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ON CONNECTION BETWEEN ROTA—BAXTER OPERATORS AND SOLUTIONS OF THE CLASSICAL YANG—BAXTER EQUATION WITH AN AD-INVARIANT SYMMETRIC PART ON GENERAL LINEAR ALGEBRA.

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Abstract: In the paper, we find the connection between solutions of the classical Yang—Baxter equation with an ad-invariant symmetric part and Rota—Baxter operators of special type on a real general linear algebra $gl_n(\mathbb{R})$. Using this connection, we classify solutions of the classical Yang—Baxter equation with an ad-invariant symmetric part on $gl_2(\mathbb{C})$ using the classification of Rota—Baxter operators of nonzero weight on $gl_2(\mathbb{C})$ and a classification of Rota—Baxter operators of weight 0 on $sl_2(\mathbb{C})$.

Keywords: Lie bialgebra, Rota—Baxter operator, classical Yang—Baxter equation, general linear Lie algebra.

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1 Introduction.

Let A be an arbitrary algebra over a field F, $\lambda \in F$. A map $R : A \to A$ is called a Rota—Baxter operator of weight λ if for all $x, y \in A$

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy). \tag{1}$$

Rota—Baxter operators for associative algebras first appeared in the paper by G. Baxter as a tool for studying integral operators that appear in the theory of probability and mathematical statistics [1]. For a long period of time, Rota Baxter operators had been intensively studied in combinatorics and probability theory mainly. For basic results and the main properties of Rota—Baxter algebras, see [2].

Independently, in 80-th Rota—Baxter operators of weight 0 on Lie algebras naturally appeared in the papers of A.A. Belavin, V.G. Drinfeld [9] and M.A. Semenov-Tyan-Shanskii [3] while studying solutions of the classical Yang—Baxter equation. It was mentioned that for any quadratic Lie algebra (L, ω), the standard technique of multilinear algebra gives a one-to-one correspondence between skew-symmetric solutions of the classical Yang—Baxter equation on L and Rota—Baxter operators $R : L \to L$ of weight 0, satisfying $R^* = -R$ (R^* is the adjoint to R operator with respect to the form ω). Recall that skew-symmetric solutions of the classical Yang—Baxter equation on a Lie algebra L induce on L the structure of a (triangular) Lie bialgebra.

In the case of Rota—Baxter operators of a nonzero weight, we have a correspondence (up to multiplication by a nonzero scalar) between structures of a factorizable Lie bialgebra (L, δ_r), $r \in L \otimes L$, on a Lie algebra L and Rota—Baxter operators of weight 1 satisfying

$$R + R^* + id = 0, \tag{2}$$

where R^* is the adjoint map with respect to some nondegenerate associative bilinear form ω (defined by r) [10],[8]. In particular, if L is a simple complex finite-dimensional Lie algebra, then any Lie bialgebra structure on L is either triangular or factorizable, that is, defined by a Rota—Baxter operator of a special type (see [11]). If L is a real simple finite-dimensional Lie algebra, then there may be a structure of a coboundary Lie bialgebra on L that is not factorizable, but becomes factorizable in the complexification $L \otimes_{\mathbb{R}} \mathbb{C}$ of the algebra L (such bialgebra structures are called almost-factorizable) [12],[13]. Note that if L is not simple, then the connection between Rota—Baxter operators of nonzero weight and solutions of the classical Yang—Baxter equation is not straightforward (see [5]).

It is worth noting that for many varieties of algebras (associative, Jordan, alternative ect.) all structures of corresponding bialgebras on semisimple finite-dimensional algebras are triangular (since they are unital, see, for example, [19] for Jordan algebras). This means that Rota—Baxter operators satisfying (2) do not seem to be interesting in these varieties (it is known that there are no Rota—Baxter operators of weight 1 on $M_n(F)$ satisfying (2)).

