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ON THE RECONSTRUCTION OF THE ABSORPTION COEFFICIENT FOR THE 2D ACOUSTIC SYSTEM

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Abstract: We consider the coefficient inverse problem for the 2D system of acoustics. Our goal is to recover the coefficient of acoustic attenuation by using the additional information of the wave-field in the number of receivers. We obtain the gradient of the cost functional and implement the numerical algorithm for solving the inverse problem, based on a optimization approach. We provide the numerical results of recovering the absorption coefficient and study its influence on the efficiency of reconstructing other parameters of the system. By taking into account the absorption of the sounding wave we aim to bring the mathematical model closer to the applications, related to the ultrasound tomography of the human tissue.

Keywords: tomography; first-order hyperbolic system; inverse problem; gradient descent method; acoustic attenuation.

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1 Introduction

In this article we deal with numerical solution of coefficient inverse problem, that corresponds to the problems of ultrasound tomography. The problems of developing methods and algorithms to use the ultrasound to recognize the genesis of breast cancer on its early stages are studied extensively lately [17, 18, 50, 21, 48, 52, 45, 26, 49, 54, 55, 57]. On the mathematical level such problems are usually considered as the inverse problems, when one has to recover the parameters of the model (that in our case describes the propagation of the ultrasound through the object of investigation) by using some measurements data [14, 33, 27, 28]. The inverse problems are known for their ill-posedness and large amount of computational resources, required for the numerical solution. Since that, the goal is to use the numerical algorithm, that utilize the provided data in an effective manner.

The mathematical models for the ultrasound acoustics usually has the form of either the second-order equation or the first-order system of PDE equations. The models, based on the second-order wave equation [56, 51, 53] are usually easier to study, and therefore, there are more ways to efficiently solve the direct problem but there is no guarantee that the calculated solution is close to the physical one. The first-order system of acoustics, that we consider in this paper, requires more computational resources for solving. Its advantage relies on its close connection to the physics of wave's propagation, since the equations can be derived straight from the conservation laws. We mention [3, 29, 34], where authors investigated inverse problems for system of hyperbolic partial differential equations.

Since we use the model, that based on the two-dimensional system of acoustic equations, its parameters, that we aim to recover, are the density of the medium, the speed of waves propagation and the coefficient of acoustic attenuation, or absorption coefficient. The reconstruction of several coefficients in the partial differential equations with a finite number of observations is very challenging.

The dynamic inverse problem of recovering two coefficients (velocity and the density) was investigated firstly in [4]. The problem was reformulated as minimizing of cost functional by gradient method. For the first time, the gradient of the cost functional with respect to both coefficients was calculated by solving direct and adjoint problems. The uniqueness of the solution was proved.

In [6] authors considered the reconstruction of three coefficients depended on space variables in the dynamical isotropic system of elasticity from two boundary measurements and proved the uniqueness and a Hölder stability using Carleman estimates in Sobolev spaces of negative order.

It was proved the local well-posedness of the 2D coefficient inverse problem for hyperbolic equation in L_2 [5]. The properties of the Frechet derivative of the operator of the inverse problem was studied. The strong convergence rate of the Landweber iteration was obtained.

It was applied the quasi-solution to 2D inverse coefficient problem for hyperbolic equation [13]. Instead of a compact set, the ball was considered where the radius is a regularization parameter. Moreover, this constant allows to estimate the strong convergence rate for many well-known gradient methods for solving coefficient inverse problems and to decrease crucially the number of iterations. The gradient of the functional was obtained using the solution of direct and adjoint problems.

The inverse problem of determining several coefficients in the scalar hyperbolic equation by data given on the part of the boundary was investigated in [12]. The Carleman estimate was obtained to prove the uniqueness and a Lipschitz stability estimate for the coefficient inverse problem.

Inverse problem of recovering several coefficients in MaxwellB's equations was investigated in [9, **?**, 11] by a finite number of measurements. For the coefficient inverse problem it was proved the Lipschitz stability estimate using the Carleman estimate.

It was presented the hybrid algorithm for robust breast ultrasound tomography, which utilizes the complementary strengths of time-of flight and diffraction tomography resulting in a direct, fast, robust and accurate high resolution method of reconstructing the sound speed through the breast [46].

The inverse problem for dynamic Maxwell's equations in an isotropic and inhomogeneous medium was investigated [42] of determining the dielectric permiabillity and conductivity from a finite number of interior measurements. Lipschitz stability estimate for the inverse problem by applying the of Carleman estimate was proved.

The continuation and coefficient inverse problem of recovering dielectric permeability and conductivity in application to ground penetrating radar was investigated in [20]. The inverse problems were reformulated as optimization problems. To minimize the cost functional gradient method was applied.

It was investigated the acceleration of the ray-based transmission reconstruction by a GPU-based implementation of the iterative numerical optimization [43].

The acoustic tomography based on the conservation laws and the system of acoustic equation of the first order was firstly proposed in [35].

The coefficient inverse problem of the recovering of the magnetic permeability and dielectric permittivity of the 3D MaxwellB's system by data of the electric field given on the part of the boundary was considered in [22]. The authors applied the Carleman estimates to get the theoretical stability.

