qquad (2.6)
\end{aligned}
$$

Using (2.3)-(2.6) in (2.2), and then multiplying the result equality by $\frac{v-u}{16}$ we get the desired result. □

Theorem 1. *Under the assumptions of Lemma 1. If $|\mathcal{H}'|$ is s-convex in the second sense for some fixed $s \in (0,1]$, then we have*

$$\begin{aligned}
&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\
\leq\ & \tfrac{b-a}{16}\left(B\left(\alpha+1,s+1\right)\left(\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right)\right.\\
&+\left(\tfrac{32(s+1)+2\alpha}{15(s+1)(\alpha+s+1)}\right)\left(\left|f'\left(\tfrac{3u+v}{4}\right)\right|+\left|f'\left(\tfrac{u+3v}{4}\right)\right|\right)\\
&\left.+\left(2B\left(\alpha+1,s+1\right)+\tfrac{4}{15(s+1)}\right)\left|f'\left(\tfrac{u+v}{2}\right)\right|\right),
\end{aligned}$$

where $\mathcal{S}_{\alpha}\mathcal{H}$ is defined by (2.1) and $B(.,.)$ is the beta function.

Proof. From Lemma 1, modulus and s-convexity of $|\mathcal{H}'|$, we have

$$\begin{aligned}
&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\
\leq\ & \tfrac{b-a}{16}\left(\int_0^1 w^{\alpha}\left|\mathcal{H}'\left((1-w)u+w\tfrac{3u+v}{4}\right)\right|dw\right.\\
&+\int_0^1\left((1-w)^{\alpha}+\tfrac{2}{15}\right)\left|\mathcal{H}'\left((1-w)\tfrac{3u+v}{4}+w\tfrac{u+v}{2}\right)\right|dw\\
&+\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)\left|\mathcal{H}'\left((1-w)\tfrac{u+v}{2}+w\tfrac{u+3v}{4}\right)\right|dw\\
&\left.+\int_0^1(1-w)^{\alpha}\left|\mathcal{H}'\left((1-w)\tfrac{u+3v}{4}+wv\right)\right|dw\right)\\
\leq\ & \tfrac{b-a}{16}\left(\int_0^1 w^{\alpha}\left((1-w)^{s}\left|\mathcal{H}'(u)\right|+w^{s}\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|\right)dw\right.\\
&+\int_0^1\left((1-w)^{\alpha}+\tfrac{2}{15}\right)\left((1-w)^{s}\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|+w^{s}\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|\right)dw\\
&+\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)\left((1-w)^{s}\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|+w^{s}\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|\right)dw\\
&\left.+\int_0^1(1-w)^{\alpha}\left((1-w)^{s}\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|+w^{s}\left|\mathcal{H}'(v)\right|\right)dw\right)\\
=\ & \tfrac{b-a}{16}\left(\left|\mathcal{H}'(u)\right|\int_0^1 w^{\alpha}(1-w)^{s}dw+\left|\mathcal{H}'(v)\right|\int_0^1(1-w)^{\alpha}w^{s}dw\right.\\
&+\left(\int_0^1 w^{\alpha+s}dw+\int_0^1\left((1-w)^{\alpha}+\tfrac{2}{15}\right)(1-w)^{s}dw\right)\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|\\
&+\left(\int_0^1\left((1-w)^{\alpha}+\tfrac{2}{15}\right)w^{s}dw+\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)(1-w)^{s}dw\right)\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|\\
&\left.+\left(\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)w^{s}dw+\int_0^1(1-w)^{\alpha+s}dw\right)\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|\right)\\
=\ & \tfrac{b-a}{16}\left(B\left(\alpha+1,s+1\right)\left(\left|\mathcal{H}'(u)\right|+\left|\mathcal{H}'(v)\right|\right)\right.\\
&+\left(\tfrac{32(s+1)+2\alpha}{15(s+1)(\alpha+s+1)}\right)\left(\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|+\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|\right)\\
&\left.+\left(2B\left(\alpha+1,s+1\right)+\tfrac{4}{15(s+1)}\right)\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|\right),
\end{aligned}$$

