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BALANCING AND MODIFIED PELL NUMBERS OF THE FORM $-2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5}$

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Abstract: In this paper, we investigate the solutions of Balancing and Modified Pell numbers of the form $-2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5}$, where a_1, a_2, a_3, a_4, a_5, are non-negative integers with $0 \leq \max\{a_1, a_2, a_3, a_4\} \leq a_5$.

Keywords: Exponential Diophantine equation, Linear forms in logarithms, Modified Pell number, Balancing number.

1 Introduction

The balancing sequence $\{B_r\}_{r\geq 1}$ is derived from Panda and Behera's [13] simple Diophantine equation

$$1 + 2 + \ldots + (r-1) = (r+1) + (r+2) + \cdots + (r+t),$$

A balancer corresponding to r is represented by the integer t. For any $r \geq 2$ the recurrence of the Balancing sequence is determined by $B_r = 6B_{r-1} - B_{r-2}$, where $B_0 = 0$ and $B_1 = 1$. For any $s \geq 2$, the Modified Pell sequence is defined as $q_s = 2q_{s-1} + q_{s-2}$, where $q_0 = q_1 = 1$. Many recent studies have focused on Diophantine equations linked to binary recurrence sequences and perfect powers, either for specified unique bases or the same bases for different powers. Fibonacci and Lucas numbers of the pattern $2^a + 3^b +$

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5^c, where a, b, c, are non-negative integers with $c \geq \max\{a, b\} \geq 0$, were discovered by Marques and Togb? Bravo [4]. Bravo and Luca [12] discussed how to solve the Diophantine equation $F_n + F_m = 2^a$ where n, m and a are all positive integers. Qu, Zeng, and Cao [5] proved Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c + 7^d$ where a, b, c, d are non-negative integers with $d \geq \max\{a, b, c\}$.
Furthermore, Chi Chim and Ziegler[6] demonstrated the sum of Fibonacci numbers as a sum Powers of 2. Irmak and He [7]] discovered the solution to the s-th power of the Fibonacci number of the form $2^a + 3^b + 5^c$, where a, b, c and s are positive integers with $1 \leq \max\{a, b\} \leq c$. Recently, Qu and Zeng[8] demonstrated that Pell and Pell-Lucas Numbers of the Form $-2^a - 3^b + 5^c$, where a, b, c are non-negative integers with $c \geq \max\{a, b\} \geq 0$. In 2023, Bhoi and Ray [9] find all solutions of the Diophantine equation $B_{n_1} + B_{n_2} = 2^{a_1} + 2^{a_2} + 2^{a_3}$, for positive integers n_1, n_2, a_1, a_2, a_3. In this study, we tend to demonstrate all solutions of the Balancing and Modified Pell numbers of the type $-2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5}$, using the same technique as in the previous results.

2 Auxiliary results

2.1. Balancing sequence. We have the Balancing sequence's Binet formula is

$$B(r) = \frac{\alpha^r - \beta^r}{4\sqrt{2}} \text{ for all } r \geq 0, \tag{1}$$

where $\beta = 3 - \sqrt{8}$ and $\alpha = 3 + \sqrt{8}$ are roots of the polynomial

$$f(x) = x^2 - 6x + 1.$$

The inequality holds true for the all-natural integer

$$\alpha^{r-1} \leq B_r \leq \alpha^r. \tag{2}$$

2.2. Modified Pell sequence. The Binet formula of the Modified Pell sequence is

$$q_s = \frac{\lambda^s + \delta^s}{\lambda + \delta} \text{ for all } s \geq 0, \tag{3}$$

where $\delta = 1 - \sqrt{2}$ and $\lambda = 1 + \sqrt{2}$ are roots of the polynomial

$$f(x) = x^2 - 2x - 1.$$

The inequality can be easily shown by induction, and it holds for the all-natural number s.

$$\lambda^{s-1} \leq q_s \leq \lambda^s. \tag{4}$$

2.3. Linear forms in logarithmic.

Definition 1. *(Absolute logarithmic height)*
Let γ be an algebraic number in the number field $\mathbb{K}$ of degree d with minimal

polynomial on $\mathbb{Z}$

$$c_0x^d + c_1x^{d-1} + \ldots + c_d = c_0 \prod_{i=1}^{d} \left(x - \gamma^{(i)}\right),$$

where each $\gamma^{(i)}$ *is a conjugate of* γ, *and* c_i *'s are relatively primes to each other with* $c_0 > 0$. *Then the logarithmic height of* γ *is defined by:*

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right). \tag{5}$$

The logarithmic height's following characteristics can be demonstrated:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{6}$$

$$h\left(\eta\gamma^{\pm 1}\right) \leq h(\eta) + h(\gamma), \tag{7}$$

$$h\left(\eta^k\right) = |k| h(\eta). \tag{8}$$

In order to demonstrate Theorems 2 and 3, we use Baker and Wüstholz [11] lower bound for linear forms in algebraic number logarithms. Using a modified version of the Matveev result [1], we are the following [3, Theorem 9.4].

