

ARTINIAN \mathbf{M} -COMPLETE, \mathbf{M} -REDUCED AND
MINIMALLY \mathbf{M} -COMPLETE ASSOCIATIVE RINGSL.M. MARTYNOV, T.V. PAVLOVA 

Abstract: The paper studies the modification of the concepts of completeness and reducibility proposed by the first of the authors (2021) for associative rings. Artinian \mathbf{M} -complete, \mathbf{M} -reduced and minimally \mathbf{M} -complete associative rings are characterized.

Keywords: associative ring; artinian ring; finite ring; simple ring; skew field; field; nilpotent ring; complete ring; reduced ring.

1 Introduction

In the theory of abelian groups, the notions of complete (divisible), reduced, and periodic (in particular, primary) group are of great importance. In [1] (also see [2] and [3]), some analogs of these notions were defined for arbitrary varieties of algebras. In the mentioned papers, the concepts of (atomic) complete, (atomic) reduced, and solvable algebra [4] (see also [5]) were defined by means of the atoms of these varieties and the Malcev products for these atoms [6]. Furthermore, the notions of periodic and primary algebra were defined by using of notions of (atomic) completeness, (atomic) reducibility, and solvability. In particular, an algebra is called periodic if any of its monogenic (i. e., one generated) subalgebras is finitely reduced. Note that a group or semigroup is periodic as an universal algebra (in our sense) if and only if it is periodic as a group or semigroup in the ordinary sense.

It is different for associative rings. In the theory of associative rings, a ring is called periodic if its multiplicative semigroup is periodic (in the ordinary sense).

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Any finite non-prime field is a periodic ring in the ordinary sense. On the other hand, such the field is monogenic, but it is not a finitely reduced ring. That is, it is not a periodic algebra in the sense of the papers [1], [2], and [3]. To remove this difference, the paper [7] suggests modifying the concepts of complete, reducible, periodic, and primary associative ring. This is done by using a special set \mathbf{M} of subvarieties of the variety As of associative rings, where \mathbf{M} is union the set of lattice atoms of subvarieties of the variety As and the set of all varieties, each of which is generated by some finite non-prime field. In this case, \mathbf{M} -periodic rings are rings with finite monogenic subrings (i.e., there is an analogy with groups and semigroups). Moreover, any finite field is both \mathbf{M} -periodic and \mathbf{M} -primary. Thus, the modification of the concepts discussed, given in [7], is more natural for associative rings.

In [7], properties of \mathbf{M} -periodic and \mathbf{M} -primary associative rings are studied. In addition, the paper [7] characterizes the \mathbf{M} -periodic, \mathbf{M} -primary, and \mathbf{M} -reduced varieties of associative rings. From the results of the paper [7] (in particular, Remark 5.17) it follows that the class \mathcal{MC} of all \mathbf{M} -complete rings of As is closed with respect to homomorphic images, extensions and direct sums in As . Furthermore, the class \mathcal{MR} of all \mathbf{M} -reduced rings of As is closed with respect to subrings, direct products, and extensions in As . Besides, the variety As is transversal (in sense of [6]) with respect to any variety belonging to \mathbf{M} .

If we replace the set $At(L(As))$ by the set \mathbf{M} in the paper [8], the main result of [8] will change. From the modified result, we obtain that any ring R belonging to As contains the largest \mathbf{M} -complete subring $C_{\mathbf{M}}(R)$; $C_{\mathbf{M}}(R)$ is a two-sided ideal of the ring R ; the factor ring $R/C_{\mathbf{M}}(R)$ is an \mathbf{M} -reduced ring. All of the above means that the basic properties of the modified concepts of completeness and reducibility for associative rings are saved. Further, let the mapping $r_{\mathbf{M}} : As \rightarrow As$ be such that $r_{\mathbf{M}}(R) = C_{\mathbf{M}}(R)$ for all $R \in As$. From the above, we obtain that $r_{\mathbf{M}}$ is a radical in the sense of Kurosh and Amitsur (see, for example, [9], p. 91 or [10], p. 27). Here, \mathcal{MC} is a radical class and \mathcal{MR} is a semisimple class.

We say that the radical $r_{\mathbf{M}}$ is the \mathbf{M} -complete radical and the ideal $C_{\mathbf{M}}(R)$ of R is the \mathbf{M} -complete radical of the ring R . Note that $C_{\mathbf{M}}(R)$ contains any \mathbf{M} -complete subring of the ring R . Therefore, $r_{\mathbf{M}}$ is a *strict radical* in the sense of Kurosh [11] (see also [10], p. 148).

In the papers [13, 14, 15, 16, 17], the complete radical of an associative ring was studied. It is easy to verify that analogous main results of these papers also hold for the \mathbf{M} -complete radical. In the papers [18, 19, 20, 21, 23], the structure of complete and reduced associative rings was studied. The main results of these papers are significantly modified if we replace the concepts of completeness and reducibility by the concepts of \mathbf{M} -completeness and \mathbf{M} -reducibility.

Recall that in the theory of abelian groups, the concept of a complete group coincides with the concept of a divisible group. Any minimal divisible abelian group is isomorphic to the (additive) quasi-cyclic group \mathbb{C}_{p^∞} , where p is a prime, or to the additive group \mathbb{Q}^+ of the field \mathbb{Q} of rational numbers. Minimal divisible abelian groups have significant importance since any divisible abelian group is a direct sum of minimal divisible abelian groups (see, for example, [12], Theorem 23.1, p. 124). An associative ring R is called minimal \mathbf{M} -complete if it is a non-null \mathbf{M} -complete ring, and all proper subrings of the ring R are \mathbf{M} -reduced rings.

The aim of this paper is to characterize \mathbf{M} -complete, \mathbf{M} -reduced associative artinian rings, and minimal \mathbf{M} -complete finite associative rings. Note that the class of artinian rings contains all finite rings.

Nevertheless in this paper, we do not limit to artinian rings and provide some results that are valid for all associative rings. We will use the results from the papers [18, 19, 20, 21, 22, 23, 24] on the study of complete and reduced associative rings and modify them for the concepts of \mathbf{M} -completeness and \mathbf{M} -reducibility. If the obtained results have fundamental changes, then proofs are provided. The modified formulations of the statements are given only with reference to a similar statement if the changes are insignificant. We will provide and prove several lemmas before formulating and proving the main results of the paper. Some lemmas are of independent interest.

First let us give some definitions, notations, and facts about associative rings.