where we have used the facts that

$$\int_0^1 w^{\alpha}(1-w)^s\,dw=\int_0^1 (1-w)^{\alpha} w^s dw = B(\alpha+1,s+1), \tag{2.7}$$

$$\int_0^1 w^{\alpha+s}dw=\int_0^1 (1-w)^{s+\alpha}\,dw=\frac{1}{s+\alpha+1}, \tag{2.8}$$

$$\int_0^1 \left((1-w)^{\alpha}+\frac{2}{15}\right)(1-w)^s\,dw=\int_0^1 \left(w^{\alpha}+\frac{2}{15}\right)w^s dw=\frac{17(s+1)+2\alpha}{15(s+1)(\alpha+s+1)} \tag{2.9}$$

and

$$\int_0^1 \left((1-w)^{\alpha}+\frac{2}{15}\right)w^s dw=\int_0^1 \left(w^{\alpha}+\frac{2}{15}\right)(1-w)^s\,dw=B(\alpha+1,s+1)+\frac{2}{15(s+1)}. \tag{2.10}$$

The proof is finished. □

Corollary 1. *In Theorem 1, if we take $s=1$, then we obtain*

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3u+v}{4}\right)-\mathcal{H}\left(\frac{u+v}{2}\right)+8\mathcal{H}\left(\frac{u+3v}{4}\right)\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\ \leq\ &\frac{v-u}{16}\left(\frac{1}{(\alpha+1)(\alpha+2)}\left(|\mathcal{H}'(u)|+|\mathcal{H}'(v)|\right)+\frac{30+2(\alpha+1)(\alpha+2)}{15(\alpha+1)(\alpha+2)}\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|\right.\\ &\left.+\frac{32+\alpha}{15(\alpha+2)}\left(\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|\right)\right).\end{aligned}$$

Corollary 2. *In Theorem 1, if we take $\alpha=1$, then we obtain*

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3u+v}{4}\right)-\mathcal{H}\left(\frac{u+v}{2}\right)+8\mathcal{H}\left(\frac{u+3v}{4}\right)\right)-\frac{1}{v-u}\int_u^v\mathcal{H}(z)\,dz\right|\\ \leq\ &\frac{v-u}{16}\left(\frac{1}{(s+1)(s+2)}\left(|\mathcal{H}'(u)|+|\mathcal{H}'(v)|\right)+\frac{38+4s}{15(s+1)(s+2)}\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|\right.\\ &\left.+\frac{32s+34}{15(s+1)(s+2)}\left(\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|\right)\right).\end{aligned}$$

Corollary 3. *If we take $\alpha=s=1$, then Theorem 1 becomes*

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3u+v}{4}\right)-\mathcal{H}\left(\frac{u+v}{2}\right)+8\mathcal{H}\left(\frac{u+3v}{4}\right)\right)-\frac{1}{v-u}\int_u^v\mathcal{H}(z)\,dz\right|\\ \leq\ &\frac{17(v-u)}{120}\left(\frac{5|\mathcal{H}'(u)|+22\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|+14\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|+22\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|+5|\mathcal{H}'(v)|}{68}\right).\end{aligned}$$

Theorem 2. *Under the assumptions of Lemma 1. If $|\mathcal{H}'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ and $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then we have*

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3u+v}{4}\right)-\mathcal{H}\left(\frac{u+v}{2}\right)+8\mathcal{H}\left(\frac{u+3v}{4}\right)\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\ \leq\ &\frac{v-u}{16}\left(\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{H}'(u)|^q+\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q+|\mathcal{H}'(v)|^q}{s+1}\right)^{\frac{1}{q}}\right)\right.\\ &+\left(\frac{17^p}{15^p}\cdot{}_2F_1\left(-p,1,\frac{1}{\alpha}+1;\frac{15}{17}\right)\right)^{\frac{1}{p}}\\ &\left.\times\left(\left(\frac{\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q+\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),\end{aligned}$$

where $\mathcal{S}_{\alpha}\mathcal{H}$ is defined by (2.1) and ${}_2F_1(.,.,.;.)$ is the hypergeometric function.