Theorem 1. *Let* $\mathbb{L}$ *be a real algebraic number field of degree* D *over* $\mathbb{Q}$. *Let* $\gamma_1, \ldots, \gamma_t \in \mathbb{L}$ *be a positive real algebraic numbers and* $b_1, b_2, \ldots, b_t$ *be nonzero integers, with*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$\log|\Lambda| > G\left(A_1 \ldots A_t\right)(1 + \log D)(1 + \log B),$$

where $G = (-1.4)\left(30^{t+3}\right)\left(t^{4.5}\right)\left(D^2\right)$, *and*

$$B \geq \max\left\{|b_1|, \ldots, |b_t|\right\},$$

and

$$A_i \geq \max\left\{Dh\left(\gamma_i\right), \left|\log\left(\gamma_i\right)\right|, 0.16\right\}, 1 \leq i \leq t.$$

where $A_1, \ldots, A_t$ *are real numbers.*

Determining the distance of a real number T to the closest integer is expressed as $\|T\| = \min\{|T - n| : n \in \mathbb{Z}\}$.

The following lemma, which is an extension of the well-known Baker Davenport lemma [10] and was proven by Dujella and Petho [2], is used to lower an upper bound.

Lemma 1. *If* M *is a positive integer and* p/q *is a convergent continued fraction of the irrational number* γ *such that* $q > 6M$, *let* K, L, μ *be real* *numbers with $K > 0$ and $L > 1$ and $\varepsilon := \|\mu q\| - M\|\gamma q\|$. Then the inequality has no solution if $\varepsilon > 0$,*

$$0 < |u\gamma - r + \mu| < \frac{K}{L^w},$$

in positive integers u, r and w, with

$$u \leq M \text{ and } w \geq \frac{\log(Kq/\varepsilon)}{\log L}.$$

Theorem 2. *The non-negative integer solutions $(r, a_1, a_2, a_3, a_4, a_5)$ of the Diophantine equation*

$$B_r = -2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5}, \tag{9}$$

with $a_5 \geq \max\{a_1, a_2, a_3, a_4\}$ are $(r, a_1, a_2, a_3, a_4, a_5) \in \{(0,1,0,0,1,1),$ $(0,1,1,1,0,1), (1,0,0,0,1,1), (1,0,1,1,0,1), (2,1,0,0,0,1), (5,0,2,3,1,3)\}$.

Theorem 3. *The non-negative integer solutions $(r, a_1, a_2, a_3, a_4, a_5)$ of the Diophantine equation*

$$q_s = -2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5}, \tag{10}$$

with $a_5 \geq \max\{a_1, a_2, a_3, a_4\}$ are $(r, a_1, a_2, a_3, a_4, a_5) \in \{(0,0,0,0,1,1),$ $(0,0,1,1,0,1), (1,0,0,0,1,1), (1,0,1,1,0,1), (2,0,0,1,0,1), (3,0,0,0,0,1),$ $(6,0,2,1,1,2)\}$.

3 Proof theorem 2

3.1. Bounding r. We are combining (1) and (9), we get

$$\frac{\alpha^r}{4\sqrt{2}} - 11^{a_5} = -2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + \frac{\beta^r}{4\sqrt{2}},$$

Taking absolute values for both sides, we get

$$\left|\frac{\alpha^r}{4\sqrt{2}} - 11^{a_5}\right| < 2^{a_1} + 3^{a_2} + 5^{a_3} + 7^{a_4} + \frac{1}{8\sqrt{2}},$$

because $|\beta|^r < \frac{1}{2}$. Dividing both sides by 11^{a_5} thus

$$\left|\frac{\alpha^r 11^{-a_5}}{4\sqrt{2}} - 1\right| < \frac{2^{a_1}}{11^{a_5}} + \frac{3^{a_2}}{11^{a_5}} + \frac{5^{a_3}}{11^{a_5}} + \frac{7^{a_4}}{11^{a_5}} + \frac{1}{8\sqrt{2}11^{a_5}},$$

yields

$$\left|\frac{\alpha^r 11^{-a_5}}{4\sqrt{2}} - 1\right| < \frac{5}{11^{0.15a_5}}. \tag{11}$$