An inverse problem of finding two coefficients in an hyperbolic acoustic equation of the second order by interior data was considered in [24]. The authors applied a Carleman technique estimates to obtain the Lipschitz stability estimates and therefore unique reconstruction of both coefficients was guaranteed. Numerical experiments of recovering both coefficients by data with noise were presented.

It was built the optimized 3D ultrasound computer tomography for improved imaging of breast cancer, realizing for the first time the full benefits of a 3D system [44].

The inverse problem for a nonlinear elastic wave equation of identifying the stored energy function of a hyperelastic material from full knowledge of the displacement field was investigated [25]. The numerical solver using the attenuated Landweber method was developed.

It was considered ultrasonic quantitative imaging method for long bones based on full-waveform inversion for the 2D first-order elastodynamics system [41]. The coefficient inverse problem was solved iteratively based on the Limited-memory BroydenвЂ“FletcherвЂ“GoldfarbвЂ“Shanno method (quasi-Newton technique).

The isotropic elastic wave equation in a bounded domain with boundary was investigated [32]. The local knowledge of the Dirichlet-to-Neumann map determined uniquely the s-wave and p-wave speed locally if there are strictly convex foliation with respect to them.

It was establish an algorithm for quantitative assessment of breast density using quantitative three-dimensional transmission ultrasound imaging and it was determined how these quantitative assessments compare with both subjective and objective mammographic assessments of breast density [26].

In [40] the gradient computation based on time reversal that dramatically reduces the memory footprint at the expense of one additional wave simulation per source was proposed for ultrasound breast tomography. It was broken the dependence on the number of measurements by using source encoding to compute stochastic gradient estimates.

It was proposed the iterative framework for CT reconstruction from transmission ultrasound data which accurately and efficiently models the strong refraction effects in imaging the female breast [47].

Note that mathematical models [47, 46, 20, 40] based on 2nd-order equations do not guarantee that the calculated solution of the forward and inverse problems is close to the exact one.

For the solution of the coefficient inverse problems gradient methods [4, 20, 16, 23] and global-convergence [15, 14, 19] are applied. We should also mention the family of Newton-type methods. However, their drawback is the solution of the additional linear inverse problem, which has to be solved on each iteration. When considering the multidimensional problems, this necessity to deal with that additional linear problem tends to become too complicated.

The numerical algorithms, based on the S. K. Godunov scheme [1] is applied for solving direct problems. Such kind of methods allows us to construct the effective numerical realization of the physical process and benefits from usage of the piecewise-smooth structure of the state variables on each time step. If we use methods based on finite approximation to solve a direct problem in the case of a piecewise smooth medium, then it is necessary to add special conditions at the interface. Such conditions are very difficult to add in the case of complex media interfaces and impossible to implement when solving the inverse problem without *apriori* information about media interfaces.

The control problem of modeling the acoustic radiation pattern of source was considered in [38]. The modelling of the acoustic radiation patterns of sources could allow to improve the resolution of acoustic tomography.

The inverse wave tomography problem of reconstructing several characteristics of the scatterer in the form of spatial distributions of sound velocity, medium density, absorption coefficient and power index of its frequency dependence, as well as the flow velocity vector was considered [61, 60].

2 Problem formulation

Let us consider the following 2D direct problem [1] as a basic model of the framework in a form of conservation laws of impulse in direction x and y and mass:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0, \tag{1}$$

$$\frac{\partial p}{\partial t} + \sigma p + \rho c^2 \nabla \cdot \mathbf{u} = F(x, y, t), \tag{2}$$

$$p|_{(x,y)\in\partial\Omega} = 0, \tag{3}$$

$$\mathbf{u}, p|_{t=0} = 0. \tag{4}$$

The equations (1)-(4) are considered in the domain $\Omega = \big\{(x, y) \in [0, L] \times [0, L]\big\}$, $t \in (0, T)$ and represents the propagation of the acoustic wave through the domain. The solution of the (1)—(4) is the velocity vector $\mathbf{u}(x, y, t) = \big(u(x, y, t), v(x, y, t)\big)$ and the acoustic pressure $p = p(x, y, t)$. The properties of the medium and described by the functions which $c(x, y)$ is the speed of the wave propagation, $\rho(x, y)$ is the density of the medium and $\sigma(x, y)$ is the coefficient of attenuation. Right-hand side of the equation (2) represents the sounding pulse, which is located in the subdomain and has the time-form of the Puzyrev wavelet (ν_0 is a frequency):

$$I(t, \nu_0) = \sin(\nu_0 t) \mathrm{e}^{-\left(\frac{\nu_0}{\pi} t\right)^2}. \tag{5}$$

Well-posedness of the direct problems for linear hyperbolic systems were investigated in [10]. The numerical simulation of nonlinear acoustic wave propagation in a liquid medium based on the Navier-Stokes equations was investigated [59].

