CERTAIN FUNCTORS FOR p-GROUPS OF CLASS TWO WITH ELEMENTARY ABELIAN DERIVED SUBGROUP OF ORDER p^2 AND ELEMENTARY ABELIAN ABELIANIZATION

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ABSTRACT. Let G be a finite d-generator p-group of class two such that G/G' is elementary abelian and $G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. The aim of the present work is to characterize the exact structure of some functors including the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square of G. We also give the corank of G.

1. Introduction and Motivation

For a given group G, the center, the derived subgroup, and the Frattini subgroup of G are denoted by $Z(G)$, G', and $\Phi(G)$, respectively. Let p be a prime number. The subgroup $\langle x^p \mid x \in G\rangle$ of G is denoted by G^p. Let $exp(G)$ be used to denote the exponent of G. Let $\mathbb{Z}_n^{(r)}$ denote the direct sum of r-copies $\mathbb{Z}_n$, in which $\mathbb{Z}_n$ is the finite cyclic group of order n.

The concept of the non-abelian tensor square $G \otimes G$ of a group G is a special case of the non-abelian tensor product of two arbitrary groups that was introduced by Brown and Loday in [5]. It is easy to check that the map

$$\kappa : G \otimes G \to G'$$
$$g \otimes g' \mapsto [g, g']$$

for all $g, g' \in G$ is an epimorphism. Let $J_2(G)$ be the kernel of κ, and let $\bigtriangledown(G)$ be a subgroup of $G \otimes G$ generated by the set $\{g \otimes g \mid g \in G\}$. It is known (see [5]) that both $J_2(G)$ and $\bigtriangledown(G)$ are central subgroups of $G \otimes G$. The quotient group

$$G \wedge G = \frac{G \otimes G}{\bigtriangledown(G)}$$

is called the non-abelian exterior square of G. The element $(g \otimes g') \bigtriangledown (G)$ in $G \wedge G$ is denoted by $g \wedge g'$ for all $g, g' \in G$. The epimorphism κ induces the epimorphism $\kappa' : G \wedge G \to G'$ given by $g \wedge g' \mapsto [g, g']$ for all $g, g' \in G$.

The concept of the Schur multiplier $\mathcal{M}(G)$ of a group G was introduced by Schur while he was studying on the projective representation of groups. The kernel of the map κ' is isomorphic to the Schur multiplier of G (for more information, see [5]).

Many authors found the structure of the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square for some classes of groups such as finite abelian groups and extra-special p-groups (see [8, 12]).

Recall that a group G is called capable if $G \cong E/Z(E)$ for some group E. Following

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[3], the epicenter $Z^*(G)$ of a group G is defined as the smallest central subgroup K of G such that G/K is capable. In particular, G is capable if and only if $Z^*(G) = 1$. A finite p-group G is called special of rank k if $G' = Z(G) = \Phi(G)$ and $Z(G)$ is an elementary abelian p-group of rank k. It is obvious that $exp(G) \leq p^2$. Special p-groups of rank one are extra-special p-groups. Capable extra-special p-groups were classified by Beyl, Felgner, and Schmid in [3]. Later, it is shown [10, Proposition 3] that if G is a finite capable p-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$, then $p^5 \leq |G| \leq p^7$. Also, he classified all finite capable special p-groups of rank two and exponent p. In the following, the classification of all finite capable special p-groups of rank two and exponent p is given.

Theorem 1.1. *(See* [10, Section 2, pages 246-247] *and* [9, Theorem 1.4(c)]*) Let G be a finite special p-group of rank two with $exp(G) = p$. Then G is capable if and only if G is isomorphic to one of the following p-groups:*

$G_1 = \Phi_4(1^5) = \langle a, a_1, a_2, b_1, b_2 | [a_i, a] = b_i, a^p = a_i^p = b_i^p = 1, 1 \leq i \leq 2\rangle$.