2 Basic definitions, designations and preliminary information

Further in the paper, by a *ring* we mean an associative ring (not necessarily with the unity), by an *ideal* we mean a two-sided ideal. By $|M|$ denote the cardinal number of a set M . Positive integers are denoted by k, l, m, n (sometimes with subscripts), and primes are denoted by p, q . By R^+ denote the additive group of the ring R . A ring with zero multiplication will be called an abelian ring. An abelian ring with an additive group R^+ is denoted by R^0 . By O denote the zero ideal of the ring R . A *simple ring* is a non-zero ring that has no ideal besides the O and itself. The smallest $n \in \mathbb{N}$ is said to be the *characteristic* of a ring R , if $nR = O$ and is denoted by $\text{char } R$. If there is no such n , then $\text{char } R = 0$ is assumed.

The set of natural numbers is denoted by \mathbb{N} , the set of primes is denoted by \mathbb{P} . By \mathbb{Z} we denote the ring of integers, and by \mathbb{Q} we denote the field of rational numbers. Furthermore, let \mathbb{Z}_n denote the the ring of residue classes modulo $n > 1$. The finite field (Galois field) of p^m elements is denoted by \mathbb{F}_{p^m} . A *prime field* is a field which has no proper subfields. Any prime field is isomorphic to the field \mathbb{Q} of rational numbers or the finite field \mathbb{F}_p of p elements.

If M is a nonempty subset of the ring R , then by $\langle M \rangle$, (M) denote the subring, the ideal of R generated by M , respectively. The subring generated by $a \in R$ is called *monogenic* and denoted by $\langle a \rangle$. An element $e \in R$ with a property $e^2 = e$ is called an *idempotent* of R . The idempotent $e \in R$ is called *basic*, if $\sigma(e)$ is the unity of the factor ring $R/J(R)$, where $J(R)$ is the Jacobson radical of the ring R and σ is the natural homomorphism of R to $R/J(R)$. A ring R is called *idempotent*, if $R^2 = R$, where $R^2 = \langle a \cdot b \mid a, b \in R \rangle$.

We denote by $M_n(R)$ the ring of square $n \times n$ matrices over a ring R . For a commutative ring R with unity, by $R[x]$ denote the ring of polynomials in x over R . By $\mathbb{Z}\langle X \rangle$ we denote a free (in As) ring over the infinite countable set $X = \{x_1, x_2, \dots\}$, i. e. the ring of polynomials with integer coefficients in non-commuting variables of X with zero free terms. The *identity* is a formal equality of the form $f(x_1, x_2, \dots, x_n) = 0$, where $f(x_1, x_2, \dots, x_n) \in \mathbb{Z}\langle X \rangle$.

Suppose φ is the natural homomorphism of $\mathbb{Z}_{p^k}[x]$ onto $\mathbb{Z}_p[x]$ and $f(x) \in \mathbb{Z}_{p^k}[x]$ is a unitary polynomial of degree m such that $\varphi(f(x))$ is an irreducible polynomial over \mathbb{Z}_p ; then the factor ring $\mathbb{Z}_{p^k}[x]/(f(x))$ is called a *Galois ring* of characteristic

p^k and order p^{km} . The Galois ring up to isomorphism is defined by the numbers p , k , n and is denoted by $GR(p^k, m)$. It is obvious that $GR(p^k, 1) \cong \mathbb{Z}_{p^k}$ and $GR(p, m) \cong \mathbb{F}_{p^m}$. Galois rings play a special role in the structural theory of finite associative rings.

Let $\text{var } \Sigma$ denote the variety of rings defined by a system Σ of ring identities and $\text{var } \mathcal{K}$ be the least variety of rings containing a class \mathcal{K} of rings (in other words, $\text{var } \mathcal{K}$ is the variety generated of \mathcal{K}). The free monogenic ring in As will be denoted by $\mathbb{Z}\langle x \rangle$. Below we use the following important notations:

$$\mathcal{Z}_n^0 = \text{var}\{nx = 0, xy = 0\} = \text{var } \mathbb{Z}_n^0;$$

$$\mathcal{F}_{p^m} = \text{var}\{px = 0, x^{p^m} = x\} = \text{var } \mathbb{F}_{p^m}.$$

Let \mathcal{V} is a variety of rings. A ring R is called \mathcal{V} -complete if R has no homomorphisms onto non-zero rings from \mathcal{V} . Equivalently, the $\mathcal{V}(R) = R$, where $\mathcal{V}(R)$ is the *verbal ideal* of R (i. e., $\mathcal{V}(R)$ is the least ideal in the set of all ideals I of the ring R , such that the factor ring R/I belong \mathcal{V}). A ring is called \mathcal{V} -solvable if it has no non-zero \mathcal{V} -complete subrings.

Let \mathbf{M} be the union of two sets \mathbf{Z} and \mathbf{F} varieties of rings, where $\mathbf{Z} = \{\mathcal{Z}_p^0 \mid p \in \mathbb{P}\}$, $\mathbf{F} = \{\mathcal{F}_{p^m} \mid p \in \mathbb{P}, m \in \mathbb{Z}_+\}$, i. e., $\mathbf{M} = \mathbf{Z} \cup \mathbf{F}$. Note that \mathbf{M} contains the set $At(L(As))$, where $At(L(As))$ consists of the varieties \mathcal{Z}_p^0 and \mathcal{F}_p for all prime p (see, for example, [25]). A ring R is called \mathbf{M} -complete if R is \mathcal{M} -complete for every $\mathcal{M} \in \mathbf{M}$. We call a ring R \mathbf{M} -reduced if R has no non-trivial \mathbf{M} -complete subrings. By analogy, the concepts of \mathbf{Z} -complete (\mathbf{F} -complete) ring and of \mathbf{Z} -reduced (\mathbf{F} -reduced) ring are defined. We point out the connection between the concepts of completeness and \mathbf{M} -completeness, as well as the concepts of reducibility and \mathbf{M} -reducibility. Obviously, any \mathbf{M} -complete ring is complete. But the converse statement, generally speaking, is incorrect. For example, any non-minimal finite field F is complete, while F is \mathbf{M} -reduced. On the other hand, any reduced ring is \mathbf{M} -reduced.

Recall that if the variety \mathcal{V} is given by the identity system Σ , then the \mathcal{V} -verbal $\mathcal{V}(R)$ of a ring R coincides with the ideal of R generated by the values in R of all polynomials that are the left-hand sides of the identities of Σ . For varieties \mathcal{Z}_p^0 , \mathcal{F}_{p^m} and a ring R , we indicate formulas to calculate the corresponding verbals: $\mathcal{Z}_p^0(R) = pR + R^2$, $\mathcal{F}_{p^m}(R) = pR + R_{p^m}$, where R_{p^m} is the ideal generated by the set $\{r^{p^m} - r \mid r \in R\}$. It is clear that the ring R is \mathbf{M} -complete (\mathbf{M} -reduce) if and only if $\mathcal{Z}_p^0(R) = R$ ($\mathcal{Z}_p^0(R) = O$) and $\mathcal{F}_{p^m}(R) = R$ ($\mathcal{F}_{p^m}(R) = O$) for any p and m .