We suppose, that the exceeded pressure, generated by the sounding wave is registered in the N receivers, each located in the subdomain Ω_k, $k = 1, \ldots, N$. The inverse problem is to recover functions $c(x, y)$, $\sigma(x, y)$ and $\rho(x, y)$ in (1)—(4) using the additional information (6) of the following form:

$$p(x, y, t) = f_k(x, y, t), \qquad (x, y) \in \Omega_k, \quad k = 1, \ldots, N. \tag{6}$$

The uniqueness and conditional stability of recovering two coefficients of a hyperbolic equation were investigated in [8, 7].

We reformulate inverse problem in the (1)—(4), (6) in operator form

$$A(\mathbf{q}) = \mathbf{f}, \qquad A : L_2((0, L) \times (0, L)) \to L_2((0, L) \times (0, L) \times (0, T)). \tag{7}$$

Here vectors $\mathbf{q}$ and $\mathbf{f}$ are as follows

$$\mathbf{q} = \mathbf{q}(x, y) = (q_1, q_2, q_3) = (c^2, \rho, \sigma),$$

$$\mathbf{f} = \mathbf{f}(x, y, t) = (f_1(x, y, t), f_2(x, y, t), \ldots, f_N(x, y, t)).$$

The properties of operator A was investigated in [5].

Let us reduce the inverse problem (1)—(4), (6) to minimization problem of the following cost functional:

$$\mathbf{J}(\mathbf{q}) = ||A(\mathbf{q}) - \mathbf{f}||^2_{L_2} = \sum_{k=1}^{N} \int_0^T \int_{\Omega_k} \big[p(x, y, t; \mathbf{q}) - f_k(x, y, t)\big]^2 dxdydt \to \min_{\mathbf{q}}. \tag{8}$$

We use the gradient-based approach to minimize the functional (8) by considering the following iteration scheme:

$$\mathbf{q}^{(n+1)} = \mathbf{q}^{(n)} - \alpha \mathbf{J}'(\mathbf{q}^{(n)}).$$

Here $\alpha \in (0, ||A||^{-2})$ is descent parameter, $\mathbf{J}'(\mathbf{q}^{(n)})$ is the gradient of the functional. Let us note [5] that

$$\mathbf{J}'(\mathbf{q}) = 2 \left[A'(\mathbf{q})\right]^* (A(\mathbf{q}) - \mathbf{f}). \tag{9}$$

Here $[A'(\mathbf{q})]^*$ is adjoint of Frechet derivative of the operator A.

The optimization of CPU-time and RAM memory for solving dynamic inverse problems using gradient-based approach was considered in [37]. It was proposed a new approach for computing a gradient in the descent method in order to use as much inverse problem data as possible on each descent iteration [30].

The gradient of the functional can be computed as follows [4, 29]. Let us introduce the adjoint problem [13, 20]:

$$\frac{\partial \Psi_1}{\partial t} + \frac{1}{\rho}\frac{\partial \Psi_3}{\partial x} = 0; \tag{10}$$

$$\frac{\partial \Psi_2}{\partial t} + \frac{1}{\rho}\frac{\partial \Psi_3}{\partial y} = 0; \tag{11}$$

$$\frac{\partial \Psi_3}{\partial t} + \sigma \Psi_3 + \rho c^2 \left(\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial y}\right) = 2\rho c^2 \sum_{k=1}^{N} \theta_{\Omega_k}(x, y)\big[p(x, y, t) - f_k(x, y, t)\big]; \tag{12}$$

$$\Psi_i(x, y, T) = 0, \qquad i = 1, 2, 3; \tag{13}$$

$$\Psi_i|_{(x,y)\in\partial\Omega} = 0, \qquad i = 1, 2, 3. \tag{14}$$

Then the gradient $\mathbf{J}'(\mathbf{q}) = (J'_{q_1}(\mathbf{q}), J'_{q_2}(\mathbf{q}), J'_{q_3}(\mathbf{q}))$ has the following form [4]:

$$J'_{q_1}(\mathbf{q})(x,y) = \int_0^T \frac{\Psi_3}{c^2(x,y)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dt; \tag{15}$$

$$J'_{q_2}(\mathbf{q})(x,y) = \int_0^T \left[-u\frac{\partial \Psi_1}{\partial t} - v\frac{\partial \Psi_2}{\partial t} + \frac{\Psi_3}{\rho(x,y)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] dt; \tag{16}$$

$$J'_{q_3}(\mathbf{q})(x,y) = \int_0^T \frac{p(x,y,t)\Psi_3(x,y,t)}{\rho(x,y)c^2(x,y)} dt. \tag{17}$$

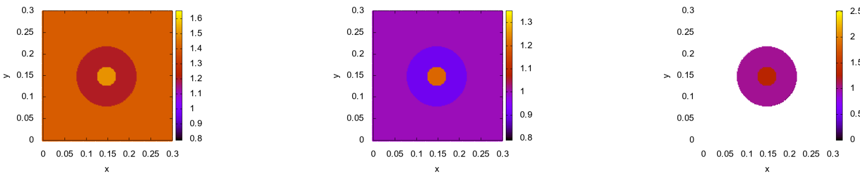
Thus, in order to make one step of the gradient descent, one has to solve the direct problem (1)—(4), using the current approximation $q_1^{(n)}$, $q_2^{(n)}$ and $q_3^{(n)}$ of the parameters, then solve the adjoint problem (10)—(14), and after that, using the solution of both problems, compute the gradient of the cost functional, using (16), (15). Since the adjoint problem (10)—(14) also has the form of the hyperbolic system we apply the Godunov-type methods to compute the solution of the adjoint problem as well.

