ON P-DERIVATIONS OF INVOLUTIVE RINGS

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Abstract. Very recently, Khan et al. [11] initiated the study of $*$-differential identities in prime ideal of an arbitrary involutive ring. On the other hand, Sandhu et al. [17] introduced the notion of P-derivation in an arbitrary ring having a prime ideal P. Our main intent in this article is to construct a bridge between these studies and investigate P-derivations satisfying $*$-differential identities in the prime ideal P of an arbitrary involutive ring. Consequently, we address some gaps left in the previous studies and obtain more general results. Some appropriate examples are also given.

1. Introduction

Throughout the article, unless particularly stated otherwise, R will always be the associative ring and its center will be denoted as $Z(R)$. Recall that a proper ideal P is called a prime ideal of R if $xRy \subseteq P$ implies that $x \in P$ or $y \in P$ for any $x, y \in R$. If the ideal $\{0\}$ is prime then R is said to be a prime ring. As usual $a \circ b$ (resp. $[a, b]$) represents the anti-commutator $ab + ba$ (resp. commutator $ab - ba$). A 2-ordered anti-automorphism $*$ of R is called an involution on R. The symbol $H(R)$ (resp. $S(R)$) denotes the collection of all symmetric (resp. skew-symmetric) elements where the element x such that $x^* = x$ is called symmetric and element x such that $x^* = -x$ is called skew-symmetric. An involution is called of first kind if $Z(R) \subseteq H(R)$ otherwise it is called of the second kind. A derivation on R is an additive map δ from R to itself such that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. Let R be an arbitrary ring and P be a prime ideal of R. Then a mapping $d : R \to R$ is called a P-derivation of R if it satisfies $d(x + y) - d(x) - d(y) \in P$ and $d(xy) - d(x)y - xd(y) \in P$ for all $x, y \in R$ (see [17]). Note that every derivation is a P-derivation but the converse is not generally true. For example, for any non-zero prime ideal P of any ring R (involutive), fix $p \in P$, the map $d(x) = p$ for all $x \in R$ is P-derivation but not a derivation on R.

It is evident that the algebra of derivations and generalized derivations play an important role in the examination of $*$-differential identities and $*$-functional identities. Almost three decades ago, Brešar et al. [8] initiated the investigation of certain additive mappings in involutive prime rings. Since then several authors have obtained significant amount of results specifically in a prime ring equipped an involution $*$ that possesses special mappings (such as derivations, generalized derivations, endomorphisms etc.) and satisfy adequate $*$-differential identities. For a detailed cross section, the reader may check [5, 6], [11–14], [15, 16], [18] and references therein. In order to extend the theory developed on prime rings with certain mappings, recently some algebraists have studied differential identities in a prime ideal P of an arbitrary ring R and observed the structure of the factor ring R/P; for such works, the reader can see [3, 4, 9]. Very recently, in [11] Khan et al. extended a classical theorem of Herstein with a pair of derivations on prime ideals of an involutive ring. Their main result is stated as follows:

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Let R be a ring with involution $*$ of the second kind, P a prime ideal of R such that $S(R) \cap Z(R) \nsubseteq P$ and char$(R/P) \neq 2$. If d_1 and d_2 are derivations of R satisfying the condition $[d_1(x), d_2(x^*)] \in P$ for all $x \in R$, then one of the following holds:

(a) $d_1(R) \subseteq P$,
(b) $d_2(R) \subseteq P$,
(c) R/P is a commutative integral domain.

Moreover in the same vein, some results have been obtained with generalized derivations in [1] and [10]. Our intent in this article is to investigate P-derivations and structure of the factor ring R/P where R is a ring with involution $*$ which satisfies the following $*$-differential identities in P :

(i) $\overline{d_1(x)x^* - x^*d_2(x)} \in Z(R/P)$,
(ii) $\overline{d_1(x)d_2(x^*) - d_3(xx^*)} \in Z(R/P)$,
(iii) $\overline{d_1(x)d_2(x^*) - d_3(x^*x)} \in Z(R/P)$,
(iv) $[d_1(x), d_2(x^*)] - d_3([x, x^*]) \in P$,
(v) $d_1(x) \circ d_2(x^*) - d_3(x \circ x^*) \in P$.