It is clear that the \mathbf{M} -complete radical $C_{\mathbf{M}}(R)$ of a ring R is equal to the sum of all \mathbf{M} -complete subrings of R . A ring R is \mathbf{M} -complete if and only if $C_{\mathbf{M}}(R) = R$. In particular, the \mathbf{M} -complete radical $C_{\mathbf{M}}(R)$ of any ring R is a \mathbf{M} -complete ideal of R . A ring R is \mathbf{M} -reduced if and only if $C_{\mathbf{M}}(R) = O$. From a well-known fact for arbitrary radicals (see, for example, [9], Proposition 1, p. 91) it follows that the \mathbf{M} -complete radical of a ring R is the intersection of all its ideals I , such that the factor ring R/I is \mathbf{M} -reduced.

Similarly, the \mathbf{Z} -complete radical $C_{\mathbf{Z}}(R)$ and the \mathbf{F} -complete radical $C_{\mathbf{F}}(R)$ of a ring R are defined by the sets \mathbf{Z} and \mathbf{F} , respectively. Recall that radical r is called strict if the radical $r(R)$ of a ring R contains every r -radical subring A (i.e., subring with the property $r(A) = A$) of R . As above, the \mathbf{M} -complete radical is strict. It is clear that \mathbf{Z} -complete and \mathbf{F} -complete radicals are also strict. A ring is \mathbf{M} -complete if and only if it is simultaneously \mathbf{Z} -complete and \mathbf{F} -complete. Denote by $\mathfrak{M}(\mathfrak{Z}, \mathfrak{F})$ the class of all rings belonging to the varieties of rings from the set $\mathbf{M}(\mathbf{Z}, \mathbf{F})$. It

is clear that $\mathbf{M}(\mathbf{Z}, \mathbf{F})$ -complete radicals are upper radicals defined by the class $\mathfrak{M}(\mathfrak{Z}, \mathfrak{F})$ respectively. In addition, for any prime p , we will need the designation \mathfrak{F}_p for the class of rings of characteristic p from class \mathfrak{F} .

Recall that the transversality of the variety As over the subvariety \mathcal{V} means that for any ring R and an ideal I of R , $\mathcal{V}(I)$ is an ideal R . As noted, transversality of the variety As over any variety from the set \mathbf{M} is proved in the paper [7].

In additive notation, the atoms of the lattice $L(Ab)$ of subvarieties of the variety Ab of all abelian groups Ab are the varieties $\mathcal{A}_p = \text{var}\{px = 0\}$ for all primes p . Note that \mathcal{A}_p -completeness of an abelian group A means the validity of the equality $\mathcal{A}_p(A) = pA = A$. Further, the divisibility of an abelian group A is equivalent to its \mathcal{A}_p -completeness over all primes p , i. e., completeness of A . It is well known that in every abelian group A a divisible subgroup is always a direct summand, in A there is a largest divisible subgroup $C(A)$, and A is the direct sum of its complete and reduced subgroups. Moreover, as noted above, every divisible abelian group is a direct sum of some sets of isomorphic copies of the additive group \mathbb{Q}^+ of rationales and of copies of quasi-cyclic groups C_{p^∞} for some primes p .

Recall that a ring is called left artinian ring if any decreasing chain of its left ideals is stabilizes. Equivalently, the ring satisfies the minimum condition of left ideals. Further, left artinian rings will be called artinian rings. It is well known (see, for example, [28], Theorem 1, p. 63) that the Jacobson radical of an artinian ring is nilpotent. In addition, by the Wedderburn-Artin Theorem (see, for example, [28], p. 65), any artinian semisimple (in the sense of Jacobson radical) ring is isomorphic to a direct sum of finitely many full matrix rings over skew fields. It is well known that a factor ring of an artinian ring is artinian. Also, if both the ideal I and the factor ring R/I of a ring R are artinian then R is itself artinian.

In conclusion of this section, we will give some well-known statements that do not relate to the concepts of \mathbf{M} -completeness and \mathbf{M} -reducibility, but are needed for the sequel.

Theorem 1. ([26], Theorem 122.7, p. 350) *Every artinian ring R is a ring direct sum $R = S \oplus T_{p_1} \oplus \dots \oplus T_{p_k}$ of some torsion-free artinian ring S and a finite number of artinian p_i -rings T_{p_i} corresponding to various primes p_i .*

Theorem 2 (Proposition 6 [32]). *Let R be a finite ring with unity of characteristic p^k and radical $J(R)$. Then R contains a subring Q isomorphic to a direct sum of matrix rings over Galois rings such that $Q/pQ \cong R/J(R)$ and a (Q, Q) -submodule M of $J(R)$ such that $R = Q + M$ with $Q \cap M = O$.*

Theorem 3. ([34, p. 35], Theorem 1.4.3) *If additive group R^+ of an left artinian ring R is a torsion-free group, then R possesses a left unity.*

Proposition 1. ([23], Lemma 12) *An artinian ring R is finite if and only if $mR = O$ for some $m \in \mathbb{N}$ and the factor ring $R/J(R)$ is finite.*

Lemma 1. ([13], Lemma 3) *If I is an ideal of ring R and K is a field, then any homomorphism $\varphi : I \rightarrow K$ it can be extended to homomorphism $\bar{\varphi} : R \rightarrow K$.*

Lemma 2. ([16], Lemma) *For any ideal I of a ring R , it holds $M_n(R)/M_n(I) \cong M_n(R/I)$.*

3 Artinian \mathbf{M} -complete rings

The aim of this section is to obtain a characterization of \mathbf{M} -complete artinian rings. Let's first give a few lemmas. Some of them are valid for arbitrary rings.

Lemma 3. *If R is a \mathbf{M} -complete ring, then R^2 is \mathbf{M} -complete ring.*

Proof. Let R be a \mathbf{M} -complete ring. Since R is an \mathcal{Z}_p^0 -complete ring for any prime p then $\mathcal{Z}_p^0(R) = pR + R^2 = R$ for any primes p , i. e., $R = R^2 + pR$. We have $R^2 = (R^2 + pR)(R^2 + pR) = R^4 + pR^3 + p^2R^2 = R^4 + pR(R^2 + pR) = R^4 + pR^2$, i. e., $\mathcal{Z}_p(R^2) = R^2$. The last equality means that the ring R^2 is \mathcal{Z}_p^0 -complete.