$G_2 = \Phi_{12}(1^6) = E_1 \times E_1$, *in which E_1 is the extra-special p-group of order p^3 with $exp(G) = p$ and $p > 2$.*

$G_3 = \Phi_{13}(1^6) = \langle a_j, b_i | [a_i, a_{i+1}] = b_i, [a_2, a_4] = b_2, a_j^p = b_i^p = 1, 1 \leq j \leq 4, 1 \leq i \leq 2\rangle$.

$$G_4 = \Phi_{15}(1^6) = \langle a_j, b_i | [a_i, a_{i+1}] = b_i, [a_3, a_4] = b_1, [a_2, a_4] = b_2^t, a_j^p = b_i^p = 1, 1 \leq j \leq 4, 1 \leq i \leq 2\rangle,$$ *where t is non-quadratic residue modulo p.*

$$G_5 = T = \langle a_j, b_i | [a_2, a_1] = [a_5, a_3] = b_1, [a_3, a_1] = [a_5, a_4] = b_2, a_j^p = b_i^p = 1, 1 \leq j \leq 5, 1 \leq i \leq 2\rangle.$$

Several works have been done to determine the structure of the Schur multiplier for some classes of finite special p-groups such as extra-special p-groups and special p-groups of exponent p having maximum rank (see [12, 14]). Moreover, Hatui [9] obtained the order of the Schur multiplier for finite special p-groups of rank two. Here, we are interested in studying the Schur multiplier of all finite p-groups of class two with elementary derived subgroup of order p^2 and elementary abelianization. In addition, it is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ for a p-group G of order p^n. The non-negative integer $t(G)$ is called the corank of G. Some authors characterized the exact structure of a finite p-group G with small corank, for example see [2, 7]. Here, we will determine the corank of a finite p-group G of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$.

Let G be a finite d-generator p-group of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. In the same motivation, the aim of the present paper is to give a complete description of the structure of some functors, such as the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square of G. Moreover, we compute the corank of G.

To determine the functors for the group G, we consider two cases for G, which is capable or non-capable. At first, by considering the non-capable group G, we will have the following result.

Theorem A. *Let G be a non-capable d-generator p-group of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. Then the following results hold:*

(*i*) *$Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)\right)}$, $t(G) = 2d+1$, $G \wedge G \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+2\right)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2)}$. In this case, $G/Z^*(G) \cong E_1 \times \mathbb{Z}_p^{(d-2)}$.*

(*ii*) *$Z^*(G) = G'$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-2\right)}$, $t(G) = 2d+3$, $G \wedge G \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)\right)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-2)}$. In this case, $G/Z^*(G) \cong \mathbb{Z}_p^{(d)}$.*

Now, we assume that G is capable. Obviously, $G^p \subseteq \Phi(G) = G'$ and $exp(G) \leq p^2$, and so $G^p = 1$, $G_p \cong \mathbb{Z}_p$, or $G_p = G'$. It is shown that if $G_p \cong \mathbb{Z}_p$, then G is non-capable in Theorem 2.1(vi). Therefore, we will characterize the same functors in two cases as follows:

Theorem B. *Let G be a capable d-generator p-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$ and $exp(G) = p$. Then one of the following cases holds:*

(*i*) *$G \cong \Phi_4(1^5) \times \mathbb{Z}_p^{(d-3)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+3\right)}$, $t(G) = 2d-2$, $G \wedge G \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+5\right)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+5)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2+3)}$.*

(*ii*) *$G \cong H \times \mathbb{Z}_p^{(d-4)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+2\right)}$, $t(G) = 2d-1$, $G \wedge G \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+4\right)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+4)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2+2)}$, where $H \cong \Phi_{12}(1^6)$, $H \cong \Phi_{13}(1^6)$, or $H \cong \Phi_{15}(1^6)$.*

(*iii*) *$G \cong T \times \mathbb{Z}_p^{(d-5)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)}$, $t(G) = 2d$, $G \wedge G \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+1\right)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+1)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$.*