There is a standard method for classification of skew-symmetric solutions of a classical Yang—Baxter equation on a given algebra A (of an arbitrary variety): it is known that these solutions are in one-to-one correspondence with pairs (B, ω), where B is a subalgebra in A and ω is a symplectic form on B (see [9]). At the same time, in the case of simple Lie algebras, there is a description of factorizable Lie bialgebra structures that uses so-called admissible triples $(\Gamma_1, \Gamma_2, \tau)$, some additional structure consisting of Γ_1 and Γ_2, two subsets of the set of simple roots Γ and a map $\tau : \Gamma_1 \to \Gamma_2$ satisfying some compatibility conditions (see [10], [18]). The description says that (up to the choice of a Cartan subalgebra) there is a correspondence between structures of factorizable Lie bialgebra on a simple complex Lie algebra $\mathfrak{g}$ and admissible triples.

If $\mathfrak{g} = \mathfrak{g}_0 \oplus Ft$ is a reductive Lie algebra ($\mathfrak{g}_0$ is a semisimple Lie algebra, $t \in Z(\mathfrak{g})$), then a different approach to the description of Lie bialgebra structures on $\mathfrak{g}$ was suggested in [4]. It was proved that any Lie bialgebra structure on $\mathfrak{g}$ is coboundary and has a form

$$\delta(x) = \delta_0(x) + [H, x] \wedge t,$$

for all $x \in \mathfrak{g}_0$ and $\delta(t) = 0$. Here $\delta_0 : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a Lie bialgebra structure on $\mathfrak{g}_0$ and $H \in ker(\delta_0)$. Note that the condition $H \in ker(\delta_0)$ implies that ad_H is at the same time a derivation and a coderivation of the bialgebra $(\mathfrak{g}, \delta_0)$. Thus, at a starting point, in order to obtain the classification one need the classification of Lie bialgebra structures on $\mathfrak{g}_0$. Using this technique, in [4] it was given the classification of Lie bialgebra structures on $gl_2(\mathbb{R})$) (up to the action of $\mathrm{Aut}(\mathfrak{g})$).

However, there is no standard method for classification of all structures of quasitriangular bialgebras that may be used for an arbitrary variety of algebras. For example, if M is a simple finite-dimensional complex Malcev algebra, the classification of quasitriangular Malcev bialgebra structures on M from [7] was obtained by considering some specific information concerning the classical double (Drinfeld's double) $M \oplus M^*$.

Note that conjugated tensors induce structures of isomorphic bialgebras but inverse is not true: isomorphic coboundary (or quasitriangular) bialgebra structures on an algebra A may be induced by non-conjugated elements of $A \otimes A$, that is, the problem of classification of non-skew-symmetric solutions of CYBE is more general.

The main aim of the paper is to suggest a new approach to the problem of classification of solutions of the classical Yang—Baxter equation with an ad-invariant symmetric part (skew-symmetric or not-skew-symmetric). In recent years, Rota—Baxter operators on many important classes of algebras have been described ([15],[16],[17], ect.). Usually, the description is made up to the action of the group of automorphisms. The natural question is, if we can use these results to classify solutions of the classical Yang—Baxter equation on these classes of Lie (Malcev, ect.) algebras? Unfortunately, if the description of Rota—Baxter operators was made up to an automorphism, then we can't use it directly since conjugate operators do not necessarily give conjugate tensors. In the current paper, we first obtain a correspondence between Rota—Baxter operators of special type an solutions of the classical Yang—Baxter equation with nonzero ad-invariant symmetric part on a complex or real general linear algebra $gl_n(F)$ ($F = \mathbb{R}$,$\mathbb{C}$). Then, we use this result, the classification of the Rota—Baxter operators on $gl_2(\mathbb{C})$ obtained in [6] and the classification of Rota—Baxter operators of weight 0 on $sl_2(\mathbb{C})$ obtained in [21] to classify (up to the action of $\mathrm{Aut}(gl_2(\mathbb{C}))$ and the multiplication by a nonzero scalar) solutions of CYBE on $gl_2(\mathbb{C})$ with an ad-invariant symmetric part.