3 Numerical results

We apply the MUSCL–Hancock method for the direct and adjoint problem solution. This method was introduced in [2] as a variant of the MUSCL scheme. This scheme is full time and space second-order accurate. The computational domain is $[0, 0.3] \times [0, 0.3]$ meters. The grid parameters are chosen as follows: $N_x = N_y = 300$, the CFL condition is equal to 0.5. As for the structure of the model, we considered the simplified version, where outer region is filled with water, and the object is part the regular tissue and part inclusion, that corresponds to the higher values of parameters. The values of the attenuation, density and the speed of sound are presented in the following table.

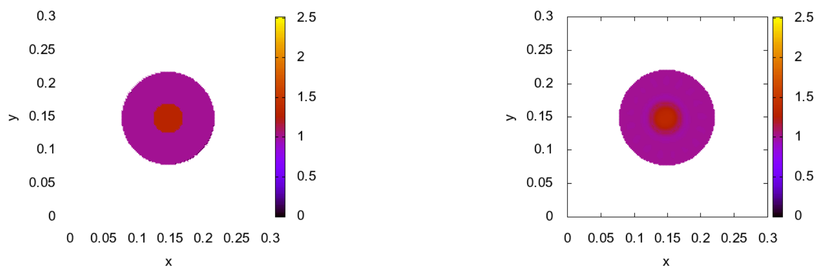
	σ	$\rho, 10^3 kg/m^3$	$c, km/s$
Outer medium(filler)	0	1.0	1.4
Regular tissue	1	0.9	1.2
Inclusion	1.5	1.2-1.3	1.5-1.6

In order to simulate the tomograph's data acquisition, we consider the system of 8 transducers (sources and receivers), placed uniformly around the object on the circle of the radius $R = 0.115$ m (the center of the circle is the center of the computational domain $x_c = 0.15, y_c = 0.15$). The size of each transducer is $r = 0.02$ m. One of the transducers (the upper one) is used as generator of the sounding wave, and the others works as the receivers (the registered data is averaged over the area of each receiver). This represents the setup of the tomograph, in which the source and the receivers are placed on the boundary of the tomograph, and the studied object is located on the inside.



$c(x, y)$ $\rho(x, y)$ $\sigma(x, y)$

FIG. 1. Test 1. The true model

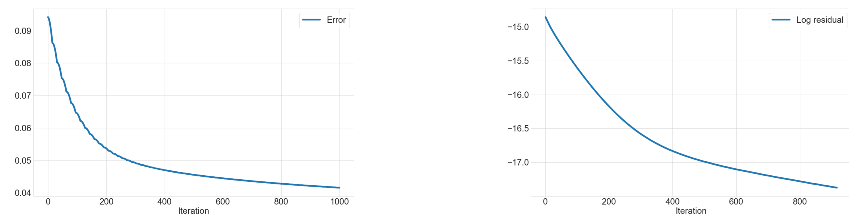


$\sigma(x, y)$ - exact $\sigma(x, y)$ - reconstructed

FIG. 2. Test 1. Reconstruction of the attenuation

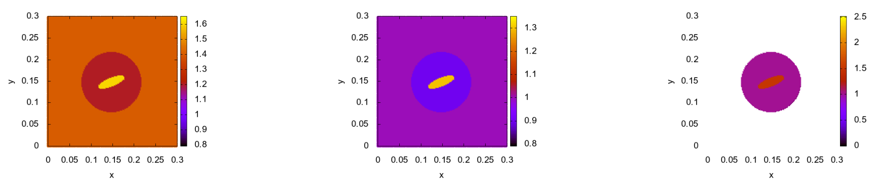
The water area outside the sources and receivers is the fictional domain, used to simulate the non-reflecting condition on the boundary of the tomograph.

We conducted two tests study the influence of the absorption on the efficiency of the inverse problem solution. The model of the first test is presented on the figure 1. During this test we considered the density and the speed of sound to be known, and considered only the problem of reconstructing the absorption coefficient. The result of the reconstruction is presented on the figure 2. The numerical solution was obtained after 1000 iterations by gradient method, where the descent parameter was chosen by trial-and-error method after several experiments. The initial guess based on the assumption, that there is irregular tissue in the object. It is shown, that numerical solution provides sufficient accuracy during the test, as the inclusion is clearly visible and the values of the absoprtion coefficient σ are close to exact ones. The figure 3 illustrates the behaviour of the error of the method and of the cost functional during the iterations. We should mention, that the even the initial values of the cost functional are relatively small. On the one hand, it depends on the basic value of the pressure during the experiments. On the other hand, it indicates that changes in the absorption coefficient has smaller influence on the residual. In order to study this influence, we considered the second series of tests. The model for the second test is presented on the figure 4. This time we considered both three parameters σ, ρ and c to be unknown. The problem, therefore, was to obtain all three coefficients by solving the inverse