2. Preliminaries

We shall use the following lemmas frequently to develop our main results:

Lemma 2.1. *[11, Lemma 2.1] Let R be a ring and P a prime ideal of R. If d is a derivation of R such that $[d(x), x] \in P$ for all $x \in R$, then $d(R) \subseteq P$ or R/P is a commutative integral domain.*

Lemma 2.2. *Let R be an arbitrary ring with prime ideal P. If $d : R \to R$ is a P-derivation of R, then d maps $Z(R)$ into $Z(R/P)$.*

Proof. Let $z \in Z(R)$. Then we have

$$d(rz) - d(r)z - rd(z) \in P \text{ for all } r \in R. \tag{2.1}$$

and

$$d(zr) - d(z)r - zd(r) \in P \text{ for all } r \in R. \tag{2.2}$$

Comparing (2.1) and (2.2), we get $[d(z), r] \in P$ for all $r \in R$. It forces that $d(z) \in Z(R/P)$. □

Lemma 2.3. *Let R be an arbitrary involutive ring with involution $*$ and P be a prime ideal of R. If $S(R) \cap Z(R) \nsubseteq P$, then $*$ is involution of the second kind.*

Proof. Suppose that the involution $*$ is not of the second kind, that means $S(R) \cap Z(R) = \{0\} \subseteq P$, which is a contradiction. It proves our claim. □

The converse of the above lemma is not true, for e.g. let $R = \{[a_i]_{2\times2} \times [b_i]_{2\times2} : a_i \in \mathbb{C}, b_i \in \mathbb{R}\}$ and $P = \{[a_i]_{2\times2} \times \{0\} : a_i \in \mathbb{C}\}$. Let us define $*$ as $(A, B)^* = (A^*, B^*)$, where A^* denotes the transpose conjugate of A. Note that $*$ is the involution of the second kind but $S(R) \cap Z(R) \subseteq P$.

3. Main Results

Throughout, R will be an associative involutive ring with involution $*$, P be a prime ideal of R such that $S(R) \cap Z(R) \nsubseteq P$ and char$(R/P) \neq 2$; unless otherwise mentioned.

Theorem 3.1. *Let $d_1, d_2 : R \to R$ be P-derivations of R such that*

$$\overline{d_1(x)x^* - x^*d_2(x)} \in Z(R/P)$$

for all $x \in R$. Then

(i) $d_1(R) \subseteq P$, $d_2(R) \subseteq P$ *or*
(ii) R/P *is a commutative integral domain.*

Proof. Assume that
$$\overline{d_1(x)x^* - x^*d_2(x)} \in Z(R/P) \text{ for all } x \in R.$$
Polarization of the above expression yields that
$$\overline{d_1(x)y^* + d_1(y)x^* - x^*d_2(y) - y^*d_2(x)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.1}$$
For some $0 \neq h \in H(R) \cap Z(R)$, changing x by xh in (3.1) and using Lemma 2.2, we find
$$\overline{xy^*d_1(h) - y^*xd_2(h)} \in Z(R/P) \text{ for all } x, y \in R.$$
For $k \in S(R) \cap Z(R)$, taking k^2 in place of h in the last relation, to get
$$2\overline{(xy^*d_1(k) - y^*xd_2(k))k} \in Z(R/P) \text{ for all } x, y \in R.$$
Using $\mathrm{char}(R/P) \neq 2$, we have
$$[(xy^*d_1(k) - y^*xd_2(k)), r]k \in P \text{ for all } x, y, r \in R.$$
Invoking Brauer's trick, it implies that either $[xy^*d_1(k) - y^*xd_2(k), r] \in P$ for all $x, y, r \in R, k \in S(R) \cap Z(R)$ or $k \in P$ for all $k \in S(R) \cap Z(R)$. But $S(R) \cap Z(R) \not\subseteq P$, thus we obtain
$$\overline{xy^*d_1(k) - y^*xd_2(k)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.2}$$
Substituting yk for y in (3.1) and using (3.2), one can find that
$$\overline{-d_1(x)y^* + d_1(y)x^* - x^*d_2(y) + y^*d_2(x)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.3}$$
Adding (3.1) and (3.3), we conclude that
$$\overline{d_1(y)x^* - x^*d_2(y)} \in Z(R/P) \text{ for all } x, y \in R.$$
It can be easily obtained from the last relation that
$$[d_1(y)x - xd_2(y), r] \in P \text{ for all } x, y, r \in R. \tag{3.4}$$
Replacing x by $xd_2(y)$ in (3.4) and using it, we arrive at
$$(d_1(y)x - xd_2(y))[d_2(y), r] \in P \text{ for all } x, y, r \in R,$$
this implies that
$$(d_1(y)x - xd_2(y))R[d_2(y), r] \subseteq P \text{ for all } x, y, r \in R,$$
In view of primeness of P, it follows that for each $y \in R$, either $d_1(y)x - xd_2(y) \in P$ for all $x \in R$ or $[d_2(y), r] \in P$ for all $r \in R$. An application of Brauer's trick yields that either
$$d_1(y)x - xd_2(y) \in P \text{ for all } x, y \in R \text{ or}$$
$$[d_2(y), r] \in P \text{ for all } y, r \in R.$$
Let us consider $d_1(y)x - xd_2(y) \in P$ for all $x, y \in R$. Replacement of x with xt in this equation forces $x[d_2(y), t] \in P$ for all $x, y, t \in R$. Primeness of P implies $[d_2(y), t] \in P$ for all $y, t \in R$, which is nothing but our other case.