As a result of the inequality (2) and equation (9), we now want to determine the relationship between r and a_5.

$$\alpha^{r-1} \leq B_r < 11^{a_5},$$

which yields $a_5 > 0.74r - 0.74$ and $r < 1.37a_5 + 1$. If $a_5 \leq 15$, then $r < 22$. In order to find the solutions to Theorem 2, we use Maple programming to search in $0 \le a_5 \le 15$ and $0 \le r < 22$. We suppose that $a_5 > 15$ for the last supposition. The inequality (2), on the other hand, is where we have

$$\begin{aligned} \alpha^r &> -2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5} \\ &> -11^{0.3a_5} - 11^{0.5a_5} - 11^{0.7a_5} - 11^{0.85a_5} + 11^{a_5} \\ &> 11^{a_5}\left(-11^{-0.7a_5} - 11^{-0.5a_5} - 11^{-0.3a_5} - 11^{-0.15a_5} + 1\right) \\ &> 0.9 \cdot 11^{a_5}, \end{aligned}$$

This indicates that $a_5 < 0.74r + 0.04$ and $r > 1.36a_5 - 0.06$ yields $a_5 < r$.

We utilise Theorem 1, on the left hand side inequality (11), with parameters $t := 3$, we take $\gamma_1 := \alpha, \gamma_2 := 11, \gamma_3 := 4\sqrt{2}$ and $b_1 := r, b_2 := -a_5, b_3 := -1$. We know that, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$. Thus $D := [\mathbb{Q}(\sqrt{2}), \mathbb{Q}] = 2$. To show that the left hand side inequality (11), in nonzero, assume that $\Lambda_1 = 0$, which is implies that $32 \cdot 11^{2a_5} = \alpha^{2r}$, so $32 \cdot 11^{2a_5} \in \mathbb{Z}$ but $\alpha^{2r} \notin \mathbb{Z}$ for all value r which is false. Note that $h(\gamma_1) = \frac{\log(\alpha)}{2}, h(\gamma_2) = \log(11)$ and $h(\gamma_3) = \log(4) + \frac{\log(2)}{2}$. Thus we can take $A_1 := 1.77, A_2 := 4.8$ and $A_3 := 3.5$. As we only get $\max\{|r|, |-a_5|, |-1|\} = r$, then we get $B := r$. So by Theorem 1, we get

$$\frac{5}{11^{0.15a_5}} > |\Lambda_1| > \exp\{-G(1 + \log(r))(1.77)(4.8)(3.5)\}, \tag{12}$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(4)(1 + \log(2))$. Since we have $a_5 > 0.74r - 0.74$, taking logarithm for both sides (12), we get

$$0.27r - 1.9 < 2.883622871 \cdot 10^{13}(1 + \log(r)).$$

Thus

$$r < 3.9 \cdot 10^{15}. \tag{13}$$

3.2. Reducing the bound on r . The upper bound for r is given in the previous subsection 3.1. Applying Lemma 1 allows us to lower this upper bound.

Let

$$w_1 := r\log(\alpha) - a_5\log(11) - \log(4\sqrt{2}).$$

Then we have, by inequality (11),

$$|e^{w_1} - 1| < \frac{5}{11^{0.15a_5}}.$$

We know $w_1 \neq 0$, since $\Lambda_1 \neq 0$.

If $w_1 > 0$, then

$$0 < w_1 < \frac{5}{11^{0.15a_5}}.$$

And if $w_1 < 0$, since we have $\frac{5}{11^{0.15a_5}} < \frac{1}{2}$, for all $a_5 > 15$, we get $|e^{w_1} - 1| < \frac{1}{2}$. Therefore we get $e^{|w_1|} < 2$, since we have $w_1 < 0$, thus:

$$0 < |w_1| < e^{|w_1|} - 1 \le e^{|w_1|}\left|e^{|w_1|} - 1\right| < \frac{10}{11^{0.15a_5}},$$

as a result, we have

$$0 < \left| r\log(\alpha) - a_5\log(11)) - \log(4\sqrt{2}) \right| < \frac{10}{11^{0.15a_5}}.$$

Both sides dividing by log(11), we get

$$0 < \left| \frac{r\log(\alpha)}{\log(11)} - a_5 - \frac{\log(4\sqrt{2})}{\log(11)} \right| < \frac{4.2}{11^{0.15a_5}}. \tag{14}$$