Proof. From Lemma 1, modulus, Hölder's inequality and s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right| \\
\leq\ & \tfrac{b-a}{16}\left(\left(\int_0^1 w^{p\alpha}dw\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{H}'\left((1-w)u+w\tfrac{3u+v}{4}\right)\right|^q dw\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1\left((1-w)^{\alpha}+\tfrac{2}{15}\right)^p dw\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{H}'\left((1-w)\tfrac{3u+v}{4}+w\tfrac{u+v}{2}\right)\right|^q dw\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{H}'\left((1-w)\tfrac{u+v}{2}+w\tfrac{u+3v}{4}\right)\right|^q dw\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1(1-w)^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{H}'\left((1-w)\tfrac{u+3v}{4}+wv\right)\right|^q dw\right)^{\frac{1}{q}}\right) \\
\leq\ & \tfrac{b-a}{16}\left(\left(\int_0^1 w^{p\alpha}dt\right)^{\frac{1}{p}}\left(\left(\int_0^1\left((1-w)^s\left|\mathcal{H}'(u)\right|^q+w^s\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|^q\right)dw\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\int_0^1\left((1-t)^s\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|^q+t^s\left|\mathcal{H}'(v)\right|^q\right)dw\right)^{\frac{1}{q}}\right) \\
&+\left(\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)^p dw\right)^{\frac{1}{p}} \\
&\times\left(\left(\int_0^1\left((1-w)^s\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|^q+w^s\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|^q\right)dw\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\int_0^1\left((1-w)^s\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|^q+w^s\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|^q\right)dw\right)^{\frac{1}{q}}\right)\right) \\
=\ & \tfrac{v-u}{16}\left(\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|\mathcal{H}'(u)\right|^q+\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q+\left|\mathcal{H}'(v)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right. \\
&+\left(\tfrac{17^p}{15^p}\cdot{}_2F_1\left(-p,1,\tfrac{1}{\alpha}+1;\tfrac{15}{17}\right)\right)^{\frac{1}{p}} \\
&\left.\times\left(\left(\tfrac{\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q+\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),
\end{aligned}$$

where we have used

$$\int_0^1 w^{p\alpha}dw=\tfrac{1}{p\alpha+1}$$

and

$$\begin{aligned}
\int_0^1\left(w^{\alpha}+\tfrac{2}{15}\right)^p dw &= \tfrac{17^p}{15^p\alpha}\int_0^1(1-x)^{\frac{1}{\alpha}-1}\left(1-\tfrac{15}{17}x\right)^p dx \\
&= \tfrac{17^p}{15^p}\cdot{}_2F_1\left(-p,1,\tfrac{1}{\alpha}+1;\tfrac{15}{17}\right).
\end{aligned}$$