Finally we have $[d_2(R), R] \subseteq P$; by Lemma 2.1, we get R/P is a commutative integral domain or $d_2(R) \subseteq P$. With this in hand, (3.4) shows that $[d_1(y)x, r] \in P$ for all $x, y, r \in R$. In same way, it leads us to $d_1(R) \subseteq P$, as desired. ∎

Corollary 3.2. *Let $d : R \to R$ be a derivation of R such that*

$$\overline{[d(x), x^*]} \in Z(R/P)$$

for all $x \in R$. Then

(i) *$d(R) \subseteq P$ or*
(ii) *R/P is a commutative integral domain.*

Corollary 3.3. *Let R be an involutive prime ring with involution $*$ of the second kind and $char(R) \neq 2$. If $d_1, d_2 : R \to R$ are nonzero derivations of R, then the following statements are equivalent:*

(i) *$d_1(x)x^* - x^*d_2(x) \in Z(R)$ for all $x \in R$.*
(ii) *R is commutative.*

Proposition 3.4. *Let $d : R \to R$ is a P-derivation of R such that*

$$\overline{d(xx^*)} \in Z(R/P) \ (or\ \overline{d(x^*x)} \in Z(R/P))$$

for all $x \in R$. Then either R/P is a commutative integral domain or $d(R) \subseteq P$.

Proof. Assume that $\overline{d(xx^*)} \in Z(R/P)$ for all $x \in R$. Polarization of this equation gives

$$\overline{d(xy^*) + d(yx^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.5}$$

Replacing y by yh in the above expression, where $0 \neq h \in H(R) \cap Z(R)$, to obtain $\overline{(xy^* + yx^*)d(h)} \in Z(R/P)$ for all $x, y \in R$. It forces either $\overline{d(h)} = \overline{0}$ or $\overline{xy^* + yx^*} \in Z(R/P)$ for all $x, y \in R$. In the latter case if we change y with y^*, we get

$$\overline{xy + y^*x^*} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.6}$$

Taking yk for y in (3.6), where $k \in S(R) \cap Z(R) \nsubseteq P$, we find

$$\overline{xy - y^*x^*} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.7}$$

Adding (3.6) and (3.7), we conclude that $[xy, r] \in P$ for all $x, y, r \in R$. Thus, we easily arrive at R/P is commutative integral domain. On the other hand, we have $\overline{d(h)} = \overline{0}$. Putting $h = k^2$ where $k \in S(R) \cap Z(R) \nsubseteq P$, we get $\overline{d(k)} = \overline{0}$. Now, taking y^* in place of y in (3.5), we have

$$\overline{d(xy) + d(y^*x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.8}$$

Substitution of yk for y in (3.8) leads us to

$$\overline{d(xy) - d(y^*x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.9}$$

Adding (3.8) and (3.9), we obtain $\overline{d(xy)} \in Z(R/P)$ for all $x, y \in R$. Theorem 1 of [14] completes the proof. ☐

Proposition 3.5. *Let $d : R \to R$ is a P-derivation such that*

$$\overline{d_1(x)d_2(x^*)} \in Z(R/P) \ (or\ \overline{d_1(x^*)d_2(x)} \in Z(R/P))$$

for all $x \in R$. Then either R/P is a commutative integral domain or $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$.

Proof. Assume that $\overline{d_1(x)d_2(x^*)} \in Z(R/P)$ for all $x \in R$. Linearizing this relation, we find that

$$\overline{d_1(x)d_2(y^*) + d_1(y)d_2(x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.10}$$

Replacing y by yh where $0 \neq h \in H(R) \cap Z(R)$ in (3.10), we get

$$\overline{d_1(x)y^*d_2(h) + yd_1(h)d_2(x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.11}$$

Putting $h = k^2$ for some $k \in S(R) \cap Z(R) \nsubseteq P$ in (3.11), to find

$$\overline{d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.12}$$

Substituting yk for y where $k \in S(R) \cap Z(R) \nsubseteq P$, in the last expression, we obtain

$$\overline{-d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.13}$$

Now, replacing y by yk where $k \in S(R) \cap Z(R) \nsubseteq P$ in (3.10), we get

$$\overline{-d_1(x)d_2(y^*)k - d_1(x)y^*d_2(k) + d_1(y)d_2(x^*)k + yd_1(k)d_2(x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.14}$$

Using the equation (3.13) in the above relation and the condition $S(R) \cap Z(R) \nsubseteq P$, we obtain

$$\overline{-d_1(x)d_2(y^*) + d_1(y)d_2(x^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.15}$$

