



SPECTRUM OF A PROBLEM ABOUT THE FLOW OF  
A POLYMERIC VISCOELASTIC FLUID IN A  
CYLINDRICAL CHANNEL  
(VINOGRADOV-POKROVSKI MODEL)

D.L. TKACHEV   
AND E.A. BIBERDORF 

**Abstract:** We study the linear stability of a resting state for flows of incompressible viscoelastic polymeric fluid in an infinite cylindrical channel in axisymmetric perturbation class. We use structurally-phenomenological Vinogradov-Pokrovski model as our mathematical model.

We formulate two equations that define the spectrum of the problem. Our numerical experiments show that with the growth of perturbations frequency along the channel axis there appear eigenvalues with positive real part for the radial velocity component of the first spectral equation. That guarantees linear Lyapunov instability of the resting state.

**Keywords:** incompressible viscoelastic polymeric medium, rheological correlation, resting state, linearized mixed problem, Lyapunov stability.

---

TKACHEV, D.L., BIBERDORF, E.A., SPECTRUM OF A PROBLEM ABOUT THE FLOW OF A POLYMERIC VISCOELASTIC FLUID IN A CYLINDRICAL CHANNEL (VINOGRADOV-POKROVSKI MODEL).

© 2023 TKACHEV D.L., BIBERDORF E.A.

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008).

Поступила 1 января 2023 г., опубликована 31 декабря 2023 г.

## 1 Introduction

To study the flows of an incompressible viscoelastic polymeric fluid in an infinite cylindrical channel we use structural-phenomenological Vinogradov-Pokrovski model as a base [1, 2]. This model interprets polymeric medium as a suspension of polymer macromolecules moving in an anisotropic fluid consisting of, e.g., solvent and other macromolecules. The environment effects on a chosen macromolecule is approximated by the impact on a chain of brownian particles, each of which is a sufficiently large part of the macromolecule. It turns out that the formulated physical model is an effective way of describing slow relaxation processes in mediums with linear polymers.

Using a mechanical analogy we call the brownian particles "beads" and the analogue of the elastic powers between the particles "springs". In the simplest case when the macromolecule is modelled as a "dumbbell" ("dumbbell" is two beads connected by a spring), we formulate the system of differential correlations (Vinogradov-Pokrovski model):

$$\rho\left(\frac{\partial}{\partial t}v_i + v_l\frac{\partial}{\partial x_l}v_i\right) = \frac{\partial}{\partial x_k}\sigma_{il}, \quad \frac{\partial v_i}{\partial x_i} = 0, \quad (1)$$

$$\sigma_{il} = -p\delta_{il} + 3\frac{\eta_0}{\tau_0}a_{il}, \quad (2)$$

$$\frac{d}{dt}a_{il} - v_{ij}a_{jl} - v_{lj}a_{ji} + \frac{1 + (k - \beta)I}{\tau_0}a_{il} = \frac{2}{3}\gamma_{il} - \frac{3\beta}{\tau_0}a_{ij}a_{jl}, \quad (3)$$

$$I = a_{11} + a_{22} + a_{33}, \quad \gamma_{il} = \frac{v_{il} + v_{li}}{2}, \quad i, l = 1, 2, 3. \quad (4)$$

Here  $\rho$  is polymer density,  $v_i$  is  $i$ -th velocity component,  $\sigma_{il}$  is stress tensor,  $p$  is pressure;  $\eta_0$ ,  $\tau_0$  are initial values of shear viscosity and relaxation time for viscoelastic component,  $v_{ij}$  is velocity gradient tensor,  $a_{il}$  is symmetric anisotropy stress tensor;  $\gamma_{il}$  is symmetrized velocity gradient tensor, where components of the velocity gradient tensor  $\nabla \otimes v$  are calculated as follows:  $v_{il} = \frac{\partial v_i}{\partial x_l}$ ,  $i, l = 1, 2, 3$ ;  $k$  and  $\beta$  are phenomenological parameters that take into account the size and the form of a macromolecule ball. Equations (1) are motion equation and incompressibility condition, and equations (2)-(3) are rheological correlation, that connects kinematic characteristics of the flow with its thermodynamic parameters; for each component  $a_{il}$  the sum of the first three terms in the left part of equality (3) is the so called upper convective derivative or Oldroyd derivative [3],  $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}, \nabla)$  is material derivative.

Note that the accepted physical representation of a polymeric medium allows us to describe its main rheological properties: the decrease of viscosity and the first difference of normal stresses with the growth of shear velocity, the growth of stretching viscosity to a certain limit with the growth of deformation velocity.

Moreover, unlike the known models FENE-R [4], FENE-CR [5] that take into account additional physical mechanisms reflecting the behaviour features

of a studied material: boundedness and nonlinearity of a spring elongation, connected to the finite length of a macromolecule and the existence of weaves and engagements in it, which obstruct its uniform and infinite elongation (instead of a Hooke law the nonlinear law of a spring elasticity is used); or RHL-model [6] that take into account potential barriers, that slow down the transition from one equilibrium configuration to the other (additional force of an inner resistance is introduced), the Pokrovski-Vinogradov model allows us to acquire nonzero values of the second difference for normal stresses. Specifically, it tries to take into account the anisotropy effect of the chosen molecule environment that is caused by its elongation and orientation in space during the flow process of its macromolecule chains.

Rheological properties, predicted by the Pokrovski-Vinogradov model with parameters  $k = 1, 2\beta$ , that guarantee monotone of a flow curve, are qualitatively and quantitatively agree with the experimental data for melts and solutions of polymers [7, 8, 9].

A number of works [10, 11] studied the linear Lyapunov stability of Poiseuille-type flows in an infinite plane channel (the pressure drop on a segment doesn't depend on time) for the model (1)-(4), as well as for its generalization on the case of nonisometric flow of an incompressible weakly conducting polymeric fluid with the existence of a negative space charge [12, 13, 14, 15] and on the case of nonisothermic model with the additional external interaction of a uniform magnetic field [16, 17, 18, 19].

In particular, in work [10] it was proven that the Poiseuille-type flow for the model (1)-(4) is linearly unstable in a perturbation class of generalized functions from the [20, 21] Schwartz space  $S'$  of the functions of slow growth with respect to the  $x$  variable, that changes along the channel side. I.e. the solution of a linearized problem is growing as an exponential function with the power  $st$ ,  $s = i\xi\hat{u} + \sqrt[3]{Q(y)}\xi^{\frac{2}{3}} + o(\xi^{\frac{2}{3}})$ , for  $|\xi| \rightarrow \infty$  ( $\xi$  is a dual variable for variable  $x$  with respect to the Fourier transform,  $\hat{u}$  is a component of the base Poiseuille-type flow,  $Q(y) \neq 0$ , is a function, that depends on the problem parameters and its base solution, "o" is small o).

The question of stability of the resting state for nonisothermic model of the polymeric fluid flow in an infinite plane channel under the influence of an external magnetic field was studied in works [22, 23, 24]. The main result being that the resting state in the case of an absolute conductivity, i.e. vanishing of the parameter inversly proportional to the magnetic Reynolds number, and additionally vanishing of one of the dissipative coefficients is linearly unstable by Lyapunov.

For the analysis of the applicability of mathematical models for the description of real flows of polymeric fluid of special interest is the question about the stability of the resting state of the model. From the physical point of view this property is a necessary one.

The result of the work [25] was refined in the works [26, 27]. It states that the spectrum of a linearized with respect to the resting state mixed problem for the system (1)-(4) does not lie in an open right half-plane. One of the

main results of these works is that the mixed problem has solutions with more than exponential growth  $e^{Re\lambda t}$ ,  $Re\lambda > 0$ ,  $t \rightarrow +\infty$ .

In the current work we study the location of the spectrum of a linear problem about the flow of a polymeric fluid in an infinite cylindrical channel. As a base solution we again chose the resting state and as a perturbation class we choose perturbations with axial symmetry and periodic with respect to the variable varying along the channel axis.