Theorem C. *Let G be a capable d-generator p-group of class two with $\Phi(G) = G^p = G' \cong \mathbb{Z}_p^{(2)}$ and $exp(G) = p^2$. Then the following results hold:*

(*i*) *$\mathcal{M}(G) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-3\right)}$ and $t(G) = 2d$.*

(*ii*) *If $p \neq 2$, then $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)}$, $G \otimes G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(d^2-1)}$, and $J_2(G) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(d^2-3)}$*

(*iii*) *If $p = 2$, then $G \wedge G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{\left(\frac{1}{2}d(d-1)-1\right)}$, $(G \otimes G)/N \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(d^2-1)}$, and $J_2(G)/N \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(d^2-3)}$, where $N = \ker\big(\bigtriangledown(G) \to \bigtriangledown(G/G')\big)$.*

2. Preliminaries

In the present section, we give some results, which will be used throughout the next section.

The following theorem plays an essential role in the proof of main theorems.

Theorem 2.1. *Let G be a d-generator p-group of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. Then the following assertions hold:*

(*i*) *$G \otimes G$ is an abelian p-group.*

(*ii*) *Let $p \neq 2$. Then $|G \wedge G| = |\mathcal{M}(G)||G'|$, $G \otimes G \cong (G \wedge G) \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$, and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$.*

(*iii*) *If $p = 2$, then $G' = G^2 = \Phi(G)$.*

(*iv*) *If* $G^p = G'$ *and* G *is non-capable, then* $Z^*(G) = G'$, $G\otimes G \cong G/G' \otimes G/G'$, $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$, *and* $G \wedge G \cong \mathcal{M}(G) \oplus G'$. *In particular,* $G \otimes G$ *is an elementary abelian* p*-group.*

(*v*) *If* $exp(G) = p$, *then* $exp(G \otimes G) = p$.

(*vi*) *If* $G^p \cong \mathbb{Z}_p$, *then* G *is non-capable and* $G \otimes G$ *is an elementary abelian* p*-group. Moreover,* $G \wedge G \cong \mathcal{M}(G) \oplus G'$.

(*vii*) *If* G *is non-capable or* $exp(G) = p$, *then both* $\mathcal{M}(G)$ *and* $G \wedge G$ *are elementary abelian* p*-groups.*

(*viii*) *If* $exp(G) = p^2$ *and* G *is capable, then* $exp(G \wedge G) = exp(G \otimes G) = p^2$.

(*ix*) *If* G *is a capable special* p*-group of rank two and exponent* p^2, *then* $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)} \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(2)}$ *and* $\mathcal{M}(G) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-3\right)}$.

Proof. (*i*) The result follows from [1, Proposition 3.1 and Lemma 3.4].

(*ii*) Clearly, $|G \wedge G| = |\mathcal{M}(G)||G'|$. From [4, Lemma 1.2(*i*), Theorem 1.3(*iii*), and Corollary 1.4], it is concluded that $\bigtriangledown(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$, $G \otimes G \cong (G \wedge G) \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$, and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$.

(*iii*) For all $x, y \in G$, we have $[y, x] = y^{-2}x^{-2}(xy)^2$. It implies that $G' \subseteq G^2$. On the other hand, $G' = \Phi(G)$, and so $G' = G^2 = \Phi(G)$. The result holds.

(*iv*) Using part (*iii*), if $p = 2$, then $G^2 = G'$. By a similar way used in the proof of [9, Theorem 1.1(a)], $Z^*(G) = G'$ for an arbitrary prime number p. [6, Proposition 16(*iv*) and (*v*)] implies that

$$G \otimes G \cong G/G' \otimes G/G', \bigtriangledown(G) \cong \bigtriangledown(G/G'), \text{ and } G \wedge G \cong G/G' \wedge G/G'.$$

It is clear that both $G \otimes G$ and $G \wedge G$ are elementary abelian p-groups, so is $\mathcal{M}(G)$. Hence, $J_2(G) \cong \mathcal{M}(G) \oplus \bigtriangledown(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d+1)\right)}$ and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.