The proof is finished. □

Corollary 4. *If we take $s=1$, then Theorem 2 becomes*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\ \leq\;& \tfrac{v-u}{16}\left(\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|\mathcal{H}'(u)\right|^{q}+\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}+\left|\mathcal{H}'(v)\right|^{q}}{2}\right)^{\frac{1}{q}}\right)\right.\\ &+\left(\tfrac{17^{p}}{15^{p}}\cdot{}_{2}F_{1}\left(-p,1,\tfrac{1}{\alpha}+1;\tfrac{15}{17}\right)\right)^{\frac{1}{p}}\\ &\left.\times\left(\left(\tfrac{\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}+\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Corollary 5. *If we take $\alpha=1$, then Theorem 2 becomes*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{1}{v-u}\int_{u}^{v}\mathcal{H}(z)\,dz\right|\\ \leq\;& \tfrac{v-u}{16}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|\mathcal{H}'(u)\right|^{q}+\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}+\left|\mathcal{H}'(v)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right.\\ &+\left(\tfrac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\\ &\left.\times\left(\left(\tfrac{\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}+\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Corollary 6. *If we take $\alpha=s=1$, then Theorem 2 becomes*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{1}{v-u}\int_{u}^{v}\mathcal{H}(z)\,dz\right|\\ \leq\;& \tfrac{v-u}{16}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\left(\tfrac{\left|\mathcal{H}'(u)\right|^{q}+\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}+\left|\mathcal{H}'(v)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.\right.\\ &\left.+\left(\tfrac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}+\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}+\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Theorem 3. *Under the assumptions of Lemma 1. If $|\mathcal{H}'|^{q}$ is s-convex in the second sense for some fixed $s\in(0,1]$ and $q\geq 1$, then we have*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\ \leq\;& \tfrac{v-u}{16}\left(\left(\tfrac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(B(\alpha+1,s+1)\left|\mathcal{H}'(u)\right|^{q}+\tfrac{1}{\alpha+s+1}\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\right.\\ &\left.+\left(\tfrac{1}{\alpha+s+1}\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|^{q}+B(\alpha+1,s+1)\left|\mathcal{H}'(v)\right|^{q}\right)^{\frac{1}{q}}\right)+\left(\tfrac{17+2\alpha}{15(\alpha+1)}\right)^{1-\frac{1}{q}}\\ &\times\left(\left(\tfrac{17(s+1)+2\alpha}{15(s+1)(\alpha+s+1)}\left|\mathcal{H}'\left(\tfrac{3u+v}{4}\right)\right|^{q}+\tfrac{15(s+1)B(\alpha+1,s+1)+2}{15(s+1)}\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\ &\left.\left.+\left(\tfrac{15(s+1)B(\alpha+1,s+1)+2}{15(s+1)}\left|\mathcal{H}'\left(\tfrac{u+v}{2}\right)\right|^{q}+\tfrac{17(s+1)+2\alpha}{15(s+1)(\alpha+s+1)}\left|\mathcal{H}'\left(\tfrac{u+3v}{4}\right)\right|^{q}\right)^{\frac{1}{q}}\right)\right),\end{aligned}$$

where $\mathcal{S}_{\alpha}\mathcal{H}$ is defined by (2.1) and $B(.,.)$ is the beta function.

Proof. From Lemma 1, modulus, power mean inequality and s-convexity of $|\mathcal{H}'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3u+v}{4}\right)-\mathcal{H}\left(\frac{u+v}{2}\right)+8\mathcal{H}\left(\frac{u+3v}{4}\right)\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right| \\
\leq\ & \frac{v-u}{16}\left(\left(\int_0^1 w^{\alpha}dw\right)^{1-\frac{1}{q}}\left(\int_0^1 w^{\alpha}\left|\mathcal{H}'\left((1-w)u+w\frac{3u+v}{4}\right)\right|^q dw\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1\left((1-w)^{\alpha}+\frac{2}{15}\right)dw\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left((1-w)^{\alpha}+\frac{2}{15}\right)\left|\mathcal{H}'\left((1-w)\frac{3u+v}{4}+w\frac{u+v}{2}\right)\right|^q dw\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left(w^{\alpha}+\frac{2}{15}\right)dw\right)^{1-\frac{1}{q}}\left(\int_0^1\left(w^{\alpha}+\frac{2}{15}\right)\left|\mathcal{H}'\left((1-w)\frac{u+v}{2}+w\frac{u+3v}{4}\right)\right|^q dw\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1(1-w)^{\alpha}dw\right)^{1-\frac{1}{q}}\left(\int_0^1(1-w)^{\alpha}\left|\mathcal{H}'\left((1-t)\frac{u+3v}{4}+wv\right)\right|^q dw\right)^{\frac{1}{q}}\right) \\
\leq\ & \frac{v-u}{16}\left(\left(\int_0^1 w^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 w^{\alpha}\left((1-w)^s\left|\mathcal{H}'(u)\right|^q+w^s\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q\right)dw\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1\left((1-w)^{\alpha}+\frac{2}{15}\right)dw\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left((1-w)^{\alpha}+\frac{2}{15}\right)\left((1-w)^s\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q+w^s\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q\right)dw\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left(w^{\alpha}+\frac{2}{15}\right)dw\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left(w^{\alpha}+\frac{2}{15}\right)\left((1-w)^s\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q+w^s\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q\right)dw\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1(1-w)^{\alpha}dw\right)^{1-\frac{1}{q}}\left(\int_0^1(1-w)^{\alpha}\left((1-w)^s\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q+w^s\left|\mathcal{H}'(v)\right|^q\right)dw\right)^{\frac{1}{q}}\right) \\
=\ & \frac{v-u}{16}\left(\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(B(\alpha+1,s+1)\left|\mathcal{H}'(u)\right|^q+\frac{1}{\alpha+s+1}\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.\right. \\
&\left.+\left(\frac{1}{\alpha+s+1}\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q+B(\alpha+1,s+1)\left|\mathcal{H}'(v)\right|^q\right)^{\frac{1}{q}}\right)+\left(\frac{17+2\alpha}{15(\alpha+1)}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\frac{17(s+1)+2\alpha}{15(s+1)(\alpha+s+1)}\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^q+\left(B(\alpha+1,s+1)+\frac{2}{15(s+1)}\right)\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\left(B(\alpha+1,s+1)+\frac{2}{15(s+1)}\right)\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^q+\frac{17(s+1)+2\alpha}{15(s+1)(\alpha+s+1)}\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^q\right)^{\frac{1}{q}}\right)\right),
\end{aligned}$$