Note that the study of the flows of fluids of different nature in domains with cylindrical boundaries is fundamentally important not only from the point of view of a boundary geometry influencing the formation of characteristic features of the flow (alike the flows of viscous fluid (Navier-Stokes model) between the two coaxial rotating cylinders in the classic Taylor work [28]), but also to have the ability to experimentally check the picture of a flow predicted by the model.

The work is structured in the following way.

In the second paragraph we formulate the initial quasilinear and linearized with respect to the resting state models written in a cylindrical coordinate system. In the axisymmetric case the spectral problem essentially splits into two problems. The first is for determining the velocity components  $u$  (along the radius),  $w$  (along the cylinder axis) and the generalized pressure  $\hat{\Omega}$ . The second is for determining the velocity component  $v$  (along the angle coordinate).

It turns out that the first problem can be reduced to the spectral problem for the component  $u$ . In the end of the paragraph we formulate the main theorem of this work. The theorem statement is about two spectral problems.

The third paragraph is the proof of the main theorem.

Finally the last fourth paragraph is the graphical interpretation of a possible spectrum placement and graphs of the eigenfunctions for different values of the Reynolds ( $Re$ ) and Weisenberg ( $Wi$ ) numbers.

The main result of numerical experiments is that for a wide range of values of numbers  $Re$  and  $Wi$  the resting state is unstable in the class of axisymmetric flows of polymeric fluid.

## 2 Quasilinear and linearized models. Formulation of the main results

Following the monographs [1, 2, 29, 30, 31] and works [32, 33], we formulate the mathematical model for describing flows of an incompressible polymeric fluid in an infinite cylinder channel with round section (see Fig. 1).

We can write the model in a dimensionless form and in a cylindrical coordinate system as follows:

$$\operatorname{div} \mathbf{u} = \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0, \quad (5)$$

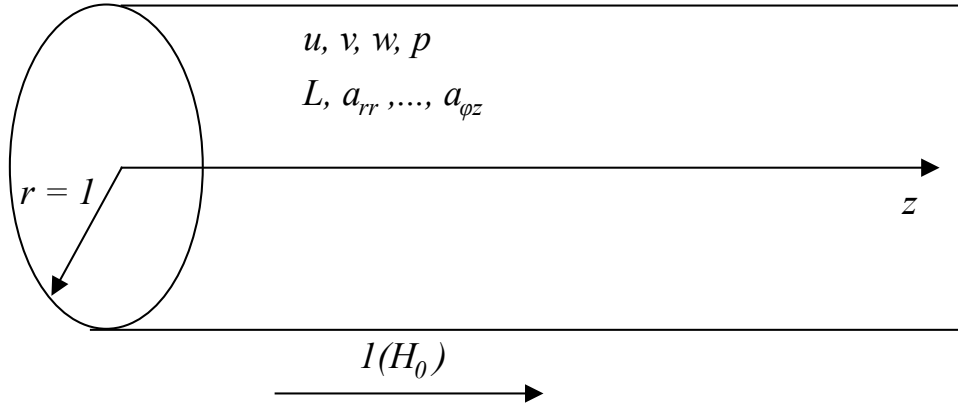


Рис. 1. Cylindrical channel and main parameters of the polymeric fluid flow

$$\frac{du}{dt} - \frac{v^2}{r} + \frac{\partial p}{\partial r} = \frac{1}{Re} \left( \frac{\partial a_{rr}}{\partial r} + \frac{1}{r} \frac{\partial a_{r\phi}}{\partial \phi} + \frac{\partial a_{rz}}{\partial z} + \frac{a_{rr} - a_{\phi\phi}}{r} \right), \quad (6)$$

$$\frac{dv}{dt} + \frac{uv}{r} + \frac{1}{r} \frac{\partial p}{\partial \phi} = \frac{1}{Re} \left( \frac{\partial a_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial a_{\phi\phi}}{\partial \phi} + \frac{\partial a_{\phi z}}{\partial z} + \frac{2a_{r\phi}}{r} \right), \quad (7)$$

$$\frac{dw}{dt} + \frac{\partial p}{\partial z} = \frac{1}{Re} \left( \frac{\partial a_{rz}}{\partial r} + \frac{1}{r} \frac{\partial a_{\phi z}}{\partial \phi} + \frac{\partial a_{zz}}{\partial z} + \frac{a_{rz}}{r} \right), \quad (8)$$

$$\frac{da_{rr}}{dt} - 2 \left( A_r \frac{\partial u}{\partial r} + \frac{a_{r\phi}}{r} \frac{\partial u}{\partial \phi} + a_{rz} \frac{\partial u}{\partial z} \right) + L_{rr} = 0, \quad (9)$$

$$\frac{da_{\phi\phi}}{dt} + 2 \left( \frac{v}{r} - \frac{\partial v}{\partial r} \right) a_{r\phi} - 2 \left( \frac{1}{r} (u + \frac{\partial v}{\partial \phi}) A_{\phi} + a_{\phi z} \frac{\partial v}{\partial z} \right) + L_{\phi\phi} = 0, \quad (10)$$

$$\frac{da_{zz}}{dt} - 2 \left( a_{rz} \frac{\partial w}{\partial r} + \frac{a_{\phi z}}{r} \frac{\partial w}{\partial \phi} + A_z \frac{\partial u}{\partial z} \right) + L_{zz} = 0, \quad (11)$$

$$\frac{da_{r\phi}}{dt} + \left( \frac{v}{r} - \frac{\partial v}{\partial r} \right) A_r + \left( a_{r\phi} \frac{\partial w}{\partial z} - a_{rz} \frac{\partial v}{\partial z} - \frac{A_{\phi}}{r} \frac{\partial u}{\partial \phi} - a_{\phi z} \frac{\partial u}{\partial z} \right) + L_{r\phi} = 0, \quad (12)$$

$$\frac{da_{rz}}{dt} - a_{rz} \left( \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} \right) - \left( A_r \frac{\partial w}{\partial r} + \frac{a_{r\phi}}{r} \frac{\partial w}{\partial \phi} + \frac{a_{\phi z}}{r} \frac{\partial u}{\partial \phi} + A_z \frac{\partial u}{\partial z} \right) + L_{rz} = 0, \quad (13)$$

$$\frac{da_{\phi z}}{dt} + \left( \frac{v}{r} - \frac{\partial v}{\partial r} \right) a_{rz} - \left( a_{\phi z} \frac{\partial u}{\partial r} + A_z \frac{\partial v}{\partial z} + a_{r\phi} \frac{\partial w}{\partial r} + \frac{A_{\phi}}{r} \frac{\partial w}{\partial \phi} \right) + L_{\phi z} = 0. \quad (14)$$

In equations (5)–(14)  $t$  is time,  $u, v, w$  are components of a velocity vector  $\mathbf{u}$  in a cylindrical coordinate system;  $p$  is hydrodynamic pressure;  $a_{rr}, \dots, a_{\varphi z}$  are components of a symmetrical anisotropy tensor  $\Pi$  of a second rank [1, 2];

$$L_{rr} = K_I a_{rr} + \beta \|a_r\|^2, \quad L_{\varphi\varphi} = K_I a_{\varphi\varphi} + \beta \|a_\varphi\|^2, \quad L_{zz} = K_I a_{zz} + \beta \|a_z\|^2,$$

$$L_{r\varphi} = K_I a_{r\varphi} + \beta(a_r, a_\varphi), \quad L_{rz} = K_I a_{rz} + \beta(a_r, a_z), \quad L_{\varphi z} = K_I a_{\varphi z} + \beta(a_\varphi, a_z),$$

$$a_r = (a_{rr}, a_{r\varphi}, a_{rz}), \quad a_\varphi = (a_{r\varphi}, a_{\varphi\varphi}, a_{\varphi z}), \quad a_z = (a_{rz}, a_{\varphi z}, a_{zz}),$$

$$A_r = a_{rr} + Wi^{-1}, \quad A_\varphi = a_{\varphi\varphi} + Wi^{-1}, \quad A_z = a_{zz} + Wi^{-1},$$

$$K_I = Wi^{-1} + \bar{k}I/\beta, \quad I = a_{rr} + a_{\varphi\varphi} + a_{zz}, \quad \bar{k} = k - \beta,$$

the square of the vector norm  $\|\cdot\|^2$  is the sum of its components,  $k, \beta$ ,  $0 < \beta < 1$  are phenomenological parameters of a rheological model [1, 2],  $Re = (\rho u_H l)/\eta_0$  is the Reynolds number,  $Wi = (\tau_0 u_H)/l$  is the Weissenberg number,  $\rho (= const)$  is the density of the medium,  $\eta_0, \tau_0$  are initial values of shear viscosity and relaxation time [1, 2],  $l$  is the characteristic length,  $u_H$  is the characteristic velocity,  $\Delta_{r,\varphi,z} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$  is the Laplace operator,  $\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \varphi} + w \frac{\partial}{\partial z}$ .