2 Motivation and preliminary results.

Let F be a field. Given a vector space V over F, denote by $V \otimes V$ its tensor square over F. Define the linear mapping τ on $V \otimes V$ by $\tau(\sum_i a_i \otimes b_i) = \sum_i b_i \otimes a_i$. We will identify the subspace of skew-symmetric tensors (that is, tensors $r \in V \otimes V$ satisfying $\tau(r) = -r$) with the exterior product $V \wedge V$, that is, for all $x, y \in V$ put

$$x \wedge y := x \otimes y - y \otimes x.$$

Let L be a Lie algebra with a product $[\cdot,\cdot]$. A Lie algebra L acts on $L^{\otimes n}$ by

$$[x_1 \otimes x_2 \otimes \ldots \otimes x_n, y] = \sum_i x_1 \otimes \ldots \otimes [x_i, y] \otimes \ldots \otimes x_n$$

for all $x_i, y \in L$. Note then for all $x \in L$

$$[L \wedge L, x] \subset L \wedge L.$$

Definition 1. *An element $r \in L^{\otimes n}$ is called L-invariant (or ad-invariant) if $[r, y] = 0$ for all $y \in L$.*

Definition 2. *A bilinear symmetric form ω on a Lie algebra L is called invariant if $\omega([a, b], c) = \omega(a, [b, c])$ for all $a, b, c \in L$.*

Definition 3. *Let L be a Lie algebra and ω be an symmetric invariant non-degenerate form on L. Then the pair (L, ω) is called a quadratic Lie algebra.*

Given a quadratic Lie algebra (L, ω), for every element $r = \sum_i a_i \otimes b_i \in L \otimes L$ we may define a linear map $R : L \to L$ as

$$R(a) = \sum_i \omega(a_i, a) b_i, \tag{3}$$

$a \in L$. By $R^* : L \to L$ define the dual map with respect to the form ω:

$$\omega(R(a), b) = \omega(a, R^*(b))$$

for all $a, b \in L$.

Definition 4. [14] *Let L be a Lie algebra with a comultiplication $\delta : L \to L \wedge L$. The pair (L, δ) is called a Lie bialgebra if and only if (L, δ) is a Lie coalgebra and δ is a 1-cocycle, i.e., it satisfies*

$$\delta([a,b]) = [\delta(a), b] + [a, \delta(b)] =$$
$$= \sum([a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] + [a, b_{(1)}] \otimes b_{(2)} + b_{(1)} \otimes [a, b_{(2)}]), \quad (4)$$

for all $a, b \in L$. Here we use the Sweedler notation: for any $x \in L$ put $\delta(x) = \sum x_{(1)} \otimes x_{(2)}$

There is an important type of Lie bialgebras. Let L be a Lie algebra and $r = \sum\limits_i a_i \otimes b_i \in L \otimes L$. Define a comultiplication δ_r on L by

$$\delta_r(a) = [r, a] = \sum_i [a_i, a] \otimes b_i + a_i \otimes [b_i, a]$$

for all $a \in L$. It is easy to see that δ_r is a 1-cocycle.

The dual algebra L^* of the coalgebra (L, δ_r) is anticommutative if and only if $r + \tau(r)$ is L-invariant. Also, L^* satisfies the Jacobi identity if and only if the element $C_L(r)$, defined as

$$C_L(r) = \sum_{ij} [a_i, a_j] \otimes b_i \otimes b_j - a_i \otimes [a_j, b_i] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j],$$

is L-invariant.

Definition 5. *We say that an element $r = \sum_i a_i \otimes b_i \in L \otimes L$ is a solution of the classical Yang—Baxter equation (CYBE) on L if*

$$C_L(r) = 0. \quad (5)$$

A solution $r \in L \otimes L$ of CYBE is called skew-symmetric, if $r \in L \wedge L$, i.e., $r + \tau(r) = 0$.

Remark 1. Let $r = \sum a_i \otimes b_i \in L \otimes L$ be a solution of CYBE and $\varphi \in \mathrm{Aut}(L)$, then $r_1 = \sum \varphi(a_i) \otimes \varphi(b_i)$ is also a solution of CYBE on L. In this case, we will say that tensors r and r_1 are conjugate. Moreover, if the symmetric part of r is L-invariant, then so is the symmetric part of r_1. Therefore, it is possible to find solutions of CYBE up to the action of $\mathrm{Aut}(L)$.