We show now that R^2 is \mathcal{F}_{p^m} -complete for any p and m . Since R is an \mathcal{Z}_p^0 -complete ring for any prime p , then for each x of R we will find elements a, a_i, b_i ($i = 1, \dots, n$) of R such that $x = pa + \sum_{i=1}^n a_i b_i$. Then $x^{p^m} - x = (pa + \sum_{i=1}^n a_i b_i)^{p^m} - (pa + \sum_{i=1}^n a_i b_i) = pz + \left((\sum_{i=1}^n a_i b_i)^{p^m} - (\sum_{i=1}^n a_i b_i) \right)$ for some $z \in R$. Therefore $R_{p^m} \subseteq pR + (R^2)_{p^m}$. It follows that $(R_{p^m})^2 \subseteq (pR + (R^2)_{p^m})(pR + (R^2)_{p^m}) \subseteq pR^2 + (R^2)_{p^m}$. Considering the last inclusion and the fact that the ring R is \mathcal{F}_{p^m} -complete, we get that $R^2 = (pR + R_{p^m})(pR + R_{p^m}) \subseteq pR^2 + (R_{p^m})^2 \subseteq pR^2 + (R^2)_{p^m} = \mathcal{F}_{p^m}(R^2)$, i. e., R^2 is \mathcal{F}_{p^m} -complete for any p and m . \square

Corollary 1. *If R is minimally \mathbf{M} -complete ring and $R^2 \neq O$ then $R^2 = R$.*

Lemma 4. *A simple ring R is either \mathbf{M} -complete and does not belong to the set \mathfrak{M} , or \mathbf{M} -reduced and it is isomorphic to a field \mathbb{F}_{p^m} for some p, m .*

Proof. A simple ring R is a ring with nonzero multiplication, so $R \in \mathbf{F}$ for some prime p . As is well know, any non-zero ring of any variety from \mathbf{F} is a subdirect product of finite fields (see, for example, [4]). \square

Lemma 5. *Any nil ring R is a \mathcal{F} -complete ring.*

Proof. The homomorphic image of a nil ring is a nil ring and therefore cannot be a non-zero ring of a variety \mathcal{F}_{p^m} for any p, m . \square

Lemma 6. ([19], Lemma 3) *If the ideal I of a ring R is contained in the kernel of any homomorphism of R onto rings from a variety \mathcal{V} of rings, then R is a \mathcal{V} -complete ring if and only if R/I is a \mathcal{V} -complete ring.*

Repeating the proof of Lemma 5 of the article [14] almost verbatim, one can verify the validity of the following statement.

Lemma 7. *A nilpotent ring R is \mathbf{M} -complete if and only if its additive group R^+ is divisible.*

We omit the proof of the following statement, analogous to Lemma 2 from [19]. It corresponds almost verbatim to the proof of this lemma and uses the results mentioned in Section 1 on the Jacobson radical of artinian rings, semisimple artinian rings, the structure of divisible abelian groups and Lemma 7.

Lemma 8. *For a ring R , the following conditions are equivalent:*

- 1) $R^+ \cong \bigoplus_{i=1}^n C_{p_i^\infty}$;
- 2) R is a \mathbf{M} -complete abelian artinian ring;
- 3) R is a \mathbf{M} -complete artinian nilpotent ring.

The main result of this section is modification of two statements of Theorems 1 and 2 of [19].

Theorem 4. *An artinian ring R is \mathbf{M} -complete if and only if the following conditions hold for its ideal R^2 :*

- 1) R^2 is an idempotent artinian ring and if $R^2 \neq O$ then $R^2/J(R^2) \cong \bigoplus_{i=1}^k M_{n_i}(K_i)$, where K_i is a skew field, $M_{n_i}(K_i) \not\cong \mathbb{F}_{p^m}$ for any prime p , $m \in \mathbb{N}$ and $i = 1, \dots, k$;
- 2) if $R^2 \neq R$ then $R/R^2 \cong \bigoplus_{j=1}^n C_{p_j^\infty}^0$.

Proof. Let R be a \mathbf{M} -complete artinian ring. From Theorem 1 it follows that R is the direct sum of its ideals S and T , where S is \mathbf{M} -complete torsion-free artinian ring and T is \mathbf{M} -complete artinian periodic ring. Therefore, it is enough to consider both rings separately.

By the Theorem 3, the ring S possesses a left unity. Therefore, $S^2 = S$. Since S is \mathbf{M} -complete ring, it follows that the factor ring $S/J(S)$ is \mathbf{M} -complete ring. From the Wedderburn-Artin Theorem and Lemma 4 it follows that factor ring $S/J(S)$ is isomorphic to a direct sum of finitely many full matrix rings over skew fields and does not contain summands isomorphic to a finite field \mathbb{F}_{p^m} for any prime p and $m \in \mathbb{N}$.

Further, consider a decreasing chain of ideals in T : $T \supseteq T^2 \supseteq T^3 \supseteq \dots$. Since T is an artinian ring, then $T^n = T^{n+1}$ for some n , i. e., ideal T^n is idempotent ring. Then $\bar{T} = T/T^n$ is a \mathbf{M} -complete artinian nilpotent ring, and by Lemma 8, \bar{T} is an abelian ring. Hence $xy \in T^n$ for all $x, y \in T$ and, therefore, $T^2 = T^n$, i. e., T^2 will be an idempotent ring. In addition, by Lemma 8, $T/T^2 \cong \bigoplus_{j=1}^n C_{p_j^\infty}^0$.

Let us show that ideal T^2 is artinian ring. Let I be a left ideal of T^2 . The group T^+ is periodic, then for any $i \in I$ there exists $m \in \mathbb{N}$ such that $mi = 0$. By Lemma 8, $T/T^2 \cong \bigoplus_{i=1}^k C_{p_i^\infty}^0$, it follows that there exists $t_1 \in T$ for any $t \in T$ such that $mt_1 = \bar{t}$ in factor ring $\bar{T} = T/T^2$. Since $t - mt_1 \in T^2$, we have $ti = (t - mt_1 + mt_1)i = (t - mt_1)i + mt_1i = (t - mt_1)i \in I$. This means that I is left ideal of T .

Thus T^2 is artinian and \mathbf{M} -complete ring by Lemma 3. Therefore $T^2/J(T^2)$ is \mathbf{M} -complete ring also. It follows that $T^2/J(T^2) \cong \bigoplus_{i=1}^k M_{n_i}(K_i)$, where K_i is skew field, $M_{n_i}(K_i) \not\cong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$.

Conversely, let for the ideal of the R^2 artinian ring R , the conditions 1) and 2) of the theorem are satisfied. Four cases are possible:

(i) $R = R^2 = O$, then R is \mathbf{M} -complete ring by definition.

(ii) $R \neq R^2 = O$ and $R/R^2 \cong \bigoplus_{j=1}^n C_{p_j^\infty}^0$. Then $R = R/R^2$ is \mathbf{M} -complete ring by Lemma 8.