Adding (3.10) and (3.15) in order to get $\overline{d_1(y)d_2(x)} \in Z(R/P)$ for all $x, y \in R$. It can be seen as $[d_1(y)d_2(x), r] \in P$ for all $x, y, r \in R$. Replacing x by xr, we conclude that $[d_1(y)xd_2(r), r] \in P$ for all $x, y, r \in R$. Substituting $d_1(t)x$ for x in the last expression, one can find $[d_1(y), r]d_1(t)xd_2(r) \in P$ for all $x, y, r, t \in R$.Thus applying Brauer's trick, for each $r \in R$ we get either $[d_1(y), r]d_1(t) \in P$ for all $y, t \in R$ or $d_2(r) \in P$. Hence, it assures that either $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$. ☐

Theorem 3.6. *Let $d_1, d_2, d_3 : R \to R$ be P-derivations of R such that*

$$\overline{d_1(x)d_2(x^*) - d_3(xx^*)} \in Z(R/P)$$

for all $x \in R$. Then

(i) *$d_1(R) \subseteq P$, $d_3(R) \subseteq P$ or*
(ii) *$d_2(R) \subseteq P$, $d_3(R) \subseteq P$ or*
(iii) *R/P is a commutative integral domain.*

Proof. Assume that

$$\overline{d_1(x)d_2(x^*) - d_3(xx^*)} \in Z(R/P) \text{ for all } x \in R.$$

A linearization of this relation obtains

$$\overline{d_1(x)d_2(y^*) + d_1(y)d_2(x^*) - d_3(xy^*) - d_3(yx^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.16}$$

For some $0 \neq h \in H(R) \cap Z(R)$, replacing y by yh in (3.16) and using Lemma 2.2, to get

$$\overline{d_1(x)y^*d_2(h) + yd_1(h)d_2(x^*) - (xy^* + yx^*)d_3(h)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.17}$$

For some $k \in S(R) \cap Z(R)$ substitution of k^2 for h in (3.17) leads us to

$$\overline{2(d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*) - (xy^* + yx^*)d_3(k))k} \in Z(R/P) \text{ for all } x, y \in R.$$

Our assumption of $\text{char}(R/P) \neq 2$ and $S(R) \cap Z(R) \nsubseteq P$ yields

$$\overline{d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*) - (xy^* + yx^*)d_3(k)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.18}$$

Replace yk instead y in (3.18) and using $S(R) \cap Z(R) \nsubseteq P$ to obtain

$$-\overline{d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*) - (-xy^* + yx^*)d_3(k)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.19}$$

Replace yk instead y in (3.16) to obtain

$$\begin{aligned}&\overline{-d_1(x)d_2(y^*)k - d_1(x)y^*d_2(k) + d_1(y)kd_2(x^*) + yd_1(k)d_2(x^*) - (-d_3(xy^*)} \\ &\qquad \overline{+d_3(yx^*))k - (-xy^* + yx^*)d_3(k)} \in Z(R/P) \text{ for all } x, y \in R.\end{aligned} \tag{3.20}$$

Subtracting (3.19) and (3.20) and using the assumption of char$(R/P) \neq 2$, we can find that

$$\overline{-d_1(x)d_2(y^*)k + d_1(y)kd_2(x^*) - (-d_3(xy^*) + d_3(yx^*))k} \in Z(R/P) \text{ for all } x, y \in R.$$

It implies that

$$\overline{-d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_3(xy^*) - d_3(yx^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.21}$$

Add (3.16) and (3.21) to obtain

$$\overline{d_1(y)d_2(x^*) - d_3(yx^*)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.22}$$

That is

$$[d_1(y)d_2(x) - d_3(yx), r] \in P \text{ for all } x, y, r \in R. \tag{3.23}$$

Replace x by k^2 where $k \in S(R) \cap Z(R) \nsubseteq P$ in (3.23) to obtain

$$[2d_1(y)d_2(k) - d_3(yk) - yd_3(k), r]k \in P \text{ for all } y, r \in R. \tag{3.24}$$

Replace x by k in (3.23) and thereby compare with (3.24) to obtain

$$[d_3(y), r]k \in P \text{ for all } y, r \in R.$$

But $k \notin P$ for all $k \in S(R) \cap Z(R)$, hence an application of Lemma 2.1 implies $d_3(R) \subseteq P$. Thus, given condition and Proposition 3.5 yields to the desired result. □

Corollary 3.7. *Let $d, g : R \to R$ be P-derivations of R such that*

$$\overline{d(x)d(x^*) - g(xx^*)} \in Z(R/P)$$

for all $x \in R$. Then

(i) *$d(R) \subseteq P$, $g(R) \subseteq P$ or*
(ii) *R/P is a commutative integral domain.*

Corollary 3.8. *Let R be an involutive prime ring with involution $*$ of the second kind and char$(R) \neq 2$. If $d_1, d_2, d_3 : R \to R$ are derivations of R such that*

$$d_1(x)d_2(x^*) - d_3(xx^*) \in Z(R)$$

for all $x \in R$ then one of the following is true:

(i) *$d_1 = 0 = d_3$;*
(ii) *$d_2 = 0 = d_3$;*
(iii) *R is commutative.*

Theorem 3.9. *Let $d_1, d_2, d_3 : R \to R$ be P-derivations of R such that*

$$\overline{d_1(x)d_2(x^*) - d_3(x^*x)} \in Z(R/P)$$

for all $x \in R$. Then

(i) *$d_1(R) \subseteq P$, $d_3(R) \subseteq P$ or*
(ii) *$d_2(R) \subseteq P$, $d_3(R) \subseteq P$ or*
(iii) *R/P is a commutative integral domain.*

Proof. Assume that

$$\overline{d_1(x)d_2(x^*) - d_3(x^*x)} \in Z(R/P) \text{ for all } x \in R.$$

A linearization of the this relation takes us to

$$\overline{d_1(x)d_2(y^*) + d_1(y)d_2(x^*) - d_3(x^*y) - d_3(y^*x)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.25}$$

For $0 \neq h \in H(R) \cap Z(R)$, substituting yh for y in (3.25) and using Lemma 2.2, we get

$$\overline{d_1(x)y^*d_2(h) + yd_1(h)d_2(x^*) - x^*yd_3(h) - y^*xd_3(h)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.26}$$

For $k \in S(R) \cap Z(R)$, replacing h by k^2 in (3.26), we get

$$\overline{2(d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*) - x^*yd_3(k) - y^*xd_3(k))k} \in Z(R/P) \text{ for all } x, y \in R.$$

Using $\mathrm{char}(R/P) \neq 2$ and $S(R) \cap Z(R) \nsubseteq P$, we have

$$\overline{d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*) - x^*yd_3(k) - y^*xd_3(k)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.27}$$

Replacing y by yk in (3.27) to find

$$\overline{-d_1(x)y^*d_2(k) + yd_1(k)d_2(x^*) - x^*yd_3(k) + y^*xd_3(k)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.28}$$

Now, taking yk instead of yk in (3.25) where $k \in S(R) \cap Z(R) \nsubseteq P$, we obtain

$$\begin{aligned} &\overline{-d_1(x)d_2(y^*)k - d_1(x)y^*d_2(k) + d_1(y)d_2(x^*)k + yd_1(k)d_2(x^*) - d_3(x^*y)k} \\ &\qquad \overline{-x^*yd_3(k) + d_3(y^*x)k + y^*xd_3(k)} \in Z(R/P) \text{ for all } x, y \in R. \end{aligned} \tag{3.29}$$

Comparing (3.28) and (3.29), we obtain

$$\overline{-d_1(x)d_2(y^*)k + d_1(y)d_2(x^*)k - d_3(x^*y)k + d_3(y^*x)k} \in Z(R/P) \text{ for all } x, y \in R.$$

Since $S(R) \cap Z(R) \nsubseteq P$, we have

$$\overline{-d_1(x)d_2(y^*) + d_1(y)d_2(x^*) - d_3(x^*y) + d_3(y^*x)} \in Z(R/P) \text{ for all } x, y \in R. \tag{3.30}$$

Adding (3.25) and (3.30), we conclude

$$d_1(y)d_2(x^*) - d_3(x^*y) \in Z(R/P) \text{ for all } x, y \in R.$$

It implies that

$$[d_1(y)d_2(x) - d_3(xy), r] \in P \text{ for all } x, y, r \in R. \tag{3.31}$$

Replace y by k^2 in (3.31) to obtain

$$[2d_1(k)d_2(x) - xd_3(k) - d_3(xk), r]k \in P \text{ for all } x, r \in R. \tag{3.32}$$

Replace y by k in (3.31) and compare with (3.32) to obtain

$$[d_3(x), r]k \in P \text{ for all } y, r \in R.$$

Since $S(R) \cap Z(R) \nsubseteq P$, by Lemma 2.1 we get $d_3(R) \subseteq P$. Thus, given the given hypothesis and Proposition 3.5 leads us to the desired result. □

Corollary 3.10. *Let $d, g : R \to R$ be P-derivations of R such that*

$$\overline{d(x)d(x^*) - g(x^*x)} \in Z(R/P)$$

for all $x \in R$. Then

(i) *$d(R) \subseteq P$, $g(R) \subseteq P$ or*
(ii) *R/P is a commutative integral domain.*

Corollary 3.11. *Let R be an involutive prime ring with involution $*$ of the second kind and $\mathrm{char}(R) \neq 2$. If $d_1, d_2, d_3 : R \to R$ are derivations of R such that*

$$d_1(x)d_2(x^*) - d_3(x^*x) \in Z(R)$$

for all $x \in R$ then one of the following is true:

(i) *$d_1 = 0 = d_3$;*

(ii) $d_2 = 0 = d_3$;
(iii) *R is commutative.*

Theorem 3.12. *Let $d_1, d_2, d_3 : R \to R$ be P-derivations of R such that*

$$[d_1(x), d_2(x^*)] - d_3([x, x^*]) \in P$$

for all $x \in R$. Then
(i) $d_1(R) \subseteq P$, $d_3(R) \subseteq P$ *or*
(i) $d_2(R) \subseteq P$, $d_3(R) \subseteq P$ *or*
(ii) R/P *is a commutative integral domain.*

Proof. Assume that $[d_1(x), d_2(x^*)] - d_3([x, x^*]) \in P$ for all $x \in R$. A linearization of this relation gives

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] - d_3([x, y^*]) - d_3([y, x^*]) \in P \text{ for all } x, y \in R. \tag{3.33}$$

Changing x by xh in (3.33), where $0 \neq h \in H(R) \cap Z(R)$ and using Lemma 2.2, we get

$$d_1(h)[x, d_2(y^*)] + d_2(h)[d_1(y), x^*] - d_3(h)[x, y^*] - d_3(h)[y, x^*] \in P \text{ for all } x, y \in R. \tag{3.34}$$

Putting $h = k^2$ where $k \in S(R) \cap Z(R) \not\subseteq P$ in (3.34), we find

$$d_1(k)[x, d_2(y^*)] + d_2(k)[d_1(y), x^*] - d_3(k)[x, y^*] - d_3(k)[y, x^*] \in P \text{ for all } x, y \in R.$$

Now, replacing x by xk in the above relation and using the fact that $k \notin P$, to have

$$d_1(k)[x, d_2(y^*)] - d_2(k)[d_1(y), x^*] - d_3(k)[x, y^*] + d_3(k)[y, x^*] \in P \text{ for all } x, y \in R. \tag{3.35}$$

Writing xk instead of x in (3.33), we observe that

$$[d_1(x), d_2(y^*)]k + [x, d_2(y^*)]d_1(k) - [d_1(y), d_2(x^*)]k - [d_1(y), x^*]d_2(k)$$
$$-d_3([x, y^*])k - [x, y^*]d_3(k) + d_3([y, x^*])k + [y, x^*]d_3(k) \in P \text{ for all } x, y \in R.$$

Equation (3.35) reduces it to

$$([d_1(x), d_2(y^*)] - [d_1(y), d_2(x^*)] - d_3([x, y^*]) + d_3([y, x^*]))k \in P \text{ for all } x, y \in R.$$

It forces

$$[d_1(x), d_2(y^*)] - [d_1(y), d_2(x^*)] - d_3([x, y^*]) + d_3([y, x^*]) \in P \text{ for all } x, y \in R. \tag{3.36}$$

Adding (3.33) and (3.36), we obtain $[d_1(x), d_2(y^*)] - d_3([x, y^*]) \in P$ for all $x, y \in R$. Changing y by y^* to get

$$[d_1(x), d_2(y)] - d_3([x, y]) \in P \text{ for all } x, y \in R. \tag{3.37}$$

For any $t \in R$, let us write yt for y in the above expression, we get

$$d_2(y)[d_1(x), t] + [d_1(x), y]d_2(t) - [x, y]d_3(t) - d_3(y)[x, t] \in P \text{ for all } x, y, t \in R.$$

In particular for $x = t$, we obtain

$$d_2(y)[d_1(t), t] + [d_1(t), y]d_2(t) - [t, y]d_3(t) \in P \text{ for all } x, y \in R.$$

Replacing y by wy in the last equation, we get

$$d_2(w)y[d_1(t), t] + [d_1(t), w]yd_2(t) - [t, w]yd_3(t) \in P \text{ for all } x, y, w \in R.$$

Now, taking $w = t$ in order to find

$$d_2(t)y[d_1(t), t] + [d_1(t), t]yd_2(t) \in P \text{ for all } x, y, w \in R.$$

Which is equivalent to

$$\overline{d_2(t)y[d_1(t),t] + [d_1(t),t]yd_2(t)} = \overline{0} \text{ for all } x, y, w \in R.$$

Applying a result of Brešar [7, Lemma 4], we conclude that $\overline{[d_1(t),t]Rd_2(t)} = \overline{0}$ for all $t \in R$. Primeness of R/P forces $\overline{[d_1(t),t]d_2(t)} = \overline{0}$ for all $t \in R$. A consequence of Corollary 3.6 of [2] yields that either $\overline{d_1(R)} = \overline{0}$ or $\overline{d_2(R)} = \overline{0}$ or R/P is commutative. That means $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$ or R/P is commutative.

Suppose that any one of $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$ holds. Then the equation (3.37) yields $d_3([x,y]) \in P$ for all $x, y \in R$. Taking yx instead of y to obtain $[x,y]d_3(x) \in P$ for all $x, y \in R$. It implies that for each $x \in R$, either $[x,y] \in P$ for all $y \in R$ or $d_3(x) \in P$. Thus applying Brauer's trick we get either $[R,R] \subseteq P$ or $d_3(R) \subseteq P$. But the former case is exempted, so we left with $d_3(R) \subseteq P$, as desired. □

The following result gives a generalization of the result [11, Theorem 2.4].