The system (5)–(14) is written in a dimensionless form: variables  $t, r, z, u, v, w, p, a_{rr}, \dots, a_{\varphi z}$  are related to  $l/u_H, l, u_H, \rho u_H^2, \frac{Wi}{3}$  correspondingly.

We state that the no-slip condition hold on the boundary  $r = 1$

$$\mathbf{u} = 0, \tag{15}$$

and on the cylinder axis, i.e. for  $r = 0$ , there are boundedness conditions on all the unknown variables  $u, v, w, p, a_{rr}, \dots, a_{\varphi z}$ .

As a base solution we choose the resting state

$$\mathbf{u} = 0, \quad p = p_0 - const, \quad a_{rr} = 0, \dots, a_{r\varphi} = 0.$$

Linearizing the boundary problem (5)–(15) with respect to the chosen solution results in a following problem (small perturbations of the components of the solution are written the same as initial variables)

$$Ru + \frac{1}{r} v_\varphi + w_z = 0, \tag{16}$$

$$u_t + \Omega_r = \frac{1}{r} (\alpha_{r\varphi})_\varphi + (\alpha_{rz})_z + \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{r}, \tag{17}$$

$$v_t + \frac{1}{r} \Omega_\varphi = (\alpha_{r\varphi})_r + \frac{1}{r} (\alpha_{\varphi\varphi} - \alpha_{rr})_\varphi + (\alpha_{\varphi z})_z + \frac{2\alpha_{r\varphi}}{r}, \tag{18}$$

$$w_t + \Omega_z = (\alpha_{rz})_r + \frac{1}{r} (\alpha_{\varphi z})_\varphi + (\alpha_{zz} - \alpha_{rr})_z + \frac{\alpha_{rz}}{r}, \tag{19}$$

$$\Lambda \alpha_{rr} = 2\mathfrak{x}^2 u_r, \tag{20}$$

$$\Lambda \alpha_{\varphi\varphi} = \frac{2}{r} (u + v_\varphi) \mathfrak{x}^2, \tag{21}$$

$$\Lambda \alpha_{zz} = 2\mathfrak{x}^2 u_z, \tag{22}$$

$$\Lambda \alpha_{r\varphi} = \frac{\mathfrak{x}^2}{r} u_\varphi - \mathfrak{x}^2 \left( \frac{v}{r} - v_r \right) \tag{23}$$

$$\Lambda \alpha_{rz} = \mathfrak{x}^2(w_r + u_z), \quad (24)$$

$$\Lambda \alpha_{\varphi z} = \mathfrak{x}^2(v_z + \frac{w_\varphi}{r}). \quad (25)$$

$$u = 0, \quad \text{for } r = 1.$$

Here  $\alpha_{rr} = \frac{a_{rr}}{Re}, \dots, \alpha_{\varphi z} = \frac{a_{\varphi z}}{Re}, \mathfrak{x}^2 = \frac{1}{WRe}, R = \frac{\partial}{\partial r} + \frac{1}{r}, \Lambda = \frac{\partial}{\partial t} + \frac{1}{W}, \Omega = p - \alpha_{rr}.$

**Замечание 1.** For the function  $\Omega$ , i.e. generalized "pressure", the following holds

$$\begin{aligned} D_0 \Omega = & \left( \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} - \frac{1}{r} \frac{\partial}{\partial r} \right) (\alpha_{\varphi\varphi} - \alpha_{rr}) + \\ & + 2R \left( \frac{\partial}{\partial z} \alpha_{rz} + \frac{1}{r} \frac{\partial}{\partial \varphi} \alpha_{r\varphi} + \frac{2}{r} \frac{\partial}{\partial \varphi} \left( \frac{\partial}{\partial z} \alpha_{\varphi z} + \frac{1}{r} \alpha_{r\varphi} \right) \right), \end{aligned} \quad (26)$$

$$\text{where } D_0 = \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} = R^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

We will be looking for a solution of the problem (16)–(25) in the special form:

$$\begin{aligned} u(t, r, \varphi, z) &= u(r) \exp\{\lambda t + inz + im\varphi\}, \dots, \\ \alpha_{\varphi z}(t, r, \varphi, z) &= \alpha_{\varphi z}(r) \exp\{\lambda t + inz + im\varphi\}, \end{aligned} \quad (27)$$

where  $\lambda = \eta + i\xi, \xi, \eta \in R^1, n, m \in Z$  are some parameters.

Then for the components of the anisotropy tensor under the additional condition

$$\lambda \neq -\frac{1}{Wi} \quad (28)$$

we get the following:

$$\begin{cases} \alpha_{rr} = \frac{u'}{\lambda}, & \alpha_{\varphi\varphi} = \frac{u + imv}{r\bar{\lambda}}, & \alpha_{zz} = \frac{inu}{\bar{\lambda}}, \\ \alpha_{r\varphi} = \frac{1}{2\bar{\lambda}} \left\{ \frac{imu}{r} + v' - \frac{v}{r} \right\}, & \alpha_{rz} = \frac{w' + inu}{2\bar{\lambda}}, \\ \alpha_{\varphi z} = \frac{i(nv + \frac{1}{r}mw)}{2\bar{\lambda}}, & \bar{\lambda} = \frac{\lambda + Wi^{-1}}{2\mathfrak{x}^2}. \end{cases} \quad (29)$$

The other four equations of the system (16)–(24) are rewritten as follows:

$$Ru + i\left(\frac{m}{r}v + nw\right) = 0, \quad (30)$$

$$\Omega' = -\left(\frac{m^2 + 2}{2r^2\bar{\lambda}} + \frac{n^2}{2\bar{\lambda}} + \lambda\right)u + \frac{1}{r\bar{\lambda}}u' + \frac{im}{2\bar{\lambda}r}\left(v' - \frac{3v}{r}\right) + \frac{in}{2\bar{\lambda}}w', \quad (31)$$

$$R^2v - \left(n^2 + \frac{1 + 3m^2}{r^2} + 2\lambda\bar{\lambda}\right)v = \frac{2nm}{r}w - \frac{4im}{r^2}u + \frac{2im\bar{\lambda}}{r}\Omega, \quad (32)$$

$$R^2w - \left(n^2 + \frac{m^2}{r^2} + 2\lambda\bar{\lambda}\right)w = 2in\bar{\lambda}\Omega + 2mnv + 2n^2u - \frac{2inu}{r}. \quad (33)$$