where we have used (2.7)-(2.10). The proof is finished. □

Corollary 7. *In Theorem 3, if we take $s=1$, then we obtain*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^{\alpha}}\mathcal{S}_{\alpha}\mathcal{H}\right|\\ \leq\;& \tfrac{v-u}{16}\left(\tfrac{1}{\alpha+1}\left(\left(\tfrac{\left|\mathcal{H}'(u)\right|^{q}+(\alpha+1)\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}}{\alpha+2}\right)^{\frac{1}{q}}+\left(\tfrac{(\alpha+1)\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}+\left|\mathcal{H}'(v)\right|^{q}}{\alpha+2}\right)^{\frac{1}{q}}\right)\right.\\ &+\left(\tfrac{17+2\alpha}{15(\alpha+1)}\right)\left(\left(\tfrac{\left(\alpha^{2}+18\alpha+17\right)\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}+\left(\alpha^{2}+3\alpha+17\right)\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}}{(\alpha+2)(17+2\alpha)}\right)^{\frac{1}{q}}\right.\\ &\left.\left.+\left(\tfrac{\left(\alpha^{2}+3\alpha+17\right)\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}+\left(\alpha^{2}+18\alpha+17\right)\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}}{(\alpha+2)(17+2\alpha)}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Corollary 8. *In Theorem 3, if we take $\alpha=1$, then we obtain*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{1}{v-u}\int_{u}^{v}\mathcal{H}(z)\,dz\right|\\ \leq\;& \tfrac{v-u}{32}\left(\left(\tfrac{2\left|\mathcal{H}'(u)\right|^{q}+2(s+1)\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}}{(s+1)(s+2)}\right)^{\frac{1}{q}}+\left(\tfrac{2(s+1)\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}+2\left|\mathcal{H}'(v)\right|^{q}}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right.\\ &+\tfrac{19}{15}\left(\left(\tfrac{2(17s+19)\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}+2(19+2s)\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}}{19(s+1)(s+2)}\right)^{\frac{1}{q}}\right.\\ &\left.\left.+\left(\tfrac{2(19+2s)\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}+2(17s+19)\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}}{19(s+1)(s+2)}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Corollary 9. *If we take $\alpha=s=1$, then Theorem 3 becomes*