(iii) $R = R^2 \neq O$ and $R/J(R) = R^2/J(R^2) \cong \bigoplus_{i=1}^k M_{n_i}(K_i)$, where K_i is skew field, $M_{n_i}(K_i) \not\cong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$. In this case, R is non-zero idempotent artinian ring and $R/J(R)$ is \mathbf{M} -complete ring. Then, $J(R)$ is \mathbf{M} -complete ring by Lemma 5. This means that $J(R)$ is contained in the kernel of any homomorphism onto rings from a variety \mathcal{F}_{p^m} for any p , m . Then by Lemma 6,

$R/J(R)$ is a \mathcal{F} -complete ring if and only if R is a \mathcal{F} -complete ring. The ring $R = R^2$ is \mathcal{Z} -complete also. Hence, R is \mathbf{M} -complete ring.

(iv) $R \neq R^2$ and $R^2 \neq O$, where $R/R^2 \cong \bigoplus_{j=1}^n C_{p_j}^0$ and ideal R^2 is idempotent

artinian ring. Then $R^2/J(R^2) \cong \bigoplus_{i=1}^k M_{n_i}(K_i)$, where K_i is skew field, $M_{n_i}(K_i) \not\cong \mathbb{F}_{p^m}$ for any prime p and $m \in \mathbb{N}$. \mathbf{M} -completeness of R^2 is proved similar to case (iii). The factor ring R/R^2 is \mathbf{M} -complete by Lemma 8. In this case, from Lemma 2.2 of [7] it follows that an extension of \mathbf{M} -complete ring R^2 by \mathbf{M} -complete ring R/R^2 is a \mathbf{M} -complete ring. Besides, it is known that an extension of artinian ring by artinian ring is artinian also. \square

A special case of Theorem 4 is a modification of the result on complete finite rings in [18].

Corollary 2. *A finite non-zero ring R is \mathbf{M} -complete if and only if the following conditions hold:*

- 1) $R^2 = R$;
- 2) $R/J(R)$ is \mathbf{M} -complete ring and $R/J(R) \cong \bigoplus_{i=1}^n M_{n_i}(\mathbb{F}_{p_i^{m_i}})$, where $n_i > 1$ for all $i = 1, 2, \dots, n$, $m_i \in \mathbb{N}$ and prime p_i .

4 Artinian \mathbf{M} -reduced rings

The aim of this section is to characterization of \mathbf{M} -reduced artinian rings.

The following statements are modifications of Lemmas [20] of reduced rings for \mathbf{M} -reduced rings. Their proofs are easy to obtain, if replace the field \mathbb{F}_p by the field \mathbb{F}_{p^m} for any p and m .

Lemma 9. ([20], Lemma 1) *For an artinian nilpotent ring R , the following conditions are equivalent:*

- 1) $mR = O$ for some $m \in \mathbb{N}$;
- 2) R is finite ring;
- 3) R is \mathbf{M} -reduced ring.

From Lemma 9 it follows that all nilpotent \mathbf{M} -reduced artinian rings are finite.

Lemma 10. ([20], Lemma 1) *Any artinian \mathbf{M} -reduced ring has a characteristic $m > 0$.*

From Lemma 10 and Theorem 1 it follows that it is sufficient to characterize artinian \mathbf{M} -reduced rings of characteristic p^k .

Lemma 11. ([20], Lemma 2) *Any ring R of characteristic p^k , where $k \in \mathbb{N}$ and p is prime number, is \mathcal{Z}_q^0 -complete and \mathcal{F}_{q^m} -complete for any prime $q \neq p$ and $m \in \mathbb{N}$.*

Lemma 12. ([20], Lemma 2) *A ring R of characteristic p^k for prime p and $k \in \mathbb{N}$, is a \mathbf{M} -complete ring if and only if the ring R/pR is \mathbf{M} -complete.*

The following lemma describes the \mathbf{M} -complete radical of an artinian ring of characteristic p^k for prime number p and $k \in \mathbb{N}$.

Lemma 13. *For an artinian ring R of characteristic p^k , where p is a prime and $k \in \mathbb{N}$, there exists $m, n \in \mathbb{N}$ such that the \mathcal{F} -complete radical $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R)$, and the \mathbf{M} -complete radical $C_{\mathbf{M}}(R) = \mathcal{F}_{p^m}^n(R)$, where n is the idempotent degree of the verbal $\mathcal{F}_{p^m}(R)$, i. e. $\mathcal{F}_{p^m}^n(R) = \mathcal{F}_{p^m}^{n+1}(R)$.*

Proof. Consider the set of verbals $\mathcal{F}_{p^d}(R)$ of the ring R , where $d \in \mathbb{N}$. Note that for any $t \in \mathbb{N}$ and $x \in R$,

$$x^{p^{td}} - x = \left(x^{p^d} - x\right) \cdot \sum_{i=0}^{s-1} x^{p^{td} - p^d - i(p^d - 1)}, \text{ where } s = \sum_{j=0}^{t-1} p^{(t-j)d}.$$

This means that if h is divisible by d , then $\mathcal{F}_{p^h}(R) \subseteq \mathcal{F}_{p^d}(R)$.

Since R is an artinian ring, R contains the minimal verbal $\mathcal{F}_{p^m}(R)$. At the same time $\mathcal{F}_{p^m}(R) \subseteq \mathcal{F}_{p^d}(R)$ for all $d \in \mathbb{N}$ (assuming that this is not the case, we get that the verbal $\mathcal{F}_{p^m}(R)$ is not minimal, since it contains the verbal $\mathcal{F}_{p^l}(R) \subseteq \mathcal{F}_{p^m}(R) \cap \mathcal{F}_{p^d}(R) \neq \mathcal{F}_{p^m}(R)$, where l is the least common multiple for d and m).

By the Lemma 11, the ring R of the characteristic p^k is \mathcal{F}_{q^t} -complete for any prime number $q \neq p$ and $t \in \mathbb{N}$. It follows from the Lemma 1 that any ideal \mathcal{F}_{q^t} -complete ring, in particular, the ideal $\mathcal{F}_{p^m}(R)$, is also \mathcal{F}_{q^t} -complete ring. From the same lemma and the fact that $\mathcal{F}_{p^m}(R) \subseteq \mathcal{F}_{p^k}(R)$ for all $k \in \mathbb{N}$ it follows that $\mathcal{F}_{p^m}(R)$ is \mathcal{F}_{p^k} -complete ring for all $k \in \mathbb{N}$. Therefore, $\mathcal{F}_{p^m}(R)$ is \mathcal{F} -complete ring, that is, $\mathcal{F}_{p^m}(R) \subseteq C_{\mathbf{F}}(R)$, from where $\mathcal{F}_{p^m}(R) = C_{\mathbf{F}}(R)$.