From (17), (26) and (29) it follows that

$$\begin{aligned} d_0\Omega &= \left(-\frac{m^2}{r^2} - \frac{1}{r}\frac{d}{dr}\right)\left(\frac{u + imv}{r\bar{\lambda}} - \frac{u'}{\bar{\lambda}}\right) + \\ &+ R\left(in\frac{w' + inu}{\bar{\lambda}} + \frac{im}{r}\frac{\frac{inu}{r} + v' - \frac{v}{r}}{\bar{\lambda}}\right) + \\ &+ \frac{im}{r}\left(-n\frac{nv + \frac{1}{r}mw}{\bar{\lambda}} + \frac{1}{r}\frac{\frac{inu}{r} + v' - \frac{v}{r}}{\bar{\lambda}}\right) - n^2\left(\frac{inu}{\bar{\lambda}} - \frac{u'}{\bar{\lambda}}\right), \end{aligned} \quad (34)$$

$$\tilde{\Omega}' = \frac{imv'}{2\bar{\lambda}} - \frac{inw'}{2\bar{\lambda}} \quad \text{for } r = 1, \quad (35)$$

where

$$\tilde{\Omega} = \Omega - \frac{u}{r\bar{\lambda}} - \frac{in}{\bar{\lambda}}w. \quad (36)$$

Here

$$d_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - n^2 = R^2 - \frac{m^2}{r^2} - n^2.$$

In an axisymmetric case, when  $m = 0$ , which is the main interest to us, the system (30)–(33) is simplified by splitting into two independent subsystems. That and the boundary conditions (25) leads us to the following two spectral boundary problems:

$$\begin{cases} (d_1 - 2\lambda\bar{\lambda})v = 0, \\ v = 0 \quad \text{for } r = 1, \\ |v(0)| < \infty. \end{cases} \quad (37)$$

$$\begin{cases} Ru + inw = 0, \\ (d_0 - 2\lambda\bar{\lambda})w - 2in\bar{\lambda}\hat{\Omega} - 2n^2u + n^2w = 0, \\ \hat{\Omega}' + \frac{2\lambda\bar{\lambda} + n^2}{2\bar{\lambda}}u = 0, \\ u = w = 0 \quad \text{for } r = 1, \\ |u(0)| < \infty, \quad |w(0)| < \infty. \end{cases} \quad (38)$$

Here

$$d_1 = R^2 - n^2 - \frac{1}{r^2}, \quad d_0 = R^2 - n^2, \quad \hat{\Omega} = \tilde{\Omega} + \frac{in}{2\bar{\lambda}}w,$$

function  $\tilde{\Omega}$  can be represented through  $\Omega$  due to (35). The correlation (34) takes the following form:

$$d_0\tilde{\Omega} = \frac{in^3}{\bar{\lambda}}(w - u).$$

Assume the parameter  $n$  is non-zero. Then by differentiating the second equation from the system (38) and replacing  $\hat{\Omega}'$  by using the third equation from the system (38) we can also replace  $w$  using the first equation and get to the spectral problem for one component  $u$ :

$$\begin{cases} u^{IV} + \frac{2}{r}u''' + u'' \left( -\frac{3}{r^2} - 2\lambda\bar{\lambda} \right) + u' \left( \frac{3}{r^3} - \frac{2\lambda\bar{\lambda}}{r} + 2in^3 \right) + \\ + u \left( -\frac{3}{r^4} + \frac{2\lambda\bar{\lambda}}{r^2} + n^2(2\lambda\bar{\lambda} + n^2) \right) = 0, \\ |u(0)| < \infty, \quad \left| \left( u' + \frac{1}{r}u \right) \right|_{r=0} < \infty, \\ u(1) = 0, \quad u'(1) + u(1) = 0. \end{cases} \quad (39)$$

When  $n = 0$  the problem (38) is significantly simplified. Firstly (up to a constant)

$$\mathbf{u} = 0, \quad \Omega = p = 1, \quad (40)$$

and secondly, the component  $w$  is the solution to the following boundary problem

$$\begin{cases} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - 2\lambda\bar{\lambda} \right) w = 0, \\ w(1) = 0, \\ |w(0)| < \infty. \end{cases} \quad (41)$$

By introducing a new variable

$$\xi = i\sqrt{2\lambda\bar{\lambda}}r \quad (42)$$

into the equation of the boundary problem (41) and by using the boundary condition for  $r = 0$  we get [34]

$$w = J_0(\xi) = J_0(i\sqrt{2\lambda\bar{\lambda}}r), \quad (43)$$

where  $J_0$  is the Bessel function of the zeroth order. The second boundary condition leads to the equality

$$2\lambda\bar{\lambda} = -\mu_k^2, \quad k = 1, 2, \dots \quad (44)$$

where  $\mu_k$  are roots of the equation

$$J_0(\mu) = 0.$$

Using their symmetry about the origin we assume  $\mu_k > 0$ .

From the equation (44) we get

$$\lambda_{1,2}^{k,0} = \frac{-\frac{1}{W_i} \pm \sqrt{\frac{1}{W_i^2} - 4\mathfrak{a}^2\mu_k^2}}{2}, \quad k = 1, 2, \dots \quad (45)$$

It is obvious that  $\operatorname{Re}\lambda_{1,2}^{k,0} \leq -\sigma < 0$  for some constant  $\sigma$ .

In its turn the boundary problem (37) takes the following form:

$$\begin{cases} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - 2\lambda\bar{\lambda} \right) v = 0, \\ v(1) = 0, \\ |v(0)| < \infty. \end{cases} \quad (46)$$

By using the new variable (42) again we get the solution of the equation of the boundary problem (46)

$$v = J_1(\mathfrak{x}i) = J_1(i\sqrt{2\lambda\bar{\lambda}}r), \quad (47)$$

where  $J_1$  is the Bessel function of the first order.

Consequently

$$\lambda_{1,2}^{k,1} = \frac{-\frac{1}{W_i} \pm \sqrt{\frac{1}{W_i^2} - 4\mathfrak{a}^2\nu_k^2}}{2}, \quad k = 2, 3, \dots \quad (48)$$

where  $J_1(\nu_k) = 0$ .

In the equation (48) we exclude the case  $\nu_1 = 0$  since in that case the variable change (42) is degenerate due to (28), namely  $\lambda = 0$ .

Remembering (29) we can find other components of the spectral vector functions. Note that these vector functions tend to zero with the growth of time due to (45), (48).

This leaves us with one special case when

$$\lambda = -\frac{1}{W_i} \quad (\text{see (28)}) \quad (49)$$

Lets return to the system (16)–(25). Assume that  $n \neq 0$ . Then we get the following solution [35]

$$\mathbf{u} = 0, \quad (50)$$

$$\alpha_{r\varphi} = -\frac{in}{r^2} \int_0^r \xi^2 \alpha_{\varphi z} d\xi, \quad (51)$$

$$\begin{aligned} \alpha_{rz}(\sqrt{n}r) = & C J_1(\sqrt{n}r) + \frac{i}{\sqrt{n}} J_1(\sqrt{n}r) \int_0^r \frac{Y_1(\sqrt{n}\eta)}{W(\sqrt{n}\eta)} \left[ (\alpha_{rr} - \alpha_{zz})_\eta + \right. \\ & \left. + \frac{\alpha_{rr}\alpha_{zz}}{\eta} \right] d\eta - \frac{i}{\sqrt{n}} Y_1(\sqrt{n}r) \int_0^r \frac{J_1(\sqrt{n}\eta)}{W(\sqrt{n}\eta)} \left[ (\alpha_{rr} - \alpha_{zz})_\eta + \frac{\alpha_{rr}\alpha_{zz}}{\eta} \right] d\eta, \end{aligned} \quad (52)$$

$$\begin{aligned} W(\eta) = & [Y_1(\eta)J_1'(\eta) - Y_1'(\eta)J_1(\eta)] \Big|_{\eta=1} \frac{1}{\eta}, \\ p = & \frac{1}{in} \left[ (\alpha_{rz})' + \frac{\alpha_{rz}}{r} \right] + \alpha_{zz}, \end{aligned} \quad (53)$$

where  $\alpha_{\varphi z}, \alpha_{rr}, \alpha_{zz}, \alpha_{\varphi\varphi}$  are arbitrary functions,  $Y_1(\xi)$  is Bessel function of the second kind,  $C$  is some constant.