$$\begin{aligned}&\left|\tfrac{1}{15}\left(8\mathcal{H}\left(\tfrac{3u+v}{4}\right)-\mathcal{H}\left(\tfrac{u+v}{2}\right)+8\mathcal{H}\left(\tfrac{u+3v}{4}\right)\right)-\tfrac{1}{v-u}\int_{u}^{v}\mathcal{H}(z)\,dz\right|\\ \leq\;& \tfrac{v-u}{32}\left(\left(\tfrac{\left|\mathcal{H}'(u)\right|^{q}+2\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}}{3}\right)^{\frac{1}{q}}+\left(\tfrac{2\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}+\left|\mathcal{H}'(v)\right|^{q}}{3}\right)^{\frac{1}{q}}\right.\\ &\left.+\tfrac{19}{15}\left(\left(\tfrac{12\left|\mathcal{H}'\left(\frac{3u+v}{4}\right)\right|^{q}+7\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}}{19}\right)^{\frac{1}{q}}+\left(\tfrac{7\left|\mathcal{H}'\left(\frac{u+v}{2}\right)\right|^{q}+12\left|\mathcal{H}'\left(\frac{u+3v}{4}\right)\right|^{q}}{19}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

3. Applications

Quadrature formula

Considering the partition Θ of the points $u=e_0<e_1<...<e_n=v$ of the interval $[u,v]$, and the following quadrature formula

$$\int_{u}^{v}\mathcal{H}(u)\,du=\lambda(\mathcal{H},\Theta)+R(\mathcal{H},\Theta),$$

where

$$\lambda(\mathcal{H},\Theta)=\sum_{i=0}^{n-1}\tfrac{e_{i+1}-e_i}{15}\left(8\mathcal{H}\left(\tfrac{3e_i+e_{i+1}}{4}\right)-\mathcal{H}\left(\tfrac{e_i+e_{i+1}}{2}\right)+8\mathcal{H}\left(\tfrac{e_i+3e_{i+1}}{4}\right)\right)$$

and the associated approximation error $R(\mathcal{H},\Theta)$.

Proposition 1. *Assume $\mathcal{H}$ be a differentiable function on (u,v) with $\mathcal{H}' \in L^1[u,v]$. Let $n \in \mathbb{N}$, if $|\mathcal{H}'|$ is s-convex function in the second sense for some fixed $s \in (0,1]$, we have*

$$\begin{aligned}|R(\mathcal{H},\Theta)| &\leq \sum_{i=0}^{n-1} \frac{(e_{i+1}-e_i)^2}{16}\left(\frac{1}{(s+1)(s+2)}\left(|\mathcal{H}'(e_i)|+|\mathcal{H}'(e_{i+1})|\right)\right.\\ &+ \frac{(38+4s)\left|\mathcal{H}'\left(\frac{e_i+e_{i+1}}{2}\right)\right|}{15(s+1)(s+2)} + \left.\frac{32s+34}{15(s+1)(s+2)}\left(\left|\mathcal{H}'\left(\frac{3e_i+e_{i+1}}{4}\right)\right|+\left|\mathcal{H}'\left(\frac{e_i+3e_{i+1}}{4}\right)\right|\right)\right).\end{aligned}$$

Proof. Applying Corollary 2 on the subintervals $[e_i, e_{i+1}]$ $(i = 0,1,...,n-1)$ of the partition Θ, we get

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3e_i+e_{i+1}}{4}\right)-\mathcal{H}\left(\frac{e_i+e_{i+1}}{2}\right)+8\mathcal{H}\left(\frac{e_i+3e_{i+1}}{4}\right)\right)-\frac{1}{e_{i+1}-e_i}\int_{e_i}^{e_{i+1}}\mathcal{H}(z)\,dz\right|\\ \leq\ & \frac{e_{i+1}-e_i}{16}\left(\frac{1}{(s+1)(s+2)}\left(|\mathcal{H}'(e_i)|+|\mathcal{H}'(e_{i+1})|\right)+\frac{38+4s}{15(s+1)(s+2)}\left|\mathcal{H}'\left(\frac{e_i+e_{i+1}}{2}\right)\right|\right.\\ &+\left.\frac{32s+34}{15(s+1)(s+2)}\left(\left|\mathcal{H}'\left(\frac{3e_i+e_{i+1}}{4}\right)\right|+\left|\mathcal{H}'\left(\frac{e_i+3e_{i+1}}{4}\right)\right|\right)\right). \qquad (3.1)\end{aligned}$$