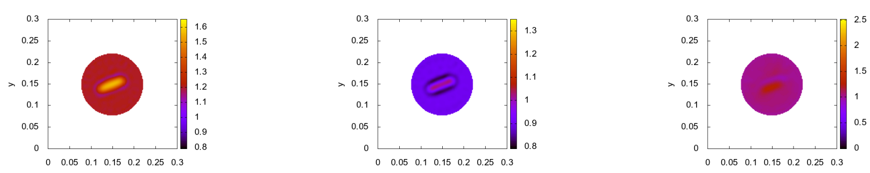
Error Residual

FIG. 3. Test 1. Behaviour of the error and the residual during the optimization



$c(x,y)$ $\rho(x,y)$ $\sigma(x,y)$

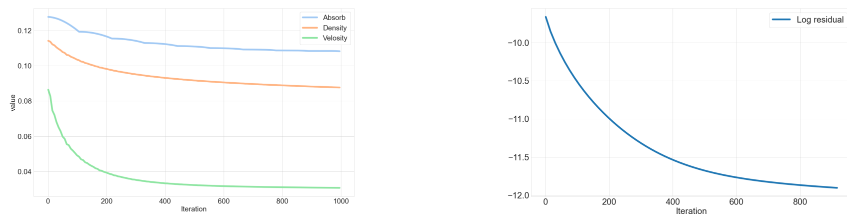
FIG. 4. Test 2. The structure of the model



$c(x,y)$ $\rho(x,y)$ $\sigma(x,y)$

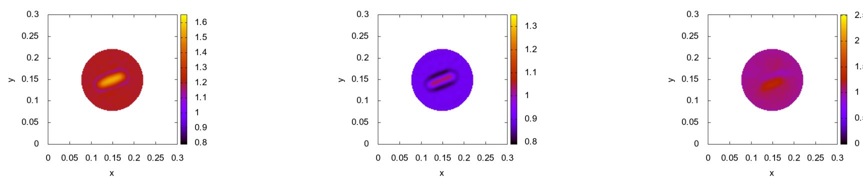
FIG. 5. Test 2. Reconstruction results

problem. The result of the reconstruction is presented on the figure 5. The numerical solution was obtained after 950 iterations by gradient method, where the descent parameter was chosen by trial-and-error method after several experiments. The initial guess based on the assumption, that there just regular tissue in the object. This time one can see the the accuracy for the each parameters is different. This is caused by the fact, that the residual (through the adjoint problem) affects the parameters with different impact. As it was shown in our previous work, such behaviour should be dealt with by adjusting the parameters of the descent independently for each component of the gradient of the functional. The figure 6 illustrates the behaviour of the error of the method and of the cost functional during the iterations. One can see, that error diminishes with different speed for every reconstructed parameter. This reflects the fact, that each parameter has a different impact on the residual. We also studied the dependence of the algorithm in case of



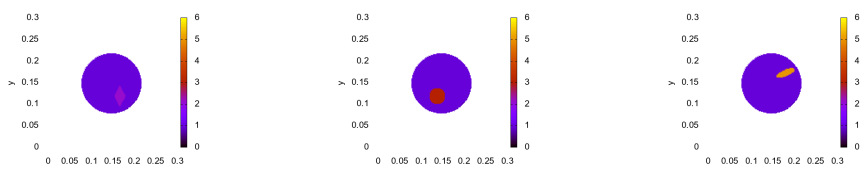
Error behaviour Residual behaviour

FIG. 6. Test 2. Behaviour of the error and the residual during the optimization



$c(x,y)$ $\rho(x,y)$ $\sigma(x,y)$

FIG. 7. Test 2. Reconstruction results for the noised data



$\sigma_1(x,y)$ $\sigma_2(x,y)$ $\sigma_3(x,y)$

FIG. 8. Test 3 - models with different absorption value inside the inclusion ($\sigma = 1.5, 3, 6$ inside the inclusion correspondingly)

exact and noisy data. We added some noise to the inverse problem data as follows:

$$f_\delta(x,y,t) = f(x,y,t)\left(1 + \alpha(x,y,t)\frac{\delta}{100}\right).$$

Here $f(x,y,t)$ are the exact data, δ is the percentage of noise level in the data, $\alpha(x,y,t)$ is a random variable uniformly distributed on the interval $(-1,1)$ for each variable x, y, and t. The last test was to study the influence on the influence of the absorption on the data, registered in the receivers. We considered three different models, illustrated by the figure 8. We have chosen three variations of the model with one inclusion. The inclusions were chosen different in form and location, but all are sufficiently large in size.

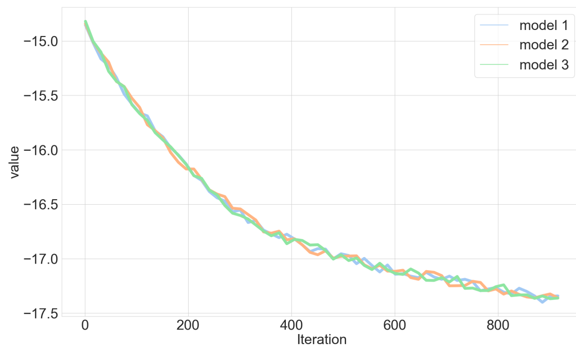


FIG. 9. Influence of the value of absorption coefficient on the residual of the functional. Blue line - first case ($\sigma = 1.5$ inside the inclusion), yellow line - second case ($\sigma = 3$ inside the inclusion), green line - third case ($\sigma = 6$ inside the inclusion)

The density and sound velocity inside the inclusions were the same, the value of absorption coefficient was chosen as $\sigma = 1.5, 3, 6$ for the first, second and third model correspondingly. Then we consider the problem of recovering the density and the speed of sound. The residual for all three cases is presented on the figure 9.