If $r \in L \wedge L$ and r is a solution of CYBE, then (L, δ_r) is said to be a *triangular* Lie bialgebra. If $r + \tau(r) \in L \otimes L$ is a nonzero L-invariant element and r is a solution of CYBE, then (L, δ_r) is called a *quasitriangular* Lie bialgebra. Triangular and quasitriangular Lie bialgebras play an important role since they lead to solutions of the quantum Yang—Baxter equation [22].

It is known that an element $r \in L \otimes L$ is a skew-symmetric solution of CYBE on a quadratic Lie algebra L if and only if the corresponding map $R : L \to L$ is a Rota—Baxter operator of weight 0 satisfying $R + R^* = 0$ [3].

If L is a semisimple Lie algebra over a field of characteristic 0, then any derivation $D : L \to M$ of L into any L-bimodule M is inner, that is, there is $m \in M$ such that $D(x) = [x, m]$ for any $x \in L$. In particular, any Lie bialgebra structure δ on L is induced by an element $r \in L \otimes L$: $\delta = \delta_r$. Similar result for reductive Lie algebras was proved in [4].

Let (L, ω) be a quadratic Lie algebra over an arbitrary field F and $r = \sum a_i \otimes b_i \in L \otimes L$. Let R be a linear map defined as in (3) and R^* be the adjoint map with respect to the form ω. In what follows, we will need the following Statement 1 and Theorem 1 from [5]. These results give the connection between solutions of CYBE and Rota—Baxter operators on L.

Statement 1. *The symmetric part $r + \tau(r)$ of r is L-invariant if and only if for all $a, b \in L$*

$$R([a, b]) + R^*([a, b]) = [R(a) + R^*(a), b]. \tag{6}$$

Theorem 1. *If r is a solution of the classical Yang—Baxter equation on L then R is a Rota—Baxter operator of weight λ if and only if for all $a, b \in L$:*

$$[R(a), b] + [R^*(a), b] + \lambda[a, b] \in ker(R). \tag{7}$$

Conversely: let $R : L \to L$ be a Rota—Baxter operator of weight λ and let $r \in L \otimes L$ be the corresponding to the map R tensor, that is, $R(a) = \sum_i \omega(a_i, a)b_i$. Then r is a solution of the classical Yang—Baxter equation if and only if R satisfies (7).

3 Connection between solutions of the CYBE and Rota—Baxter operators on $gl_n(\mathbb{R})$.

In this section, all vector spaces are assumed to be over a field F, where $F = \mathbb{R}$ or $F = \mathbb{C}$. Let $M_n(F)$ be the matrix algebra of order n over F with the multiplication xy. The multiplication in the general linear algebra $gl_n(F) = M_n(F)^{(-)}$ we will denote by $[\cdot, \cdot]$:

$$[x, y] = xy - yx,$$

$x, y \in gl_n(F)$. Recall, that $gl_2(F)$ contains a nontrivial center spanned by the identity matrix E and is not a semisimple Lie algebra. We will also consider $sl_n(F) = \{x \in gl_n(F) | \mathrm{tr}(x) = 0\}$ as a Lie subalgebra in $gl_n(F)$. Then $gl_n(F) = F\mathrm{E} \oplus sl_n(F)$, where E is the identity matrix. Note that for any $\varphi \in \mathrm{Aut}(gl_n(F))$: $\varphi(\mathrm{E}) = \theta\mathrm{E}$, for some $\theta \in F$, $\theta \neq 0$ and $\varphi(sl_n(F)) = sl_n(F)$.

We will consider $gl_n(F)$ as a quadratic Lie algebra with the trace form ω:

$$\omega(x, y) = \mathrm{tr}(xy)$$

Theorem 2. *An element $r \in gl_n(F) \otimes gl_n(F)$ is a solution of CYBE with $gl_n(F)$-invariant even part $r + \tau(r)$ if and only if the corresponding map R defined by (3) is a Rota—Baxter operator of weight λ satisfying*

$$R(x) + R^*(x) + \lambda id = 0, \tag{8}$$

and for some $\alpha \in F$

$$R(\mathrm{E}) + R^*(\mathrm{E}) + \lambda\mathrm{E} = \alpha\mathrm{E}. \tag{9}$$

Proof. If a Rota—Baxter operator R of weight λ satisfies (8) and (9), then by Statement 1 and Theorem 1, the corresponding tensor $r \in gl_n(F) \otimes gl_n(F)$ is a solution of the classical Yang—Baxter equation with $gl_n(F)$-invariant symmetric part.