Summing the inequalities (3.1) for all $i = 0,1,...,n-1$, using the triangular inequality, and then multiplying the obtained result by $(e_{i+1}-e_i)$, we get the desired result. □

Proposition 2. *Assume $\mathcal{H}$ be a differentiable function on (u,v) with $\mathcal{H}' \in L^1[u,v]$. Let $n \in \mathbb{N}$, if $|\mathcal{H}'|^q$ is s-convex function in the second sense for some fixed $s \in (0,1]$, we have*

$$\begin{aligned}|R(\mathcal{H},\Theta)| &\leq \sum_{i=0}^{n-1}\frac{(e_{i+1}-e_i)^2}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left(\left(\frac{|\mathcal{H}'(e_i)|^q+\left|\mathcal{H}'\left(\frac{3e_i+e_{i+1}}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{H}'\left(\frac{e_i+3e_{i+1}}{4}\right)\right|^q+|\mathcal{H}'(e_{i+1})|^q}{s+1}\right)^{\frac{1}{q}}\right.\\ &+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\mathcal{H}'\left(\frac{3e_i+e_{i+1}}{4}\right)\right|^q+\left|\mathcal{H}'\left(\frac{e_i+e_{i+1}}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.\\ &+\left.\left.\left(\frac{\left|\mathcal{H}'\left(\frac{e_i+e_{i+1}}{2}\right)\right|^q+\left|\mathcal{H}'\left(\frac{e_i+3e_{i+1}}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).\end{aligned}$$

Proof. Applying Corollary 5 on the subintervals $[e_i, e_{i+1}]$ $(i = 0,1,...,n-1)$ of the partition Θ, we get

$$\begin{aligned}&\left|\frac{1}{15}\left(8\mathcal{H}\left(\frac{3e_i+e_{i+1}}{4}\right)-\mathcal{H}\left(\frac{e_i+e_{i+1}}{2}\right)+8\mathcal{H}\left(\frac{e_i+2e_{i+1}}{4}\right)\right)-\frac{1}{e_{i+1}-e_i}\int_{e_i}^{e_{i+1}}\mathcal{H}(z)\,dz\right|\\ \leq\ & \frac{x_{i+1}-x_i}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{H}'(x_i)|^q+\left|\mathcal{H}'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{H}'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q+|\mathcal{H}'(x_{i+1})|^q}{s+1}\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\mathcal{H}'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q+\left|\mathcal{H}'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|\mathcal{H}'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+\left|\mathcal{H}'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\Bigg). \tag{3.2}$$

Summing the inequalities (3.2) for all $i=0,1,...,n-1$, using the triangular inequality, and then multiplying the obtained result by $(e_{i+1}-e_i)$, we get the desired result. □

Application to special means

For arbitrary real numbers $u,u_1,u_2,...,u_n,v$ we have:
The Arithmetic mean: $\boldsymbol{A}(u_1,u_2,...,u_n)=\frac{u_1+u_2+...+u_n}{n}$.

The k-Logarithmic mean: $\boldsymbol{L}_k(u,v)=\left(\frac{v^{k+1}-u^{k+1}}{(k+1)(v-u)}\right)^{\frac{1}{k}}$, $u,v>0, u\neq v$ and $k\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 3. *Let $u,v\in\mathbb{R}$ with $0<u<v$, then we have*

$$\left|8\boldsymbol{A}^2(u,u,u,v)-\boldsymbol{A}^2(u,v)+8\boldsymbol{A}^2(u,v,v,v)-15\boldsymbol{L}_2^2(u,v)\right|\leq\frac{17}{8}\left(v^2-u^2\right).$$

Proof. Apply Corollary 3 to the function $\mathcal{H}(z)=z^2$. □

References

[1] M. A. Ali, J. E. Nápoles Valdés, A. Kashuri and Z. Zhang, Fractional non conformable Hermite-Hadamard inequalities for generalized φ-convex functions. Fasc. Math. No. 64 (2020), 5–16.

[2] M. Alomari and M. Darus, On some inequalities of Simpson-type via quasi-convex functions and applications. Transylv. J. Math. Mech. 2 (2010), no. 1, 15-24.