As stated by the Theorem 1, we have

$$\gamma := \frac{\log(\alpha)}{\log(11)}, \mu := -\frac{\log\left(4\sqrt{2}\right)}{\log(11)},$$
$$K := 4.2, L := 1.4, w := a_5.$$

We recognise that γ is irrational. Let $\frac{p_k}{q_k}$ be the k th convergence of γ. Also, we can take that $M = 3.9 \cdot 10^{15}$. We obtain $q_{31} = 39964073819961229$, the denominator 31th convergent of using Maple programming, which is greater than 6M, and

$$\varepsilon := \|\mu q\| - M\|\gamma q\| = 0.31.$$

Therefore, we get $a_5 < 122$, and hence

$$r < 1.37 * (122) + 1 < 169,$$

Therefore, the equation $(r, a_1, a_2, a_3, a_4, a_5)$ has a solution in the a_5 and r range shown above. Once more, we may use Theorem 1, take $M := 169$, and perform the calculation using Maple programming to arrive at q_{11}, the denominator's 11th convergent of γ , which is more than $6M$, we obtain

$$\varepsilon := \|\mu q\| - M\|\gamma q\| = 0.06.$$

We get that $a_5 < 40$, and hence

$$r < 1.37 \cdot 40 + 1 < 56.$$

In the range, we employ Maple programming $40 \geq a_5 \geq \max\{a_1, a_2, a_3, a_4\} \geq 0$ and $r < 56$. We obtain the same set of solutions in Theorem 2. by programme inspection. The proof is so finished.

4 Proof theorem 3

4.1. Bounding s. When the equations (3) and (10) are combined, we obtain

$$\frac{\lambda^s}{2} - 11^{a_5} = -2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} - \frac{\delta^s}{2},$$

Taking absolute values for both sides and $|\delta|^s < \frac{1}{2}$, we get

$$\left| \frac{\lambda^s}{2} - 11^{a_5} \right| < 2^{a_1} + 3^{a_2} + 5^{a_3} + 7^{a_4} + \frac{1}{4},$$

Dividing both sides by 11^{a_5} thus

$$\left|\frac{\lambda^s 11^{-a_5}}{2}-1\right| < \frac{2^{a_1}}{11^{a_5}}+\frac{3^{a_2}}{11^{a_5}}+\frac{5^{a_3}}{11^{a_5}}+\frac{7^{a_4}}{11^{a_5}}+\frac{1}{4*11^{a_5}},$$

yields

$$\left|\frac{\lambda^s 11^{-a_5}}{2}-1\right| < \frac{5}{11^{0.15a_5}}. \tag{15}$$

When s and a_5 are related, we now want to estimate that relationship. Using the equation (10) and the inequality (4), we are able to arrive at

$$\lambda^{s-1} < q_s < 11^{a_5},$$

which yields $a_5 > 0.37s - 0.37$ and $s < 2.7a_5 + 1$. If $a_5 \leq 10$, then $s < 28$. In order to find the set solutions in Theorem 3, we use Maple programming to search in $0 \leq a_5 \leq 10$ and $0 \leq s < 28$. We presume that $a_5 > 10$ as our final presumption. With respect to the second inequality of (4), we have

$$\begin{aligned}
\lambda^s &> -2^{a_1} - 3^{a_2} - 5^{a_3} - 7^{a_4} + 11^{a_5}\\
&> -11^{0.3a_5} - 11^{0.5a_5} - 11^{0.7a_5} - 11^{0.85a_5} + 11^{a_5}\\
&> 11^{a_5}\left(-11^{-0.7a_5} - 11^{-0.5a_5} - 11^{-0.3a_5} - 11^{-0.15a_5} + 1\right)\\
&> 0.97 \cdot 11^{a_5},
\end{aligned}$$

which implies that $a_5 < 0.37r + 0.012$ and $s > 2.7a_5 - 0.03$, and this yields $a_5 < s$.

We apply Theorem 1, on the left hand side inequality (11), we have $t := 3$, we take $\gamma_1 := \lambda, \gamma_2 := 11, \gamma_3 := 2$ and $b_1 := s, b_2 := -a_5, b_3 := -1$. Hence, we get, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$. Thus $D := [\mathbb{Q}(\sqrt{2}), \mathbb{Q}] = 2$. According to Theorem 2, the left hand side inequality (15) is nonzero, assuming that $\Lambda_2 = 0$,which implies that $4 \cdot 11^{2a_5} = \lambda^{2s}$, so $4 \cdot 11^{2a_5} \in \mathbb{Z}$ but $\lambda^{2s} \notin \mathbb{Z}$ for all value s which is false. We have $h(\gamma_1) = \frac{\log(\lambda)}{2}, h(\gamma_2) = \log(11)$ and $h(\gamma_3) = \log(2)$. Thus, we get $A_1 := 0.88, A_2 := 4.8$ and $A_3 := 1.4$. Note that if $\max\{|s|, |-a_5|, |-1|\} = s$, then we can take $B := s$. So by Theorem 1, we obtain