In case of  $n = 0$ , the solution has a simpler form:

$$\mathbf{u} = 0, \quad (54)$$

$$\alpha_{rz} = \alpha_{r\varphi} = 0, \quad (55)$$

$$p = \alpha_{rr} + \int_0^r \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{\xi} d\xi, \quad (56)$$

$\alpha_{\varphi z}, \alpha_{rr}, \alpha_{zz}, \alpha_{\varphi\varphi}$  are arbitrary functions.

Of course, components of the anisotropy tensor in the righthand parts of the formulae (51), (52), (53) and (56) should be such that the values in the lefthand side are bounded.

**Замечание 2.** *There is no point in detailing the conditions on the components of the anisotropy tensor required to bound the solution for  $r = 0$  since it is possible that  $\lambda = -\frac{1}{Wi}$  is separated from other points of the spectrum which allows us to use the reverse Laplace transform due to (29). It is indeed so for  $n = 0$ : it is sufficient to remember the representation (45), (48) for the points of the spectrum. The general case when  $n \neq 0$  is a subject for the special analysis.*

Returning to the general problems (39), (37) lets formulate the main theorem of this work.

**Теорема 1.** *Let  $\lambda \neq -\frac{1}{Wi}$ . Then the spectral equation for the eigenvalues of the boundary problem (39) has the following form*

$$u_2'(1)u_1(1) - u_1'(1)u_2(1) = 0,$$

*the functions  $u_1(r)$  and  $u_2(r)$  are defined in (58), (59); (60), (61) correspondingly. The spectrum of the boundary problem (37) is defined by the correlation formula (64), i.e.*

$$\lambda_{1,2}^{k,3} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi} - 4\mathfrak{a}^2(\sigma_k^2 + n^2)}}{2}, \quad k = 1, 2, 3, \dots$$

where  $\sigma_k$  are roots of the Bessel function  $J_1$ .

### 3 Proof of the main theorem

We begin by analysing the boundary problem (39), namely by studying the fundamental system of the differential equation. The main idea here is the information about the basis of a defining Euler equation [36, 37]

$$\tilde{u}^{IV} + \frac{2}{r}\tilde{u}''' - \frac{3}{r^2}\tilde{u}'' + \frac{3}{r^3}\tilde{u}' - \frac{3}{r^4}\tilde{u} = 0. \quad (57)$$

The basis of the equation (57) consists of the following functions:

$$\tilde{u}_1 = r^3, \quad \tilde{u}_2 = r, \quad \tilde{u}_3 = r \ln r, \quad \tilde{u}_4 = \frac{1}{r}.$$

The function  $\tilde{u}_1$  defines the basis element of the equation from problem (39) [37]:

$$u_1 = \sum_{k=0}^{\infty} a_k r^{k+3} = a_0 r^3 + a_1 r^4 + a_2 r^5 + \dots, \quad (58)$$

where  $u_1$  is an entire function,  $-\infty < r < +\infty$ .

Coefficients  $a_i$  can be found:

$$\begin{aligned} a_0 &= 1, \quad a_1 = 0, \quad a_2 = \frac{1}{24}\lambda\bar{\lambda}, \quad a_3 = -\frac{2}{175}in^3, \\ a_k &= \frac{2\lambda\bar{\lambda}}{(k+2)(k+4)}a_{k-2} - \frac{2in^3}{(k+2)^2(k+4)}a_{k-3} - \\ &\quad - \frac{n^2(2\lambda\bar{\lambda} + n^2)}{k(k+2)^2(k+4)}a_{k-4}, \quad k = 4, 5, 6, \dots \end{aligned} \quad (59)$$

The next element of basis  $u_2$  corresponds to the function  $\tilde{u}_2$ , the other elements don't satisfy the boundary conditions for  $r = 0$ .

Here is a characteristic equation for (57):

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3) + 2\alpha(\alpha-1)(\alpha-2) - 3\alpha(\alpha-1) + 3\alpha - 3 = 0.$$

It's roots are  $\alpha_1 = 3$ ,  $\alpha_{2,3} = 1$ . They define three solutions  $\tilde{u}_1$ ,  $\tilde{u}_2$  and  $\tilde{u}_3$ , but there is a problem for calculating  $u_2$  due to the fact that the difference between roots is integer and non-negative  $\alpha_1 - \alpha_2 = 2$ . In this case, generally speaking, the method of indeterminate coefficients doesn't give us the recurrent correlation similar to (59). So we should use the Frobenius method to construct  $u_2$  [37]. But in our case the coefficient  $c_1$  before  $r^2$  equals zero and the method of indeterminate coefficients can be applied.

So,

$$u_2 = \sum_{k=0}^{\infty} c_k r^{k+1} = c_0 r + c_1 r^2 + c_2 r^3 + \dots, \quad (60)$$

$$\begin{aligned}
c_0 &= 1, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = -\frac{2}{27}in^3, \\
c_k &= \frac{2\lambda\bar{\lambda}}{k(k-2)}c_{k-2} - \frac{2in^3}{k^2(k-2)}c_{k-3} - \frac{n^2(2\lambda\bar{\lambda} + n^2)}{k^2(k-2)(k+2)}c_{k-4}, \quad k = 4, 5, 6, \dots
\end{aligned} \tag{61}$$

That means that the set of solutions of the equation (39) which satisfy the boundary conditions for  $r = 0$  has the following representation:

$$u(r) = C_1 u_1(r) + C_2 u_2(r), \tag{62}$$

where  $C_1, C_2$  are arbitrary constants,  $u_1(r), u_2(r)$  are entire functions of the form (58) with coefficients (59) or (60) with coefficients (61).

The boundary conditions for  $r = 1$  lead us to the spectral equation for the eigenvalues  $\lambda$ :

$$u'_2(1)u_1(1) - u'_1(1)u_2(1) = 0, \tag{63}$$

proving the first statement of the theorem 1.

Analysis of the spectral boundary problem (37) can be done by analogy with the analysis of the boundary problem (46). That gives us

$$\lambda_{1,2}^{k,3} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi} - 4\Re^2(\sigma_k^2 + n^2)}}{2}, \quad k = 1, 2, 3, \dots \tag{64}$$

where  $\sigma_k$  are the roots of the equation

$$J_1(\sigma_k) = 0.$$

Similar to the problem (37) (see inequality (45)) it is clear that

$$\operatorname{Re} \lambda_{1,2}^{k,3} \leq \sigma_1 < 0, \tag{65}$$

where  $\sigma_1$  is some constant. Which completes the proof of theorem 1.

**Замечание 3.** Due to the inequalities (65), the component  $v$  tends to zero with the growth of variable  $t$ .

#### 4 Numerical study of the spectrum of the problem (39)

To illustrate the results formulated in paragraph 2 we study the problem (39), using numerical methods.

It is known, that spectra of discretized operators are depend significantly on the way they are discretized. If the discretization is unfortunate, then the calculated spectrum can be significantly different from the spectrum of the initial operator.

Monograph [38] uses matrix collocation derivatives to approximate the derivatives in the segment  $[-1, 1]$  and uses Gauss-Lobatto points as nodes

$$\xi_k = \cos \left( \pi \left( 1 - \frac{k-1}{N} \right) \right), \quad k = 1, 2, \dots, N+1.$$

If the discrete analogue of the function  $u(\xi)$  is defined in this nodes  $u_k = u(\xi_k)$  and can be represented as  $u = (u_1, \dots, u_{N+1})^T$ , then its derivative  $u'(\xi_k)$  is approximated as a multiplication  $D_1 u$ , where  $D_1$  is a matrix of size  $(N+1) \times (N+1)$ , and its elements are found through the formula

$$d_{ij}^{[1]} = \begin{cases} (-1)^{l+j} c_l / (c_j (\xi_l - \xi_j)), & l \neq j, \\ -\xi_j / (2(1 - \xi_j^2)), & 2 \leq l = j \leq N, \\ (2N^2 + 1)/6, & l = j = 1, \\ -(2N^2 + 1)/6, & l = j = N + 1, \end{cases}$$

$$c_j = \begin{cases} 2, & \text{for } j = 1, N + 1, \\ 1 & \text{for } j = 2, \dots, N. \end{cases}$$

Higher order derivatives are obtained by multiplying  $D_k = D_1^k$ .