The decreasing chain of ideals $\mathcal{F}_{p^m}(R) \supseteq \mathcal{F}_{p^m}^2(R) \supseteq \mathcal{F}_{p^m}^3(R) \supseteq \dots$ of the ring R stabilizes at some step n . That is, $\mathcal{F}_{p^m}^n(R)$ is an idempotent ring, so \mathcal{Z} -complete. In addition, $\mathcal{F}_{p^m}^n(R)$ is \mathcal{F} -complete ring according to the Lemma 1. So, $\mathcal{F}_{p^m}^n(R) \subseteq C_{\mathbf{M}}(R)$. Conversely, since $C_{\mathbf{M}}(R) \subseteq \mathcal{F}_{p^m}(R)$, then $C_{\mathbf{M}}^n(R) \subseteq \mathcal{F}_{p^m}^n(R)$. Hence $C_{\mathbf{M}}(R) = C_{\mathbf{M}}^n(R) \subseteq \mathcal{F}_{p^m}^n(R)$ and therefore, $C_{\mathbf{M}}(R) = \mathcal{F}_{p^m}^n(R)$. \square

Lemma 14. *A non-nilpotent artinian ring R of characteristic p^k , where p is prime, $k \in \mathbb{N}$, is a \mathbf{M} -reduced ring if and only if R is a finite ring and $C_{\mathbf{F}}(R) = J(R)$. In addition, the factor ring $R/J(R)$ is isomorphic to a finite direct sum of fields $\mathbb{F}_{p^{k_i}}$ for $k_i \in \mathbb{N}$.*

Proof. First we shall show that $C_{\mathbf{F}}(R) = J(R)$. From the Lemma 5 it follows that $J(R)$ is \mathcal{F} -complete ring, so $J(R) \subseteq C_{\mathbf{F}}(R)$. Conversely, by Lemma 13, $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R)$ for some $m \in \mathbb{N}$. Since R is \mathbf{M} -reduced ring, then $C_{\mathbf{F}}^n(R) = \mathcal{F}_p^n(R) = O$ for some $n \in \mathbb{N}$. Thus, $\mathcal{F}_{p^m}(R)$ is a nilpotent ideal, hence $C_{\mathbf{F}}(R) = \mathcal{F}_{p^m}(R) \subseteq J(R)$.

The factor ring $R/J(R)$ is isomorphic to a direct sum of finitely many full matrix rings over skew fields. Since $\mathcal{F}_{p^m}(R) = J(R)$, the factor ring $R/J(R)$ belongs to the variety \mathcal{F}_{p^m} . Therefore, any of these summands belongs to the variety \mathcal{F}_{p^m} and is isomorphic to $\mathbb{F}_{p^{k_i}}$ for some $k_i \in \mathbb{N}$ by Lemma 4. Also, since $R/J(R)$ is a finite ring, then R is also a finite ring by the Proposition 1.

Conversely, if R is a finite ring, then $J(R)$ is \mathbf{M} -reduced ring by Lemma 9. Hence, the ring R is \mathbf{M} -reduced as an extension of \mathbf{M} -reduced ring $J(R)$ by \mathbf{M} -reduced ring $R/J(R)$. \square

The following statement, similar to Teorem 2 of [20], describes the structure of artinian \mathbf{M} -reduced rings.

Theorem 5. *An artinian ring R is a \mathbf{M} -reduced if and only if R is a finite ring with \mathcal{F} -complete radical $C_{\mathbf{F}}(R) = J(R)$ and either $R = J(R)$ or $R/J(R) \cong \bigoplus_{i=1}^n \mathbb{F}_{p_i}^{k_i}$, where p_i is prime, $k_i \in \mathbb{N}$.*

Proof. By Lemma 10, for any artinian \mathbf{M} -reduced ring R there exists $m \in \mathbb{N}$ such that $mR = O$. Let $m = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_n^{k_n}$ is the canonical representation of the

number m . Then by the Theorem 1, the ring R is a finite direct sum of its ideals R_i , where $p_i^{k_i} R_i = O$ for all $1 \leq i \leq n$.

It follows from the properties of a finite direct sum of rings that the rings R_i for all $1 \leq i \leq n$ are \mathbf{M} -reduced artinian rings. If the ring R_i is non-nilpotent, then it satisfies the conditions of Lemma 14, otherwise R_i satisfies the conditions of Lemma 9. If R_i is nilpotent, then $C_{\mathbf{F}}(R_i) = R_i$ by Lemma 5. In each case, R_i is a finite ring and $C_{\mathbf{F}}(R_i) = J(R_i)$. Thus, R is a finite ring and its \mathcal{F} -complete radical

$$C_{\mathbf{F}}(R) = \bigoplus_{i=1}^n C_{\mathbf{F}}(R_i) = \bigoplus_{i=1}^n \mathcal{F}_{p_i^{m_i}}(R_i) = \bigoplus_{i=1}^n J(R_i) = J(R).$$

Moreover, if $R \neq J(R)$, then the factor ring $R/J(R)$ is a finite direct sum of ideals isomorphic to finite fields.

Conversely, any finite ring R satisfying the conditions of the theorem is an extension of the \mathbf{M} -reduced ring $J(R)$ by the \mathbf{M} -reduced ring $R/J(R)$. This means that R is a \mathbf{M} -reduced ring. \square

5 Minimally \mathbf{M} -complete artinian rings

The aim of this section is to characterize minimally \mathbf{M} -complete artinian rings. Before proving the main result, we will give the formulations of analogs of the auxiliary statements of the articles [21] and [23] and prove some of them.

The proofs of the following several statements correspond almost verbatim to the proofs of their analogues, so we omit them.

Proposition 2. ([17], Proposition 1) *For a basic idempotent e of a non-nilpotent artinian ring R , $C_{\mathbf{M}}(eRe) = eC_{\mathbf{M}}(R)e$.*

For the \mathbf{M} -complete radical, the requirement that the idempotent e be the basic idempotent of a non-nilpotent artinian ring A is essential. For example, in the \mathbf{M} -complete ring $R = M_2(\mathbb{F}_p)$, for the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the subring $eRe \cong \mathbb{F}_p$ is \mathbf{M} -reduced, i. e. $C_{\mathbf{M}}(eRe) = O$. But, $eC_{\mathbf{M}}(R)e = eRe \neq O$.