Corollary 3.13. *Let $d_1, d_2 : R \to R$ be P-derivations of R such that*

$$[d_1(x), d_2(x^*)] \in P$$

for all $x \in R$. Then one of the following holds true

(i) $d_1(R) \subseteq P$,
(ii) $d_2(R) \subseteq P$,
(iii) *R/P is a commutative integral domain.*

Corollary 3.14. *Let $d : R \to R$ be a P-derivation of R such that*

$$d([x, x^*]) \in P$$

for all $x \in R$. Then one of the following holds true

(i) $d(R) \subseteq P$,
(ii) *R/P is a commutative integral domain.*

Corollary 3.15. *Let R be an involutive prime ring with involution $*$ of the second kind and char$(R) \neq 2$. If $d_1, d_2, d_3 : R \to R$ are derivations of R such that*

$$[d_1(x), d_2(x^*)] - d_3([x, x^*]) = 0$$

for all $x \in R$ then one of the following holds true:

(i) $d_1 = 0 = d_3$;
(ii) $d_2 = 0 = d_3$;
(iii) *R is commutative.*

Theorem 3.16. *Let $d_1, d_2, d_3 : R \to R$ be P-derivations of R such that*

$$d_1(x) \circ d_2(x^*) - d_3(x \circ x^*) \in P$$

for all $x \in R$. Then

(i) $d_1(R) \subseteq P$, $d_3(R) \subseteq P$ *or*
(ii) $d_2(R) \subseteq P$, $d_3(R) \subseteq P$.

Proof. Let $d_1(x) \circ d_2(x^*) - d_3(x \circ x^*) \in P$ for all $x \in R$. Linearizing this relation gives

$$d_1(x) \circ d_2(y^*) + d_1(y) \circ d_2(x^*) - d_3(x \circ y^*) - d_3(y \circ x^*) \in P \text{ for all } x, y \in R. \quad (3.38)$$

Replacing x by xh in (3.38) where $0 \neq h \in H(R) \cap Z(R)$ and using Lemma 2.2, we get

$$(x \circ d_2(y^*))d_1(h) + (d_1(y) \circ x^*)d_2(h) - (x \circ y^*)d_3(h) - (y \circ x^*)d_3(h) \in P. \tag{3.39}$$

Taking $h = k^2$, where $k \in S(R) \cap Z(R) \nsubseteq P$ in (3.39), we have

$$(x \circ d_2(y^*))d_1(k) + (d_1(y) \circ x^*)d_2(k) - (x^* \circ y)d_3(k) - (x \circ y^*)d_3(k) \in P. \tag{3.40}$$

Changing x into xk in the above equation, and using the assumption $S(R) \cap Z(R) \nsubseteq P$, we may infer that

$$d_1(k)(x \circ d_2(y^*)) - d_2(k)(d_1(y) \circ x^*) + d_3(k)(x^* \circ y) - d_3(k)(x \circ y^*) \in P \tag{3.41}$$

Now changing xk instead of x in the equation (3.38), where $0 \neq k \in S(R) \cap Z(R)$ and using (3.41) to get

$$k(d_1(x) \circ d_2(y^*) - d_1(y) \circ d_2(x^*) - d_3(x \circ y^*) + d_3(y \circ x^*)) \in P \tag{3.42}$$

Again our assumption $S(R) \cap Z(R) \nsubseteq P$ forces that

$$d_1(x) \circ d_2(y^*) - d_1(y) \circ d_2(x^*) - d_3(x \circ y^*) + d_3(y \circ x^*) \in P. \tag{3.43}$$

Now adding (3.38) and (3.43), and then taking y^* instead of y we have

$$d_1(x) \circ d_2(y) - d_3(x \circ y) \in P \tag{3.44}$$

For any $t \in R$ changing y into yt in 3.44 we get

(3.45)

$$-d_2(y)[d_1(x), t] + (d_1(x) \circ y)d_2(t) - y[d_1(x), d_2(t)] - (x \circ y)d_3(t) + d_3(y)[x, t] + yd_3([x, t]) \in P$$

Putting $x = t$ in the above equation

$$-d_2(y)[d_1(t), t] + (d_1(t) \circ y)d_2(t) - y[d_1(t), d_2(t)] - (t \circ y)d_3(t) \in P \tag{3.46}$$