To adapt the collocation method for the spectral problem (39), we need to take into account that the problem is stated on the segment  $0 < r < 1$ . That means it needs to be linearly transformed  $\xi = 2r - 1$  into the segment  $-1 < \xi < 1$ . Using the complex derivative formula  $d/dr = 2d/d\xi$  we get that the approximation of the problem (39) derivative has the form  $\hat{D}_1 = 2D_1$  for functions defined in nodes  $r_k = (\xi_k + 1)/2$ , where  $\xi_k$  are Gauss-Lobatto points.

Turning to discreteization of homogeneous boundary conditions we should note that the combination of the homogeneous Dirichlet and Neumann conditions can often be found in boundary problems and different applications. The mechanism for accounting for these conditions in collocation matrices is detailed in [38]. As an example the monograph uses the spectrum of the Orr-Sommerfeld operator for the plane-parallel Poiseuille flow.

In case of the problem (39) the function  $u$  satisfies homogeneous Dirichlet conditions on both boundaries  $u(0) = u(1) = 0$ . To take that into account we need to delete the first and the last row and column of the matrix  $D$ . As a result we get the matrix  $\hat{D}$  of size  $(N-1) \times (N-1)$ , which is applied to the vector  $(u_2, \dots, u_N)^T$ .

Similar to the method described in [38] for the Neumann condition on both boundaries we will represent the function in the form  $u(r) = (r-1)v(r)$ , where  $v(1) = 0$ . From the obvious formula

$$\frac{d^k}{dr^k} u = k \frac{d^{k-1}}{dr^{k-1}} v + (r-1) \frac{d^k}{dr^k} v = k \frac{d^{k-1}}{dr^{k-1}} \left( \frac{1}{r-1} u \right) + (r-1) \frac{d^k}{dr^k} \left( \frac{1}{r-1} u \right)$$

we obtain correlations on the collocation derivatives for the function that satisfies the Dirichlet and Neumann conditions on the right boundary:

$$\tilde{D}_k = k \hat{D}_{k-1} \text{diag} \left( \frac{1}{r_j - 1} \right) + \text{diag}(r_j - 1) \hat{D}_k \text{diag} \left( \frac{1}{r_j - 1} \right). \quad (66)$$

To take into account the boundedness in zero condition for the collocation derivatives (see (39)), we use the preliminary information about the form of the solutions, that satisfy them. In the third paragraph we acquired the solutions to the equation of the problem (39) in the form of power series (see (58) and (60)). From them it follows that the following equivalence relations hold:

$$1 = \lim_{r \rightarrow 0} \frac{u_1(r)}{r} = \lim_{r \rightarrow 0} \frac{u_2(r)}{r^3}.$$

or

$$u_1(r) = r(1 + r\varphi_1(r)), \quad u_2(r) = r^3(1 + r\varphi_2(r)),$$

where  $\varphi_i(0) \neq 0$ ,  $i = 1, 2$ .

Note that functions  $r$  and  $r^3$  satisfy the Euler equation

$$Lq = 0, \quad \text{where} \quad L = r^2 \frac{d^2}{dr^2} - 3r \frac{d}{dr} + 3 = 0,$$

and after we substitute the functions  $u_1$  and  $u_2$  in it the relative error has the form

$$\frac{Lu_i}{u_i} = r\Phi_i(r) \rightarrow 0 \text{ for } r \rightarrow 0.$$

Based on these facts we get that the satisfaction of boundary conditions on the left boundary for the solution  $u$  of the problem (39) is equivalent to the fact that  $u$  satisfies the equality

$$r^2 \frac{d^2 u}{dr^2} - 3r \frac{du}{dr} + 3u = 0 \text{ for } r \rightarrow 0. \quad (67)$$

We will use this equality as the boundary condition.

Let's move on to the construction of the necessary matrix bundle. We have the equality  $Lu = 0$ , that holds asymptotically in zero (67), and the equation (see (39))

$$\left[ \frac{d^4}{dr^4} + \frac{2}{r} \frac{d^3}{dr^3} - \frac{3}{r^2} \frac{d^2}{dr^2} + \left( \frac{3}{r^3} + 2in^3 \right) \frac{d}{dr} + \left( \frac{-3}{r^4} + n^4 \right) - \right. \\ \left. - 2\mu \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{1}{r^2} + n^2 \right) \right) \right] u = 0, \quad (68)$$

where  $\mu = \lambda \bar{\lambda}$ , which is true for  $0 < r < 1$ .

We discretize both differential operators using matrices (66), which already take into account the homogeneous Dirichlet and Neumann conditions. This gives us

$$\left[ \tilde{D}_2 - \text{diag} \frac{3}{r_k} \tilde{D}_1 + \text{diag} \frac{3}{r_k^2} \right] u = 0, \quad (69)$$

$$\left[ \tilde{D}_4 + \text{diag} \frac{2}{r_k} \tilde{D}_3 - \text{diag} \frac{3}{r_k^2} \tilde{D}_2 + \left( \text{diag} \frac{3}{r_k^3} + 2in^3 I \right) \tilde{D}_1 + \right. \\ \left. + \left( \text{diag} \frac{-3}{r_k^4} + n^4 I \right) - 2\mu \left( \tilde{D}_2 + \text{diag} \frac{1}{r_k} \tilde{D}_1 - \left( \text{diag} \frac{1}{r_k^2} + n^2 I \right) \right) \right] u = 0, \quad (70)$$

where in this case  $u$  is the vector such that  $u_k = u(r_k)$ ,  $I$  is the identity matrix of size  $(N-1) \times (N-1)$ .

We introduce the following matrices

$$A_0 = \tilde{D}_2 - \text{diag} \frac{3}{r_k} \tilde{D}_1 + \text{diag} \frac{3}{r_k^2}, \quad B_0 = 0, \\ A_1 = D_4 + \text{diag} \frac{2}{r_k} \tilde{D}_3 - \text{diag} \frac{3}{r_k^2} \tilde{D}_2 + \left( \text{diag} \frac{3}{r_k^3} + 2in^3 I \right) \tilde{D}_1 + \left( \text{diag} \frac{-3}{r_k^4} + n^4 I \right), \\ B_1 = 2 \left( \tilde{D}_2 + \text{diag} \frac{1}{r_k} \tilde{D}_1 - \left( \text{diag} \frac{1}{r_k^2} + n^2 I \right) \right).$$

Here  $B_0$  is a zero matrix of size  $(N-1) \times (N-1)$ . Then the discretized equations (69), (70) can be written as

$$(A_0 - \mu B_0)u = 0, \quad (A_1 - \mu B_1)u = 0.$$

correspondingly.

Since the equality (67) has an asymptotic character, we can unite both matrix equalities by changing first rows of matrices  $A_1, B_1$  into first rows of matrices  $A_0, B_0$ . Thereby we point out that the closest to zero point  $r_1$  satisfies condition (67), and every other one satisfies (68). We will denote this matrix bundle as  $A - \mu B$ .

Additionally we balance the bundle, which helps with the quality of the spectral calculations. The balancing procedure consists of the calculation of the nondegenerate matrix  $X$  such that

$$A = X\hat{A}, \quad B = X\hat{B}, \quad \hat{A}\hat{A}^* + \hat{B}\hat{B}^* = I.$$

To do this it is sufficient to make a QR decomposition

$$\text{qr} \begin{pmatrix} A^* \\ B^* \end{pmatrix} = \begin{pmatrix} \hat{A}^* & * \\ \hat{B}^* & * \end{pmatrix} \begin{pmatrix} X^* \\ 0 \end{pmatrix}.$$

This operation doesn't change the bundle spectrum and the set of right eigenvectors.