Corollary 3. ([17], Corollary 2) *Non-nilpotent minimally \mathbf{M} -complete artinian ring contains a unit.*

Lemma 15. ([21], Lemma 11) *If any decreasing chain of ideals of a ring R contained in the ideal I of this ring stabilizes at some finite step, then from \mathbf{M} -reducibility of the ring R follows \mathbf{M} -reducibility of the ring R/I .*

Corollary 4. *Any homomorphic image of an artinian \mathbf{M} -reduced ring is a \mathbf{M} -reduced ring.*

In general, Corollary 4 is not true. For example, the free ring $\mathbb{Z}\langle X \rangle$ over the infinite countable set X is an \mathbf{M} -reduced ring. But its homomorphic image is, in particular, the field \mathbb{Q} of rational numbers, which is \mathbf{M} -complete.

Corollary 5. *The homomorphic image of a minimally \mathbf{M} -complete finite ring is a minimally \mathbf{M} -complete ring.*

Lemma 16. ([23], Lemma 11) *A finite idempotent ring R of characteristic p^k is minimally \mathbf{M} -complete if and only if R/pR is minimally \mathbf{M} -complete ring.*

Lemma 17. ([21], Lemma 15) *If I is a nilpotent ideal of a ring R of characteristic p^k and K is a \mathbf{M} -reduced subring of the ring R , then the homomorphic image \overline{K} in the ring $\overline{R} = R/I$ is also a \mathbf{M} -reduced ring.*

Corollary 6. *If R is a minimally \mathbf{M} -complete artinian ring of characteristic p^k , then the factor ring $R/J(R)$ is also a minimally \mathbf{M} -complete ring.*

Lemma 18. ([21], Lemma 1) *Minimally \mathbf{M} -complete nilpotent ring is isomorphic to the ring \mathbb{Q}^0 or the ring C_p^0 for some prime p .*

Lemma 19. ([21], Lemma 3) *A skew field K of characteristic zero is minimally \mathbf{M} -complete if and only if it is isomorphic to the field \mathbb{Q} of rational numbers.*

Corollary 7. *The ring \mathbb{Z} of integers and any of its subrings are \mathbf{M} -reduced.*

Proposition 3. ([16], Proposition) *For any ring R and $n > 1$, the ring $M_n(R)$ is \mathbf{M} -complete if and only if the ring R is \mathbf{Z} -complete.*

Corollary 8. *For any idempotent ring R and $n > 1$, the ring $M_n(R)$ is \mathbf{M} -complete.*

The description of minimally \mathbf{M} -complete skew field of prime characteristic p differs significantly from the description of complete skew field of prime characteristic p obtained in Lemma 4 of [21].

Lemma 20. *A skew field K of prime characteristic p is minimally \mathbf{M} -complete if and only if K is isomorphic to an algebraic closure $\widehat{\mathbb{F}}_p$ of the field \mathbb{F}_p .*

Proof. Let K be the minimally \mathbf{M} -complete skew field of a prime characteristic p . Then K contains the field \mathbb{F}_p that obviously lies in the center of K .

Just as with the proof of Lemma 4 of [21], it is shown that the existence of an element in K that is transcendent with respect to the field of \mathbb{F}_p is impossible.

Therefore all elements of the skew field K are algebraic with respect to the field \mathbb{F}_p . It is clear that elements of K are algebraic with respect to a field \mathbb{F}_{p^m} for any $m > 1$. Recall, that the field \mathbb{F}_{p^m} is \mathbf{M} -reduced by Lemma 4. If we take into account the well-known facts that for any finite field \mathbb{F}_q in the ring $\mathbb{F}_q[x]$ there exists an irreducible polynomial of any positive degree (see, for example, [27], Corollary 2.11, p. 70) and an algebraic extension of \mathbb{F}_q containing any of its root is again a finite field, it is easy to understand that K must coincide with the union of a countable infinite strictly increasing sequence of corresponding finite fields, i. e., K is isomorphic to an algebraic closure $\widehat{\mathbb{F}}_p$ of the field \mathbb{F}_p .

Conversely, if a skew field K of prime characteristic p is an algebraic closure of the field \mathbb{F}_p , i. e., $K \cong \widehat{\mathbb{F}}_p$, then any proper non-zero subring F of K is a finite field \mathbb{F}_{p^m} for some m and therefore it is \mathbf{M} -reduced. Thus, K is minimally \mathbf{M} -complete ring. \square

The analogue of Lemma 5 of [21] for the \mathbf{M} -completeness of also has significant changes.

Lemma 21. *The full ring $M_n(K)$ of matrices over a skew field K is minimally \mathbf{M} -complete if and only if $M_n(K)$ is isomorphic either to the field \mathbb{Q} of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field \mathbb{F}_p or a ring $M_2(\mathbb{F}_p)$ for some prime p .*

Proof. Let $M_n(K)$ is a minimally \mathbf{M} -complete ring of matrices of order n over the skew field K . Being \mathbf{M} -complete, the ring $M_n(K)$ does not belong to any variety of \mathbf{M} , since the rings of the latter are \mathbf{M} -reduced.

Let $n = 1$. In this case, $M_1(K) \cong K$. If $\text{char } K = 0$, then $K \cong \mathbb{Q}$ by Lemma 19. If $\text{char } K = p$, then $K \cong \widehat{\mathbb{F}}_p$ by Lemma 20.

Let it be now $n > 1$. The ring $M_n(K)$ in this case contains a skew field K as its proper subring. Due to the minimal \mathbf{M} -completeness of the ring $M_n(K)$, the skew field K must be \mathbf{M} -reduced. By Lemma 4, being a simple ring, K must be isomorphic to a finite field \mathbb{F}_{p^m} for some p and m . It is obvious that the ring $M_n(\mathbb{F}_{p^m})$ for $n \geq 3$ contains as a proper subring isomorphic to a \mathbf{M} -complete ring $M_2(\mathbb{F}_{p^m})$, and therefore is not \mathbf{M} -minimally complete. Hence, $n = 2$. If $m > 1$ then the ring $M_2(\mathbb{F}_{p^m})$ contains the \mathbf{M} -complete proper subring $M_2(\mathbb{F}_p)$. Thus $m = 1$.