For any $w \in R$, taking wy instead of y in (3.46)

$$-d_2(w)y[d_1(t), t] + [d_1(t), w]yd_2(t) - [t, w]yd_3(t) \in P \tag{3.47}$$

Taking $w = z$, a central element here to obtain

$$d_2(z)y[d_1(t), t] \in P \tag{3.48}$$

Which implies that either $d_2(z) \in P$ for all $z \in Z$ or $[d_1(t), t] \in P$ for all $t \in R$. For the latter case, by using the Lemma 2.1 we get that either $d_1(R) \subseteq P$ or R/P is a commutative integral domain. On the other hand if $d_2(z) \in P$ for all $z \in Z$, we can have $d_1(z) \in P$ for all $z \in Z$ in the similar way. By taking $y = z \in Z(R)$ in the equation (3.44) and using 2-torsion freeness of R/P we obtain $d_3(xz) \in P$. Changing x into xr for any $r \in R$, we have $d_3(x)Rz \in P$ which implies that either $d_3(x) \in P$ for all $x \in R$ or $z \in P$ for all $z \in Z(R)$. The latter case is not possible because of the assumptions of our theorem. Thus $d_3(x) \in P$ for all $x \in R$. Thus our initial hypothesis yields $d_1(x) \circ d_2(x^*) \in P$, and hence a P-derivation analogy of Theorem 2.10 of [11] yields that either $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$. In any of these cases, by our hypothesis we have $d_3(x \circ x^*) \in P$. Invoking Lemma 2.3 of [11], we get $d_3(R) \subseteq P$, as desired.

Finally, we consider the case, when R/P is commutative. In this view we find $d_1(x)d_2(x^*) - d_3(xx^*) \in P$ for all $x \in R$. A particular case of Theorem 3.6 forces that $d_3(R) \subseteq P$, hence we get the conclusion as above. It completes the proof. □

Corollary 3.17. *Let $d_1, d_2 : R \to R$ be P-derivations of R such that*

$$d_1(x) \circ d_2(x^*) \in P$$

for all $x \in R$. Then one of the following holds true

(i) $d_1(R) \subseteq P$,
(ii) $d_2(R) \subseteq P$.

Corollary 3.18. *Let $d : R \to R$ be a P-derivation of R. Then the following are equivalent:*

(i) $d(x \circ x^*) \in P$ *for all* $x \in R$,
(ii) $d(R) \subseteq P$.

Corollary 3.19. *Let R be an involutive prime ring with involution $*$ of the second kind and char$(R) \neq 2$. If $d_1, d_2, d_3 : R \to R$ are derivations of R such that*

$$d_1(x) \circ d_2(x^*) - d_3(x \circ x^*) = 0$$

for all $x \in R$ then one of the following holds true:

(i) $d_1 = 0 = d_3$;
(ii) $d_2 = 0 = d_3$.

4. EXAMPLES

The following example demonstrates that the condition $S(R) \cap Z(R) \not\subseteq P$ used in Lemma 3.4, Lemma 3.5, Theorem 3.6, Theorem 3.9 and Theorem 3.12 can not be dropped.

Example 4.1. *Let $R = M_2(\mathbb{C}) \times M_2(\mathbb{R})$ be the ring and $P = M_2(\mathbb{C}) \times \{0\}$ prime ideal of R. Define involution $*$ on R as the standard inverse co-ordinate wise. Define $d_1 = d_2 = d_3 = d : R \to R$ as $d((A, B)) = (f_1(A), f_2(B))$, where $f_1 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is any map and $f_2 : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a derivation defined as $f_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$. One can easily notice that d is a P-derivation. It can be seen that the identities:*

(i) $\overline{d_1(x)d_2(x^*) - d_3(xx^*)} \in Z(R/P)$,
(ii) $\overline{d_1(x)d_2(x^*) - d_3(x^*x)} \in Z(R/P)$ *and*
(iii) $[d_1(x), d_2(x^*)] - d_3([x, x^*]) \in P$

are satisfied, but neither R/P is commutative nor $d_i(R) \subseteq P$. Hence the condition $S(R) \cap Z(R) \not\subseteq P$ is crucial.

The following example demonstrates that the condition $S(R) \cap Z(R) \not\subseteq P$ used in Theorem 3.16 can not be dropped.

Example 4.2. *Let $R = M_2(\mathbb{C}) \times M_2(\mathbb{R})$ be the ring and $P = M_2(\mathbb{C}) \times \{0\}$ prime ideal of R. Define involution $*$ on R as the standard inverse co-ordinate wise. Define $d_1 = 0$ and $d_2 = d_3 = d : R \to R$ as $d((A, B)) = (f_1(A), f_2(B))$, where $f_1 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is any map and $f_2 : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a derivation defined as $f_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$. One can easily notice that d is a P-derivation and the identity*

$$d_1(x) \circ d_2(x^*) - d_3(x \circ x^*) \in P$$

for all $x \in R$ is satisfied, but neither $d_2(R) \subseteq P$ nor $d_3(R) \subseteq P$. Hence the condition $S(R) \cap Z(R) \not\subseteq P$ is crucial.

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