4 Concluding remarks

In this work we considered the coefficient inverse problem of recovering three coefficients (the speed of sound, density and absorption) of the first order hyperbolic system, that describes the propagation of the 2D acoustic waves in a heterogeneous medium. We used the second order MUSCL-Hancock scheme to solve the direct and adjoint problems, and applied optimization scheme to the coefficient inverse problem.

The presented numerical results illustrates the acceptable accuracy and stability of the proposed algorithm. However, the impact of the absorption on the data is relatively low, as shown by the last numerical experiment. This leads to certain difficulties when recovering all three coefficients, because one has to adjust the descent parameters for density, speed of sound and absorption independently. There are several ways to improve the proposed method in the future work, for example, one can study different variations of gradient descent, consider the different ways to measure the data to increase the impact of absorption on the data, or introduce some connection (maybe based on statistics) between density and speed of sound inside the inclusions on one hand, and absorption on the other hand.

References

[1] S.K. Godunov, A.V. Zabrodin, M.Y. Ivanov, A.N. Kraikov, G.P. Prokopov, *Numerical Solution for Multidimensional Problems of Gas Mechanics*, Nauka: Moscow, Russia, 1976.

[2] van Leer, B.: On the relation between the upwind-differencing schemes of Godunov, Engquist-Osher and Roe. SIAM J Sci Statist Comput. **1984**, 5(1), 1–20.

[3] Romanov, V.G.; Kabanikhin, S.I. *Inverse Problems for Maxwell's Equations*; VSP: Utrecht, The Netherlands, 1994.

[4] He, S.; Kabanikhin, S.I. An optimization approach to a three-dimensional acoustic inverse problem in the time domain. *J. Math. Phys.* **1995**, *36*, 4028–4043.

[5] S.I. Kabanikhin, S.I.; Scherzer, O.; Shishlenin, M.A. Iteration methods for solving a two dimensional inverse problem for a hyperbolic equation. *J. Inverse Ill-Posed Probl.* **2003**, *11*, 87–109.

[6] O. Y. Imanuvilov, V. Isakov and M. Yamamoto. An inverse problem for the dynamical Lame system with two sets of boundary data, Comm. Pure Appl. Math. 2003. 56, 1366–1382.

[7] Romanov V.G., Glushkova D.I. The problem of determining two coefficients of a hyperbolic equation. Doklady Mathematic. 2003, 67(3), pp. 386–390

[8] Glushkova D.I., Romanov V.G. A stability estimate for a solution to the problem of determination of two coefficients of a hyperbolic equation. Siberian Mathematical Journal. 2003, 44(2), pp. 250–259

[9] S. Li, An inverse problem for Maxwell's equations in bi-isotropic media, SIAM Journal on Mathematical Analysis, 37 (2005), 1027–1043.

[10] Blokhin, A.M.; Trakhinin, Y.L. *Well-Posedness of Linear Hyperbolic Problems: Theory and Applications*; Nova Publishers, 2006.

[11] S. Li and M. Yamamoto, An inverse problem for MaxwellB's equations in anisotropic media in two dimensions, Chin. Ann. Math. Ser B 28 (2007), 35–54.

[12] M. Bellassoued, D. Jellali and M. Yamamoto, Lipschitz stability in in an inverse problem for a hyperbolic equation with a finite set of boundary data, Applicable Analysis 87 (2008), 1105–1119.

[13] Kabanikhin, S.I.; Shishlenin, M.A. Quasi-solution in inverse coefficient problems. *J. Inverse Ill-Posed Probl.* **2008**, *16*, 705–713.

[14] Beilina, L.; Klibanov, M.V. Synthesis of global convergence and adaptivity for a hyperbolic coefficient inverse problem in 3D. *J. Inverse Ill-Posed Probl.* **2010**, *18*, 85–132.

[15] Xin, J.; Beilina, L.; Klibanov, M. Globally convergent numerical methods for some coefficient inverse problems. *Comput. Sci. Eng.* **2010**, *12*, 64–76.

[16] Beilina, L. Adaptive Finite Element Method for a coefficient inverse problem for the MaxwellB's system. *Appl. Anal.* **2011**, *90*, 1461–1479.

[17] Duric, N.; Littrup, P.; Li, C.; Roy, O.; Schmidt, S.; Janer, R.; Cheng, X.; Goll, J.; Rama, O.; Bey-Knight, L.; Greenway, W. Breast ultrasound tomography: Bridging the gap to clinical practice. *Proc. SPIE* **2012**, *8320*, 832000.