$$\frac{5}{11^{0.15a_5}} > |\Lambda_2| > \exp\{-G(1+\log(s))(0.88)(4.8)(1.4)\}, \tag{16}$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(4)(1+\log(2))$. Since we have $a_5 > 0.37s - 0.37$, taking the logarithm for both sides (16), we get

$$0.13s - 1.74 < 5.734662432 \cdot 10^{12}(1+\log(s)).$$

Thus

$$s < 1.6 \cdot 10^{15}. \tag{17}$$

4.2. Reducing the bound on s. The upper bound for s, which was obtained in the previous subsection 4.1, is too large. Applying Lemma 1, we lower this upper bound.

Let

$$y_1 := s\log(\lambda) - a_5\log(11) - \log(2).$$

Then we have, by inequality (15),

$$|e^{y_1} - 1| < \frac{5}{11^{0.15a_5}}.$$

We get $y_1 \neq 0$, since $\Lambda_2 \neq 0$.
If $y_1 > 0$, then

$$0 < y_1 < \frac{5}{11^{0.15a_5}}.$$

And if $y_1 < 0$, since we have $\frac{5}{11^{0.15a_5}} < \frac{1}{2}$, for all $a_5 > 10$, we get $|e^{y_1} - 1| < \frac{1}{2}$. Therefore we get $e^{|y_1|} < 2$, since we have $y_1 < 0$, thus:

$$0 < |y_1| < e^{|y_1|} - 1 \leq e^{|y_1|}\left|e^{|y_1|} - 1\right| < \frac{10}{11^{0.15a_5}}.$$

Thus, we obtain

$$0 < |s\log(\lambda) - a_5\log(11)) - \log(2)| < \frac{10}{11^{0.15a_5}}.$$

Both sides divided by $\log(11)$, we get

$$0 < \left|\frac{s\log(\lambda)}{\log(11)} - a_5 - \frac{\log(2)}{\log(11)}\right| < \frac{4.2}{11^{0.15a_5}}. \tag{18}$$

Theorem 1 states that we have

$$\gamma := \frac{\log(\lambda)}{\log(11)}, \mu := -\frac{\log(2)}{\log(11)},$$
$$K := 4.2, , L := 1.4, w := a_5.$$

Let $\frac{p_k}{q_k}$ be the k-th convergence of the irrational number γ. We take that $M = 1.6{\cdot}10^{15}$. With the help of Maple programming, we get $q_{33} = 39964073819961229$, the denominator 33th convergent of γ, which exceeds $6M$, and

$$\varepsilon := \|\mu q\| - M\|\gamma q\| = 0.13.$$

As a result , we get $a_5 < 124$, and hence

$$s < 2.7 \cdot (124) + 1 < 336,$$

It implies that there is a solution for $(s, a_1, a_2, a_3, a_4, a_5)$ in the a_5 and s range mentioned before. Once more, we can use Maple programming to determine the denominator 11th convergent of γ, which is more than $6M$, using $M := 336$ and Theorem 1, we obtain

$$\varepsilon := \|\mu q\| - M\|\gamma q\| = 0.4.$$

We get that $a_5 < 36$, and hence

$$s < 2.7 \cdot 36 + 1 < 99.$$

We use Maple programming in the range $36 \geq a_5 \geq \max\{a_1, a_2, a_3, a_4\} \geq 0$, and $s < 99$. We obtain the same results from the programme checks for Theorem 3. As a result, the major outcome is finished

5 Conclusion

In order to find and reduce the upper bound considered above, we used a lower bound of linear forms in logarithms of algebraic number and used Lemma 1. In the Theorems 2 and 3, we found all the answers to the Diophantine equations 9 and 10.

6 Development in the Future

We point out that we could use our method to show that the Diophantine equation only has a finite number of solutions $B_r = 2^{a_1} + 3^{a_2} + 5^{a_3} + 7^{a_4} + 11^{a_5}, q_s = 2^{a_1}+3^{a_2}+5^{a_3}+7^{a_4}+11^{a_5}$, in non-negative integers a_1, a_2, a_3, a_4, a_5 with $a_5 \geq \max\{a_1, a_2, a_3, a_4\}$. Future researchers will be faced with this dilemma, which we leave open.

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