At last, we show that the ring $M_2(\mathbb{F}_p)$ is minimally \mathbf{M} -complete. Consider any proper non-zero \mathbf{M} -complete subring R of a ring $M_2(\mathbb{F}_p)$. Being finite, and therefore artinian, a semisimple factor ring $R/J(R)$ by virtue of the Wedderburn-Artin Theorem is a direct sum of a finite number of matrix rings over suitable skew fields. It is clear that in our case these skew fields must be finite fields. But then the orders of the matrices included in the decomposition of the ring must be equal to 1. Being a direct sum of \mathbf{M} -reduced fields, by Lemma 2.5 of [7], the ring $R/J(R)$ must also be \mathbf{M} -reduced. But then it is clear that the subring R is not \mathbf{M} -complete. Thus the ring $M_2(\mathbb{F}_p)$ has no proper non-zero \mathbf{M} -complete subrings and therefore it is a minimally \mathbf{M} -complete ring. \square

Note that in [22] it is indicated that Lemma 5 of [21], describing minimally complete rings $M_n(K)$ of all $(n \times n)$ -matrices over a skew field K , at the end mistakenly states that such is the ring $M_2(\mathbb{F}_p)$ for any p . That this is not the case follows from the well-known representation of finite fields by matrices (see, e.g., [27], p. 90): elements of a finite field \mathbb{F}_{p^n} of order p^n can be represented by square matrices of the order n over the field \mathbb{F}_p . Consequently, the ring $M_2(\mathbb{F}_p)$ contains a subring isomorphic to the complete field \mathbb{F}_{p^2} , and therefore is not a minimally complete ring. As a consequence of this, from the formulations of Lemma 5 and the condition 2) of Theorem of [21] rings of matrices of the form $M_2(\mathbb{F}_p)$ for all p should be excluded. Rings R for which the factor ring R/pR is isomorphic to $M_2(\mathbb{F}_p)$ are also excluded from the formulation of the condition 3) of the same theorem. The exact formulation of Theorem of [21] is Theorem 4 of [24].

Corollary 9. *A semisimple artinian ring is minimally \mathbf{M} -complete if and only if it is isomorphic either to the field \mathbb{Q} of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field \mathbb{F}_p or a ring $M_2(\mathbb{F}_p)$ for some prime p .*

Lemma 22. *A minimally \mathbf{M} -complete finite ring R of a prime characteristic is semisimple by Jacobson.*

Proof. A ring R satisfying the conditions of Lemma 22 is an algebra of finite dimension over a field \mathbb{F}_p . A ring $R/J(R)$ is minimally \mathbf{M} -complete by Corollary 5. Therefore, by Corollary 9, $R/J(R)$ is a ring $M_2(\mathbb{F}_p)$ of square matrices of order 2 over finite field \mathbb{F}_p for some p , i. e., $R/J(R)$ is a central simple algebra. In any case, we get that the algebra is a separable algebra over a field \mathbb{F}_p . The field \mathbb{F}_p is perfect, therefore, according to the Wedderburn-Maltsev Theorem (see, for example, Theorem 13.18 in [29], p. 575), $R = J(R) \oplus S$, where S is a subalgebra isomorphic

$R/J(R)$. Since the ring $R/J(R)$ is \mathbf{M} -complete, and R is minimally \mathbf{M} -complete, then $R = S$, i.e. R is a semisimple ring. \square

Corollary 10. *In a minimally \mathbf{M} -complete finite ring R of characteristic p^k , the equality $J(R) = pR$ is valid.*

Proof. In the ring R the ideal pR is nilpotent, therefore $pR \subseteq J(R)$. On the other hand, the ring R/pR is minimally \mathbf{M} -complete by Lemma 5 and semisimple by Lemma 22. Hence $J(R) \subseteq pR$, i. e., $J(R) = pR$. \square

Lemma 23. *A minimally \mathbf{M} -complete ring R of all matrices of some order over the Galois ring is isomorphic to the ring $M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$.*

Proof. For every Galois ring $GR(p^k, m)$, the factor ring $GR(p^k, m)/pGR(p^k, m) \cong GR(p, m) = \mathbb{F}_{p^m}$. Note, that any Galois ring has an unit and therefore it is an idempotent ring. By Corollary 8, the matrix ring $M_n(GR(p^k, m))$ is \mathbf{M} -complete for any $n \geq 2$. It follows that if $R = M_n(GR(p^k, m))$ be the minimally \mathbf{M} -complete ring then $n = 2$. By Lemma 2, $R/pR \cong M_2(GR(p^k, m))/M_2(pGR(p^k, m)) \cong M_2(GR(p, m)) = M_2(\mathbb{F}_{p^m})$. It follows from Lemma 21 that $R/pR \cong M_2(\mathbb{F}_p) \cong M_2(GR(p, 1))$. But then $R \cong M_2(GR(p^k, 1)) \cong M_2(\mathbb{Z}_{p^k})$.

Conversely, let $R = M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$. Then R is a finite ring for which $R^2 = R$ and $p^k R = O$. By Lemma 16, the ring R is minimally \mathbf{M} -complete if and only if the ring R/pR is minimally \mathbf{M} -complete. Since $R/pR \cong M_2(\mathbb{F}_p)$ is minimally \mathbf{M} -complete by Lemma 21, we see that the ring $R = M_2(\mathbb{Z}_{p^k})$ is minimally \mathbf{M} -complete. \square

The main result of this section is the following modification of Theorem 1, [21].

Theorem 6. 1) *Any minimally \mathbf{M} -complete nilpotent ring is isomorphic to the ring \mathbb{Q}^0 or the ring $C_{p^\infty}^0$ for some prime number p .*

2) *A simple ring with unit is minimally \mathbf{M} -complete if and only if it is isomorphic to the field \mathbb{Q} of rational numbers or the algebraic closure $\widehat{\mathbb{F}}_p$ of the field \mathbb{F}_p or a ring $M_2(\mathbb{F}_p)$ for some prime p .*

3) *A finite ring is minimally \mathbf{M} -complete if and only if it is isomorphic to a matrix ring $M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$.*

Proof. 1) This statement of the Theorem 6 is the content of Lemma 18.

2) Let a simple ring with unit is \mathbf{M} -minimally complete. Then it is artinian ring (see, for example, Corollary 4, [31], p. 196). But a simple artinian ring is isomorphic to the ring of matrices $M_n(K)$ for some skew field K and a natural number n by the Wedderburn-Artin Theorem. The rest follows from the Lemma 9.

3) Let R be a finite minimally \mathbf{M} -complete ring. It follows that the additive group of the ring R is bounded. Then by the Theorem 1, R is a ring of characteristic p^k for some prime p . The ring R is non-nilpotent by the Lemma 9. It follows that R is a ring with unity by Corollary 3.

By Corollary 6, the ring $R/J(R)$ is also minimally \mathbf{M} -complete. Then $R/J(R) \cong M_2(\mathbb{F}_p)$ by Corollary 9. By Theorem 2, the ring R contains a subring S isomorphic to the direct sum of full matrix rings over Galois rings such that $S/J(S) \cong R/J(R)$. In the ring S , the equality $J(S) = pS$ is valid. This means that $S/pS \cong M_2(\mathbb{F}_p)$. We obtain that S is a minimally \mathbf{M} -complete subring of the ring R by the Lemma 16. Therefore $R = S$.

Minimally \mathbf{M} -complete rings of all matrices some order over Galois rings are described in Lemma 23. Hence we get that $R \cong M_2(\mathbb{Z}_{p^k})$ for some prime p and $k \in \mathbb{N}$. \square

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