[3] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) Publ. Inst. Math. (Beograd) (N.S.) 23(37) (1978), 13–20.

[4] T. Chiheb, N. Boumaza and B. Meftah, Some new Simpson-like type inequalities via prequasiinvexity. Transylv. J. Math. Mech.12 (2020), no.1, 1-10.

[5] S. S. Dragomir, R. P. Agarwal and P. Cerone, On Simpson's inequality and applications. J. Inequal. Appl. 5 (2000), no. 6, 533-579.

[6] S. Erden, S. Iftikhar, P. Kumam and P. Thounthong, On error estimations of Simpson's second type quadrature formula. Math. Methods Appl. Sci. 2020, 1-13.

[7] I. Franjić and J. Pečarić, On corrected dual Euler-Simpson formulae. Soochow J. Math. 32 (2006), no. 4, 575–587.

[8] A. Kashuri, P. O. Mohammed, T. Abdeljawad, F. Hamasalh and Y. Chu, New Simpson type integral inequalities for s-convex functions and their applications. Math. Probl. Eng. 2020, Art. ID 8871988, 12 pp.

[9] A. Kashuri, B. Meftah and P.O. Mohammed, Some weighted Simpson type inequalities for differentiable s-convex functions and their applications. J. Frac. Calc. and Nonlinear Sys.1 (2021) no. 1, 75-94.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[11] B. Meftah, Fractional Ostrowski type inequalities for functions whose first derivatives are s-preinvex in the second sense. International Journal of Analysis and Applications, 15 (2017) no. 2, 146-154.

[12] B. Meftah, Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated log-preinvex. Punjab Univ. J. Math. (Lahore) 51 (2019), no. 2, 21–37.

[13] B. Meftah, M. Merad, N. Ouanas and A. Souahi, Some new Hermite-Hadamard type inequalities for functions whose nth derivatives are convex. Acta Comment. Univ. Tartu. Math. 23 (2019), no. 2, 163–178.

[14] B. Meftah, M. Benssaad, K. Kaidouchi and S. Ghomrani, Conformable fractional Hermite-Hadamard type inequalities for product of two harmonic s-convex functions. Proc. Amer. Math. Soc. 149 (2021), no. 4, 1495–1506.

[15] J. E. Nápoles Valdés, J. M. Rodríguez and J. M. Sigarreta, New Hermite–Hadamard type inequalities involving non-conformable integral operators. Symmetry, 11(2019) no. 9, 1108.

[16] M. A. Noor, K. I. Noor and S. Iftikhar, Newton inequalities for p-harmonic convex functions. Honam Math. J. 40 (2018), no. 2, 239-250.

[17] M. E. Özdemir, M. Gürbüz and Ç. Yildiz, Inequalities for mappings whose second derivatives are quasi-convex or h-convex functions. Miskolc Math. Notes 15 (2014), no. 2, 635-649.

[18] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[19] M. R. Delavar, A. Kashuri and M. De La Sen, On Weighted Simpson's $\frac{3}{8}$ Rule. Symmetry, 13 (2021), no.10, 1933.

[20] M. Z. Sarikaya, E. Set and M. E. Özdemir, On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex. J. Appl. Math. Stat. Inform. 9 (2013), no. 1, 37-45.

[21] E. Set, M. E. Özdemir and M. Z. Sarıkaya, On new inequalities of Simpson's type for quasi-convex functions with applications. Tamkang J. Math. 43 (2012), no. 3, 357-364.

[22] M. Vivas-Cortez, P. Kórus and J. E. Nápoles Valdés, Some generalized Hermite-Hadamard-Fejér inequality for convex functions. Adv. Difference Equ. 2021, Paper No. 199, 11 pp.

École des hautes études commerciales Alger. Algeria
E-mail address: selmasalah@hotmail.fr

[2]Département des Mathématiques, Faculté des mathématiques, de l'informatique et des sciences de la matière, Université 8 mai 1945 Guelma. Algeria.
E-mail address: badrimeftah@yahoo.fr