[18] Jirik, R.; Peterlik, I.; Ruiter, N.; Fousek, J.; Dapp, R.; Zapf, M.; Jan J. Sound-speed image reconstruction in sparse-aperture 3D ultrasound transmission tomography. *IEEE Trans. Ultrason. Ferroelectr. Freq. Control* **2012**, *59*, 254–264.

[19] Klibanov, M.V. Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems. *J. Inverse Ill-Posed Probl.* **2013**, *21*, 477–560.

[20] Kabanikhin, S.I.; Nurseitov, D.B.; Shishlenin, M.A.; Sholpanbaev, B.B. Inverse problems for the ground penetrating radar. *J. Inverse Ill-Posed Probl.* **2013**, *21*, 885–892.

[21] Burov, V.A.; Zotov, D.I.; Rumyantseva, O.D. Reconstruction of the sound velocity and absorption spatial distributions in soft biological tissue phantoms from experimental ultrasound tomography data. *Acoust. Phys.* **2015**, *61*, 231–248.

[22] L. Beilina, M. Cristofol, and K. Niinimaki, Optimization approach for the simultaneous reconstruction of the dielectric permittivity and magnetic permeability functions from limited observations, Inverse Problems and Imaging, 9 (2015), 1–25.

[23] Beilina, L.; Hosseinzadegan, S. An adaptive finite element method in reconstruction of coefficients in Maxwell's equations from limited observations. *Appl. Math.* **2016**, *61*, 253–286.

[24] L Beilina, M Cristofol, S Li, M Yamamoto. Lipschitz stability for an inverse hyperbolic problem of determining two coefficients by a finite number of observations. Inverse Problems. 2017. Vol. 34. No. 1. PP. 015001.

[25] J. Seydel, Th. Schuster. Identifying the stored energy of a hyperelastic structure by using an attenuated Landweber method. Inverse Problems. 2017. 33:12, pages 124004.

[26] Wiskin, J.; Malik, B.; Natesan, R.; Lenox, M. Quantitative assessment of breast density using transmission ultrasound tomography. *Med. Phys.* **2019**, *46*, 2610–2620, doi:10.1002/mp.13503.

[27] Klibanov, M.V. Travel time tomography with formally determined incomplete data in 3D. *Inverse Probl. Imaging* **2019**, *13*, 1367–1393.

[28] Klibanov, M.V. On the travel time tomography problem in 3D. *J. Inverse Ill-Posed Probl.* **2019**, *27*, 591–607.

[29] Kabanikhin, S.I.; Klyuchinskiy, D.V.; Novikov, N.S.; Shishlenin, M.A. Numerics of acoustical 2D tomography based on the conservation laws. *J. Inverse Ill-Posed Probl.* **2020**, *28*, 287–297.

[30] Klyuchinskiy D., Novikov N., Shishlenin M. A Modification of gradient descent method for solving coefficient inverse problem for acoustics equations. Computation, 2020, 8(3), 73.

[31] Kabanikhin, S.I.; Sabelfeld, K.K.; Novikov, N.S.; Shishlenin, M.A. Numerical solution of an inverse problem of coefficient recovering for a wave equation by a stochastic projection methods. *Monte Carlo Methods Appl.* **2015**, *21*, 189–203.

[32] P. Stefanov, G. Uhlmann, A. Vasy. (2018) Local recovery of the compressional and shear speeds from the hyperbolic DN map. Inverse Problems 34:1, pages 014003.

[33] Filatova, V.; Danilin, A.; Nosikova, V.; Pestov, L. Supercomputer Simulations of the Medical Ultrasound Tomography Problem. *Commun. Comput. Inf. Sci.* **2019**, *1063*, 297–308.

[34] Filatova V., Pestov L., Poddubskaya A. Detection of velocity and attenuation inclusions in the medical ultrasound tomography. Journal of Inverse and Ill-Posed Problems. 2021, 29(3), pp. 459–466

[35] Kulikov I.M., Novikov N.S., Shishlenin M.A. Mathematical modeling of ultrasonic wave propagation in a two-dimensional medium: direct and inverse problem. Siberian Electronic Mathematical News. 2015. Vol. 12. p. C.219–C.228.

[36] R. Klein, Th. Schuster, A. Wald. 2021. Sequential Subspace Optimization for Recovering Stored Energy Functions in Hyperelastic Materials from Time-Dependent Data. Time-dependent Problems in Imaging and Parameter Identification, pages 165–190.

[37] Klyuchinskiy, D.V., Novikov, N.S., Shishlenin, M.A. CPU-time and RAM memory optimization for solving dynamic inverse problems using gradient-based approach. Journal of Computational Physics. (2021) 439, No. 110374.

[38] Shishlenin M., Savchenko N., Novikov N., Klyuchinskiy D. Modeling of 2D Acoustic Radiation Patterns as a Control Problem. Mathematics, 2022, 10(7), 1116.

[39] Xinhua Guo, Yuanhuai Zhang, Jiabao An, Qing Zhang, Ranxu Wang, Xiantao Yu, Experimental investigation on characteristics of graphene acoustic transducers driven by electrostatic and electromagnetic forces. Ultrasonics. 2023. Vol. 127. 106857.