(*v*) The result holds by [1, Lemma 3.4].

(*vi*) From part (*iii*), $p > 2$. By a similar way used in the proof of [9, Theorem 1.3(a)], G is non-capable and $Z^*(G) \subseteq G'$. Hence, $G/Z^*(G) = G/G'$ or $G/Z^*(G) \cong E_1 \times \mathbb{Z}_p^{(d-2)}$, from [11, Theorem 3.1]. By a similar way used in the proof of part (*iv*) and using [6, Proposition 16(*iv*)] and [8, Corollary 2.4], we can see that $G \wedge G$ is elementary abelian, and so $G \wedge G \cong \mathcal{M}(G) \oplus G'$.

(*vii*) It is easily obtained by parts (*iv*), (*v*), and (*vi*).

(*viii*) Using part (*vi*), $G^p = G'$. Without loss of generality, assume that $G' = G^p = \langle x^p \rangle \oplus \langle y^p \rangle$, in which $x, y \in G$ and $x^p \neq y^p$. From [6, Proposition 7], consider the following exact sequence

$$G' \otimes (G/G') \to G \otimes G \xrightarrow{\tau} G/G' \otimes G/G' \to 1.$$

Put $S = \langle x^p \otimes g, y^p \otimes g_1 \mid g, g_1 \in G \rangle$. It is clear that $\ker \tau = S$. By [6, Proposition 16(*v*)], $(G \otimes G)/S \cong G/G' \otimes G/G'$ and $S \neq 1_{G \otimes G}$. For some $g \in G$, we have $x^p \otimes g \neq 1_{G \otimes G}$, so $(x \otimes g)^p = (x^p \otimes g)(g \otimes [x, g])^{\frac{-1}{2}p(p-1)} \neq 1_{G \otimes G}$, using [1, Lemma 3.4]. Thus $exp(G \otimes G) = p^2$. By a similar method, we can see that $exp(G \wedge G) = p^2$.

(*ix*) Using [9, Theorems 1.1(c), 1.3(a), and 1.5], $G^p = G'$ and $|\mathcal{M}(G)| = p^{\frac{1}{2}d(d-1)-1}$, and so $|G \wedge G| = p^{\frac{1}{2}d(d-1)+1}$. [6, Proposition 7] implies that the following sequence

$$G' \otimes (G/G') \to G \wedge G \xrightarrow{\eta} G/G' \wedge G/G' \cong \mathcal{M}(G/G') \to 1$$

is exact. Let $K = \ker \eta$. From [12, Corollary 2.2.12], $\mathcal{M}(G/G') \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)\right)}$. Since $|G \wedge G| = p^{\frac{1}{2}d(d-1)+1}$, we have $1 \neq K \cong \mathbb{Z}_p$. It is concluded that $(G\wedge G)^p = K \cong \mathbb{Z}_p$, using part $(viii)$. Therefore $G\wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)}$. We claim that $exp(\mathcal{M}(G)) = p^2$. Assume to the contrary that $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)}$. Then $G \wedge G \cong \mathcal{M}(G) \oplus \mathbb{Z}_{p^2}$, hence, $G' \cong (G \wedge G)/\mathcal{M}(G) \cong \mathbb{Z}_{p^2}$, which is impossible. Therefore, $\mathcal{M}(G) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-3\right)}$ and $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)} \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(2)}$.

□

The following lemma will be used in the proof of Theorem 2.3.