Here is a further analysis plan.

We will use the standard software to calculate the spectrum of the matrix bundle  $\hat{A} - \mu\hat{B}$  we got. Then we calculate approximate eigenvalues  $\lambda_j$  of the problem (39) as roots of the quadratic equation  $\mu = \lambda(\lambda + 1/Wi)/2\alpha^2$ . Figures 2, 3 illustrate the location of the spectrum points for large and small parameters  $Wi$  and  $Re$ . On this figures we can see that despite the several orders of magnitude difference between the parameters, the structure of the

spectrum is very similar in both cases, the significant difference being the scale of the structure. It is important to note that if for  $n = 1$  the whole spectrum lies strictly in the left half-plane, then already for  $n = 50$  several eigenvalues lie in the right half-plane.

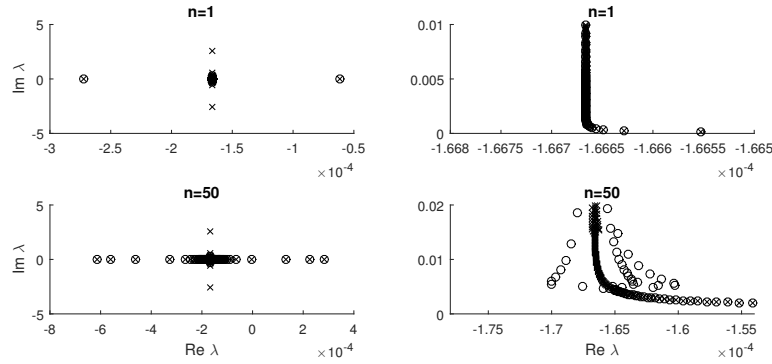


Рис. 2. Location of the spectrum points for  $Wi = 3 \cdot 10^3$ ,  $Re = 5 \cdot 10^3$  general picture to the right, local picture in the zone of concentration of points of the spectrum to the left; eigenvalues are circles for  $N = 100$ , crosses for  $N = 300$ .

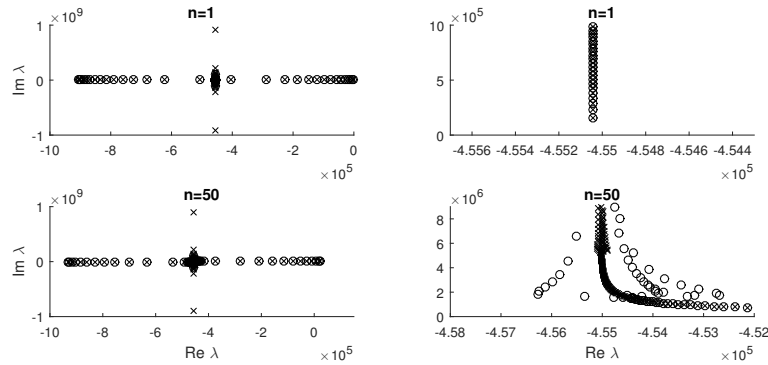


Рис. 3. Location of the spectrum points for  $Wi = 1.1 \cdot 10^{-6}$ ,  $Re = 10^{-2}$  general picture to the right, local picture in the zone of concentration of points of the spectrum to the left; eigenvalues are circles for  $N = 100$ , crosses for  $N = 300$ .

Fig. 4 shows graphs of the maximum of the real parts of eigenvalues for different  $N$  and values of  $Re$  and  $Wi$ . Graphs are in the double logarithmic scale. All graphs have a pronounced minimum which corresponds to the moment of the transition of the eigenvalue with the maximal real part through the imaginary axis.

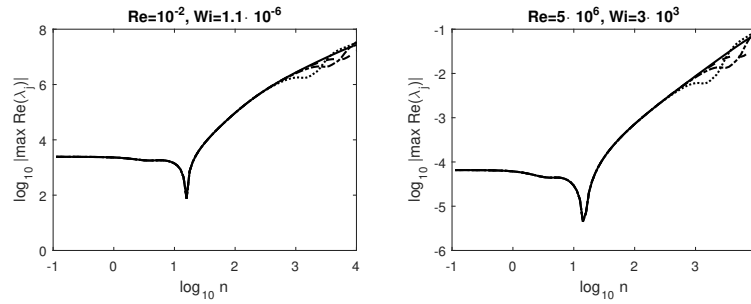


Рис. 4. Graph of the module of the maximum real part of the eigenvalue depending on  $n$ . Dotted line is  $N = 100$ , dash-dotted line –  $N = 150$ , dashed line is  $N = 200$ , solid line is  $N = 300$ .

Figures 5 and 6 show eigenvectors corresponding to eigenvalue with the maximum real part. These vectors are calculated for  $Wi = 1.1 \cdot 10^{-6}$ ,  $Re = 10^{-2}$ ,  $N = 300$ . For other values of  $Wi$  and  $Re$  there are almost no difference in eigenvectors compared to the illustrated ones.

We should note that the oscillation frequency of the eigenfunction significantly increases with the growth of  $n$ . Moreover, starting from a certain moment the fixed number of the collocation points  $N$  is not enough to represent the function that oscillates so frequently. If that happens then the numerical results for chosen  $N$  and  $n$  can't be considered correct. It seems that precisely that effect influences the difference between graphs on Fig. 4 for large  $n$  and different  $N$ .

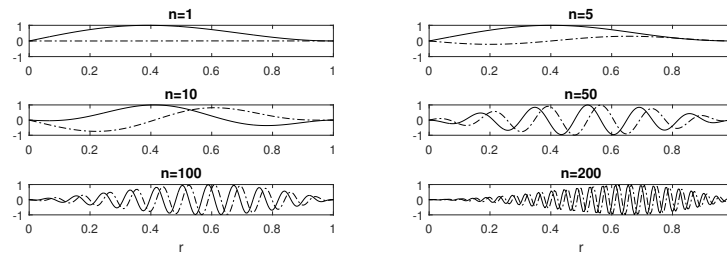


Рис. 5. Right eigenvectors, corresponding to the eigenvalue with the largest real part for  $N = 300$ . Solid line is the real part, dash-dotted line is the imaginary part.

## References

- [1] V.N. Pokrovskii, *The mesoscopic theory of polymer dynamics*, Springer Ser. Chem. Phys., **95**, Springer, Dordrecht, 2010.
- [2] Yu.A. Altukhov, A.S. Gusev, G.V. Pishnograï, *Introduction into mesoscopic theory of flowing polymeric systems*, Alt. GPA, Barnaul, 2012.

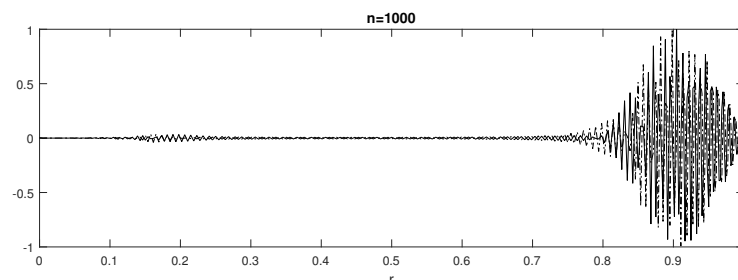


Рис. 6. Right eigenvectors, corresponding to the eigenvalue with the largest real part for  $N = 300$ ,  $n = 1000$ . Solid line is the real part, dash-dotted line is the imaginary part.