[40] Felix Lucka, Mailyn Perez-Liva, Bradley E. Treeby, Ben T. Cox. High resolution 3D ultrasonic breast imaging by time-domain full waveform inversion. Inverse Problems. 2022. 38. 025008

[41] Bernard S, Monteiller V, Komatitsch D, Lasaygues P 2017 Ultrasonic computed tomography based on full-waveform inversion for bone quantitative imaging. Phys. Med. Biol. 62, 7011–7035.

[42] Li, S., Yamamoto, M. An inverse problem for maxwell's equations in isotropic and non-stationary media. Applicable Analysis. 2013. 92(11), 2335–2356.

[43] Birk M, Dapp R, Ruiter N V, Becker J. GPU-based iterative transmission reconstruction in 3D ultrasound computer tomography. J. Parallel Distrib. Comput. 2014. 74 1730–1743

[44] Gemmeke H, Hopp T, Zapf M, Kaiser C and Ruiter N V 2017 3D ultrasound computer tomography: hardware setup, reconstruction methods and first clinical results Nucl. Instrum. Methods Phys. Res. A 873 59–65

[45] Guo R, Lu G, Qin B and Fei B 2018 Ultrasound imaging technologies for breast cancer detection and management: a review Ultrasound Med. Biol. 44 37–70

[46] Huthwaite P and Simonetti F 2011 High-resolution imaging without iteration: a fast and robust method for breast ultrasound tomography J. Acoust. Soc. Am. 130 1721–1734

[47] Li S, Jackowski M, Dione D P, Varslot T, Staib L H and Mueller K 2010 Refraction corrected transmission ultrasound computed tomography for application in breast imaging. Med. Phys. 37 2233–2246

[48] Malik B, Terry R, Wiskin J and Lenox M 2018 Quantitative transmission ultrasound tomography: imaging and performance characteristics Med. Phys. 45 3063–3075

[49] Manohar S and Dantuma M 2019 Current and future trends in photoacoustic breast imaging Photoacoustics 16 100134

[50] Marmot M G, Altman D G, Altman D G, Cameron D A, Dewar J A, Thompson S G and Wilcox M 2013 The benefits and harms of breast cancer screening: an independent review Br. J. Cancer 108 2205–2240

[51] Matthews T P, Wang K, Li C, Duric N and Anastasio M A 2017 Regularized dual averaging image reconstruction for full-wave ultrasound computed tomography IEEE Trans. Ultrason. Ferroelectr. Freq. Control 64 811–825

[52] Opielinski K J, Pruchnicki P, Szymanowski P, Szepieniec W K, Szweda H, Swis E, Jozwik M, Tenderenda M, Bulkowski M 2018 Multimodal ultrasound computer-assisted tomography: an approach to the recognition of breast lesions Comput. Med. Imaging Graph. 65 102–114

[53] Perez-Liva M, Herraiz J L, Udias J M, Miller E, Cox B T and Treeby B E 2017 Time domain reconstruction of sound speed and attenuation in ultrasound computed tomography using full wave inversion J. Acoust. Soc. Am. 141 1595–1604

[54] Sood R, Rositch A F, Shakoor D, Ambinder E, Pool K-L, Pollack E, Mollura D J, Mullen L A and Harvey S C 2019 Ultrasound for breast cancer detection globally: a systematic review and meta-analysis J. Glob. Oncol. 5 1–17

[55] Vourtsis A 2019 Three-dimensional automated breast ultrasound: technical aspects and first results Diagn. Intervent. Radiol. 100 579–592

[56] Wang K, Matthews T, Anis F, Li C, Duric N, Anastasio M, 2015 Waveform inversion with source encoding for breast sound speed reconstruction in ultrasound computed tomography IEEE Trans. Ultrason. Ferroelectr. Freq. Control 62 475B–93

[57] Zhu Q and Poplack S 2020 A review of optical breast imaging: multi-modality systems for breast cancer diagnosis Eur. J. Radiol. 129 109067

[58] Jiaze He, Jing Rao, Jacob D. Fleming, Hom Nath Gharti, Luan T. Nguyen, Gaines Morrison. Numerical ultrasonic full waveform inversion (FWI) for complex structures in coupled 2D solid/fluid media. Smart Materials and Structures. 2021. Vol. 30, Number 8. 085044.

[59] Kozelkov A.S., Krutyakova O.L., Kurulin V.V., Strelets D.Yu., Shishlenin M.A. The accuracy of numerical simulation of the acoustic wave propagations in a liquid medium based on Navier-Stokes equations. Siberian Electronic Mathematical Reports. (2021). Vol. 18, No. 2. Pp. 1238–1250.

[60] Shurup A.S. Numerical comparison of iterative and functional-analytical algorithms for inverse acoustic scattering. Eurasian Journal of Mathematical and Computer Applications. 2022, 10(1), pp. 79–99.

[61] Rumyantseva O.D., Shurup A.S., Zotov D.I. Possibilities for separation of scalar and vector characteristics of acoustic scatterer in tomographic polychromatic regime. Journal of Inverse and Ill-Posed Problems. 2021, 29(3), pp. 407–420.

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