Lemma 2.2. *Let G be a finite p-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(k)}$ and $|Z(G)| = p^m$. Then*

(i) *$exp\big(Z(G)\big) = p$ if and only if $Z(G) \cong G' \oplus \mathbb{Z}_p^{(m-k)} \cong \mathbb{Z}_p^{(m)}$,*

(ii) *$exp\big(Z(G)\big) = p^2$ if and only if G' is not a direct summand of $Z(G)$ if and only if $Z(G) \cong \mathbb{Z}_p^{(m-2t)} \oplus \mathbb{Z}_{p^2}^{(t)}$ for some t with $1 \leq t \leq k$.*

Proof. G/G' is elementary abelian, so is $Z(G)/G'$. Hence, $exp\big(Z(G)\big) \leq p^2$.

(i) One can easily check that $exp\big(Z(G)\big) = p$ if and only if $Z(G) \cong G' \oplus \mathbb{Z}_p^{(m-k)} \cong \mathbb{Z}_p^{(m)}$, as desired.

(ii) Since $exp\big(Z(G)\big) \leq p^2$, part (i) implies that $exp\big(Z(G)\big) = p^2$ if and only if G' is not a direct summand of $Z(G)$. Now, we show that G' is not a direct summand of $Z(G)$ if and only if $Z(G) \cong \mathbb{Z}_p^{(m-2t)} \oplus \mathbb{Z}_{p^2}^{(t)}$ for some t with $1 \leq t \leq k$. Without loss of generality, assume that $G' = \oplus_{i=1}^{k} \langle x_i \rangle$ such that $\langle x_j \rangle$ is not a direct summand of $Z(G)$ for all j with $1 \leq j \leq t$ and for some t with $1 \leq t \leq k$. It is easy to see that $\langle x_j \rangle \subsetneqq K_j$, in which $\mathbb{Z}_{p^2} \cong K_j \subseteq Z(G)$ and $K_j \cap K_r = 0$ for all r, j with $1 \leq r, j \leq t$ and $r \neq j$. It follows that $Z(G) \cong \mathbb{Z}_p^{(m-2t)} \oplus \mathbb{Z}_{p^2}^{(t)}$ for some t with $1 \leq t \leq k$. The converse is clear.

□

The following theorem shows that the structure of a p-group G of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$, which depends on the way G' is embedded in $Z(G)$.
Recall that the group G is a central product of H and K, if $G = HK$, H and K are subgroups of G and $[H, K] = 1$. If G is a central product of two subgroups H and K, we shall write $G = H \cdot K$.

Theorem 2.3. *Let G be a p-group of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$ and $|Z(G)| = p^m$. Then one of the following cases holds:*

(i) *$G \cong H \times \mathbb{Z}_p^{(m-2)}$, in which H is a special p-group of rank two.*

(ii) *$G \cong (H \cdot K_1) \times \mathbb{Z}_p^{(m-3)}$, in which $K_1 \cong \mathbb{Z}_{p^2}$, $K_1 \cap G' \cong \mathbb{Z}_p$, and H is a special p-group of rank two.*

(*iii*) *$G \cong \big(H\cdot(K_2\times K_3)\big)\times \mathbb{Z}_p^{(m-4)}$, in which $K_2\cong K_3\cong \mathbb{Z}_{p^2}$, $K_2\cap K_3=1$, $(K_2\times K_3)\cap G'=G'$, and H is a special p-group of rank two.*

Proof. Lemma 2.2 implies that $Z(G)\cong G'\times \mathbb{Z}_p^{(m-2)}$, $Z(G)\cong K_1\times \mathbb{Z}_p^{(m-2)}$, where $K_1\cap G'\cong \mathbb{Z}_p$ and $K_1\cong \mathbb{Z}_{p^2}$, or $Z(G)\cong K_2\times K_3\times \mathbb{Z}_p^{(m-4)}$, where $K_2\cap K_3=1$, $(K_2\times K_3)\cap G'=G'$ and $K_2\cong K_3\cong \mathbb{Z}_{p^2}$. To obtain the structure of G, we divide the proof into three cases as follows:

(*i*) Let $Z(G)=G'\times A$, where $A\cong \mathbb{Z}_p^{(m-1)}$. If $A=1$, then G is a special p-group of rank two, and the proof is complete.
Let now $A\neq 1$. Since G/G' is elementary abelian, we have $G/G'=H/G'\times (AG')/G'$ for a subgroup H of G. Thus

$$G=HA \ \text{ and } G'=H\cap AG'=(H\cap A)G'.$$

Hence, $H\cap A\subseteq G'\cap A=1$, and so $G\cong H\times A$. Since $Z(H)\times A=Z(G)=G'\times A$ and $G'=H'$, we have $Z(H)=H'=G'$. It is concluded that H is a special p-group of rank two.

(*ii*) Assume that $Z(G)\cong K_1\times A_1\times A_2$, in which $K_1\cap G'\cong \mathbb{Z}_p$, $K_1\cong \mathbb{Z}_{p^2}$, $\mathbb{Z}_p\cong A_1\subsetneq G'$, and $A_2\cong \mathbb{Z}_p^{(m-3)}$. We consider two cases as follows:
Case 1 Let $m=3$. Then $Z(G)=K_1\times A_1$. By a similar way used in the proof of part (*i*), we may obtain $G=HK_1$, in which H is a special p-group of rank two, $K_1\nsubseteq H$, $[H,K_1]=1$, and $H\cap K_1=H'\cap K_1$. It follows that $G=H\cdot K_1$ such that $K_1\cap H'\cong \mathbb{Z}_p$.
Case 2 Let $m\geq 4$. Similarly to **Case 1**, $G=P\times A_2$, $P'=G'$, and $Z(P)=K_1\times A_1$ for a subgroup P of G. **Case 1** implies that $P=H\cdot K_1$ and $K_1\cap H=H'\cap K_1\cong \mathbb{Z}_p$. Therefore $G=(H\cdot K_1)\times A_2$ and $G'=H'$, in which $K_1\cong \mathbb{Z}_{p^2}$, $K_1\cap G'\cong \mathbb{Z}_p$, $A_2\cong \mathbb{Z}_p^{(m-3)}$, and H is a special p-group of rank two. The proof is complete.

(*iii*) Let $Z(G)\cong K_2\times K_3\times \mathbb{Z}_p^{(m-4)}$, where $(K_2\times K_3)\cap G'=G'$ and $K_2\cong K_3\cong \mathbb{Z}_{p^2}$. The result follows by a similar way used in the proof of part (*ii*).

□

The following theorem shows that a capable p-group G of class two with $\Phi(G)=G'\cong \mathbb{Z}_p^{(2)}$ can be considered as a direct product of a capable special p-group of rank two and an elementary abelian p-group.

Theorem 2.4. *Let G be a d-generator p-group of class two such that $Z(G)\cong \mathbb{Z}_p^{(m)}$ and $\Phi(G)=G'\cong \mathbb{Z}_p^{(2)}$. Then G is capable if and only if $G\cong H\times \mathbb{Z}_p^{(m-2)}$, where H is a capable special p-group of rank two.*

Proof. Now, let G be capable. Then $exp\big(Z(G)\big)=p$, by [3, Proposition 1.2]. So, Theorem 2.3(*i*) implies that $G\cong H\times \mathbb{Z}_p^{(m-2)}$, where H is a special p-group of rank two. Since H/H' is elementary abelian, H/H' is capable, by [3, Proposition 7.3]. Hence, $Z^*(H)\subseteq H'$. On the other hand, by [15, Proposition 3.2], $Z^*(H)\cap H'=1$, and so $Z^*(H)=1$. Thus H is capable. The converse holds by [10, Remark (2) p.247]. □

3. Proofs of Main Theorems

Now, we are ready to a position to compute the corank, the Schur multiplier, the non-abelian exterior square, and the non-abelian tensor square of a non-capable p-group G of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$.