- [3] J.G. Oldroyd, *On the formulation of rheological equations of state*, Proc. R. Soc. Lond., Ser. A, **200** (1950), 523–541. Zbl 1157.76305
- [4] R.B. Bird, P.J. Dotson, N.L. Johnson, *Polymer solution rheology based on a finitely extensible bead-spring chain model*, J. Non-Newtonian Fluid Mech., **7** (1980), 213–235. Zbl 0432.76012
- [5] M.D. Chilcott, J.M. Ralliston, *Creeping flow of dilute polymer solutions past cylinders and spheres*, J. Non-Newtonian Fluid Mech., **29**:3 (1988), 381–432. Zbl 0669.76016
- [6] J. Remmelgas, G. Harrison, L.G. Leal, *A differential constitutive equation for entangled polymer solutions*, J. Non-Newtonian Fluid Mech., **80**:2-3 (1999), 115–134. Zbl 0951.76005
- [7] V.N. Pokrovski, *Statistical mechanics of dilute suspension*, Nauka, Moscow, 1978.
- [8] S.A. Zinovich, I.E. Golovicheva, G.V. Pyshnogra, *Influence of the molecular mass on shear and lateral viscosity of linear polymers*, J. Appl. Mech. Tech. Phys., **41**:2 (2000), 347–352. Zbl 0999.74032
- [9] G. Pyshnogra, D. Merzlikina, P. Filip, R. Pivokonsky, *Mesoscopic single and multi-mode rheological models for polymeric melts viscometric flow description*, WSEAS transactions on heat and mass transfer, **13** (2018), 49–65.
- [10] A.M. Blokhin, D.L. Tkachev, *Linear asymptotic instability of a stationary flow of a polymeric medium in a plane channel in the case of periodic perturbations*, J. Appl. Ind. Math. **8**:4 (2014), 467–478. Zbl 1340.76037
- [11] A.M. Blokhin, A.V. Yegitov, D.L. Tkachev, *Linear instability of solutions in a mathematical model describing polymer flows in an infinite channel*, Comput. Math. Math. Phys., **55**:5 (2015), 848–873. Zbl 1426.76153
- [12] Alexander Blokhin, Dmitry Tkachev, *Spectral asymptotics of a linearized problem about flow of an incompressible polymeric fluid. Base flow is analogue of a Poiseuille flow*, AIP Conf. Proc., **2027** (2018), Article ID 030028.
- [13] A.M. Blokhin, D.L. Tkachev, *Analogue of the Poiseuille flow for incompressible polymeric fluid with volume charge. Asymptotics of the linearized problem spectrum*, J. Phys. Conf. Ser., **894** (2017), Article ID 012096.
- [14] A.M. Blokhin, A.V. Yegitov, D.L. Tkachev, *Asymptotics of the spectrum of a linearized problem of the stability of a stationary flow of an incompressible polymer fluid with a space charge*, Comput. Math. Math. Phys., **58**:1 (2018), 102–117. Zbl 1444.35128
- [15] Alexander Blokhin, Dmitry Tkachev, Aleksey Yegitov, *Spectral asymptotics of a linearized problem for an incompressible weakly conducting polymeric fluid*, Z. Angew. Math. Mech., **98**:4 (2018), 589–601.

- [16] A.M. Blokhin, D.L. Tkachev, *Stability of Poiseuille-type flows for an MHD model of an incompressible polymeric fluid*, J. Hyperbolic Differ. Equ., **16**:4 (2019), 793–817. Zbl 1442.76136
- [17] A.M. Blokhin, D.L. Tkachev, *Stability of the Poiseuille-type flow for a MHD model of an incompressible polymeric fluid*, Eur. J. Mech., B. Fluids, **80** (2020), 112–121. Zbl 1436.76061
- [18] A.M. Blokhin, D.L. Tkachev, *Stability of Poiseuille-type flows for an MHD model of an incompressible polymeric fluid*, Fluid Dyn., **54**:8 (2019), 1051–1058. Zbl 1457.76072
- [19] A.M. Blokhin, D.L. Tkachev, *Stability of Poiseuille-type flows in an MHD model of an incompressible polymeric fluid*, Sb. Math., **211**:7 (2020), 901–921. Zbl 1450.76009
- [20] V.S. Vladimirov, *Generalized functions in mathematical physics*, Mir, Moscow, 1979. Zbl 0515.46034
- [21] L.Hörmander, *The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis*, Springer-Verlag, Berlin etc., 1983. Zbl 0521.35001
- [22] A.M. Blokhin, D.L. Tkachev, *MHD model of incompressible polymeric fluid. Linear instability of the resting state*, Complex Var. Elliptic Equ., **66**:6-7 (2021), 929–944. Zbl 1466.76023
- [23] A.M. Blokhin, D. Tkachev, *Linear instability of the resting state for the MHD model of an incompressible polymeric fluid*, AIP Conf. Proc, **2351** (2021), Article ID 040057.
- [24] A.M. Blokhin, D.L. Tkachev, *On Linearly Unstable Steady States of an MHD Model of an Incompressible Polymeric Fluid in the Case of Absolute Conductivity*, Sib. Adv. Math., **32** (2022), 1–12.
- [25] A.M. Blokhin, A.Yu. Goldin, *Linear stability of an incompressible polymer liquid at rest*, J. Math. Sci., New York, **230**:1 (2018), 14–24. Zbl 1413.76009
- [26] D.L. Tkachev, *Spectrum and linear Lyapunov instability of a resting state for flows of an incompressible polymeric fluid*, J. Math. Anal. Appl., **522**:1 (2023), Article ID 126914. Zbl 7655389
- [27] D.L. Tkachev, *The spectrum and Lyapunov linear instability of the stationary state for polymer fluid flows: the Vinogradov-Pokrovski model*, Sib. Math. J., **64**:2 (2023), 407–423. Zbl 1511.76029
- [28] G.I. Taylor, *Stability of a Viscous Liquid contained between Two Rotating Cylinders*, Philosophical transactions of the Royal Society A, **223**:605-615 (1923), 289–343.
- [29] L.I. Sedov, *Continuum mechanics. Vol. 1*, Moscow, Nauka, 1970. Zbl 0206.54404
- [30] L.G. Loytsyanskiy, *Mechanics of liquids and gases*, Pergamon Press, Oxford etc., 1972. Zbl 0247.76001
- [31] Shih-I-Pai, *Introduction to the theory of compressible flow*, D. Van Nostrand Co., New York, 1959. Zbl 0091.18004
- [32] A.M. Blokhin, A.S. Rudometova, *Stationary flows of a weakly conducting incompressible polymeric liquid between coaxial cylinders*, J. Appl. Ind. Math., **11**:4 (2017), 486–493. Zbl 1399.76004
- [33] A.M. Blokhin, R.E. Semenko, A.S. Rudometova, *Magnetohydrodynamic vortex motion of an incompressible polymeric fluid*, J. Appl. Ind. Math., **15**:1 (2021), 7–16. Zbl 1511.76114
- [34] M. Abramowitz, I. Stegun, *Handbook of mathematical functions*, NBS, 1964.
- [35] E. Kamke, *Manual of ordinary differential equations*, Moscow, Nauka, 1976. MR0414970
- [36] P. Hartman, *Ordinary differential equations. 2nd ed. Reprint*, Birkhäuser, Boston-Basel-Stuttgart, 1982. Zbl 0476.34002
- [37] E.A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York etc., 1955. Zbl 0064.33002
- [38] L.N. Trefethen, *Spectral methods in MATLAB*, SIAM, Philadelphia, 2000. Zbl 0953.68643

DMITRY LEONIDOVICH TKACHEV  
SOBOLEV INSTITUTE OF MATHEMATICS,  
PR. KOPTYUGA, 4,  
630090, NOVOSIBIRSK, RUSSIA  
*Email address:* [tkachev@math.nsc.ru](mailto:tkachev@math.nsc.ru)

ELINA ARNOLDVNA BIBERDORF  
SOBOLEV INSTITUTE OF MATHEMATICS,  
PR. KOPTYUGA, 4,  
630090, NOVOSIBIRSK, RUSSIA  
*Email address:* [ermolova@math.nsc.ru](mailto:ermolova@math.nsc.ru)