Proof of Theorem A. (*i*) First, we show that $Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)\right)}$. Now, let $Z^*(G) \cong \mathbb{Z}_p$. By Theorem 2.1(*iii*) and (*iv*), if $p = 2$, then $G^2 = G'$, and so $Z^*(G) = G'$. It follows that p must be odd. So, [11, Theorem 3.1] implies that $G/Z^*(G) \cong E_1 \times \mathbb{Z}_p^{(d-2)}$. We conclude that $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)\right)}$ for $p > 2$, using [3, Theorem 4.2] and [13, Main Theorem]. Clearly, $t(G) = 2d + 1$. The converse can be obtained by a similar way used in the above. Therefore, $Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)\right)}$ if and only if $t(G) = 2d+1$. By Theorem 2.1(*ii*), we can determine the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$.

(*ii*) It is obvious to see that $Z^*(G) = G'$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-2\right)}$ if and only if $t(G) = 2d + 3$, using [3, Theorem 4.2] and [12, Corollary 2.2.12]. Using Theorem 2.1(*vii*), we obtain that both $\mathcal{M}(G)$ and $G \wedge G$ are elementary abelian p-groups. Hence, $G \wedge G \cong \mathcal{M}(G) \oplus G'$. By Theorem 2.1(*ii*) and (*iv*), we determine the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$.

In what follows, we compute the corank, the Schur multiplier, the non-abelian exterior square, and the non-abelian tensor square of a capable d-generator p-group G of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$.

Proof of Theorem B. Theorem 2.4 implies that $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where H is a capable special p-group of rank two and exponent p. Using [9, Theorem 1.4(c)], let $H \cong \Phi_4(1^5)$. Then $G \cong \Phi_4(1^5) \times \mathbb{Z}_p^{(d-3)}$. By [9, Theorem 1.4(c)] and [12, Theorem 2.2.10 and Corollary 2.2.12], we get

$$\mathcal{M}(G) \cong \mathcal{M}(H) \oplus \mathcal{M}(\mathbb{Z}_p^{(d-3)}) \oplus (H/H' \otimes \mathbb{Z}_p^{(d-3)}) \cong \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)+3\right)}.$$

Hence, $t(G) = 2d+4$. Similarly, we can obtain the Schur multiplier of G when H is isomorphic to one of the p-groups $\Phi_{12}(1^6)$, $\Phi_{13}(1^6)$, $\Phi_{15}(1^6)$, or T. Using Theorem 2.1(*ii*), we may obtain the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$.

Proof of Theorem C. Theorem 2.4 implies that $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where H is a capable special p-group of rank two and exponent p^2. Suppose that H is a d_1-generator p-group. Then G is a $(d_1 + m - 2)$-generator p-group. So, $d = d_1 + m - 2$. Using Theorem 2.1(*ix*), [9, Theorems 1.1(c) and 1.5], [12, Theorem 2.2.10, and Corollary 2.2.12], we get

$$\begin{aligned}\mathcal{M}(G) &\cong \mathcal{M}(H) \oplus \mathcal{M}(\mathbb{Z}_p^{(m-2)}) \oplus (H/H' \otimes \mathbb{Z}_p^{(m-2)})\\ &\cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d_1(d_1-1)+\frac{1}{2}(m-2)(m-3)+d_1(m-2)-3\right)}\\ &\cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}(d_1+m-2)(d+m-3)-3\right)}\\ &\cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-3\right)},\end{aligned}$$

and so $\mathcal{M}(G) \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-3\right)}$. Clearly, $t(G) = 2d$. From Theorem 2.1($viii$), $exp(G \wedge G) = p^2$. By a similar way used in the proof of Theorem 2.1(ix), it is concluded that $(G \wedge G)^p \cong \mathbb{Z}_p$. So, $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{\left(\frac{1}{2}d(d-1)-1\right)} \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(2)}$. Using Theorem 2.1($ii$) and [4, Theorem 1.3($ii$)], we may obtain the structure of $G \otimes G$ and $J_2(G)$.

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