Let r be a solution of the classical Yang—Baxter equation with $gl_n(F)$-invariant symmetric part. For any $\lambda \in F$, consider a map $\theta_\lambda : gl_n(F) \to gl_n(F)$ defined as

$$\theta_\lambda(x) = R(x) + R^*(x) + \lambda x$$

for any $x \in gl_n(F)$.

Consider a set

$$I_\lambda = \{\theta_\lambda(x) |\ x \in [gl_n(F), gl_n(F)]\}.$$

Take an arbitrary $\lambda \in F$. From Statement 1, it follows that the map θ_λ satisfies

$$\theta_\lambda([x, y]) = [\theta_\lambda(x), y]$$

for all $x, y \in gl_n(F)$. In other words, θ_λ belongs to the centralizer of $gl_n(F)$. In particular, I_λ is an ideal in $gl_n(F)$ for any λ. Moreover, $I_\lambda \subset [gl_n(F), gl_n(F)] = sl_n(F)$ (as consequence, $sl_n(F)$ is θ_λ-invariant). Since $sl_n(F)$ is simple, we have two possibilities: $I_\lambda = 0$ or $I_\lambda = sl_n(F)$. We want to prove that there exists a unique $\alpha \in F$ such that $I_\alpha = 0$. The uniqueness is straightforward: if $I_{\alpha_1} = I_{\alpha_2} = 0$, then for any $x \in sl_n(F)$:

$$R(x) + R^*(x) + \alpha_1 x = R(x) + R^*(x) + \alpha_2 x$$

that is not possible if $\alpha_1 \neq \alpha_2$.

Consider the case when $F = \mathbb{R}$. It is known that the complexification of $sl_n(\mathbb{R})$ is equal to $sl_n(\mathbb{C})$, the simple complex Lie algebra (that is, $sl_n(\mathbb{R})$ is an absolutely simple real Lie algebra). From [20] it follows that any centralizer of $sl_n(\mathbb{R})$ is a scalar map. If $F = \mathbb{C}$, this result follows from the Schur's lemma. Therefore, the restriction of θ_λ to $sl_n(F)$ is equal to $\gamma\mathrm{id}$ for some $\gamma \in F$. It means that $I_{\lambda-\gamma} = \theta_{\lambda-\gamma}(sl_n(F)) = 0$.

Take the scalar $\lambda \in F$ such that $I_\lambda = 0$. From (7) it follows that I_λ is R-invariant. Now we can use Theorem 2 form [5] to get that R and R^* are Rota—Baxter operators of weight λ on the quotient algebra $gl_n(F)/I_\lambda = gl_n(F)$.

By the definition of I_λ, the condition (8) holds. Finally, (9) follows from (6) and the fact that the center of $gl_2(F)$ is spanned by E. □

Remark 2. Apart from the case of a simple complex Lie algebra, here we can't say that

$$R + R^* + \lambda id = 0.$$

In Theorem 2, we proved that for all $x \in sl_n(\mathbb{C})$: $\theta_\lambda(x) = R(x)+R^*(x)+\lambda x = 0$. But $\theta_\lambda(\mathrm{E}) \neq 0$ in general, as we will see in the next section.

In what follows, we will classify solutions of CYBE on $sl_2(\mathbb{C})$ using the results of Theorem 2. For this, we need to consider two cases: the case of a nonzero weight (in is enough to consider weight 1) and the case of weight zero.

4 Classification of solutions of CYBE with an ad-invariant symmetric part on $gl_2(\mathbb{C})$, the case of weight 1.

In this section, we will classify solutions of the CYBE with an ad-invariant symmetric part on $gl_2(\mathbb{C})$ such that the corresponding map is a Rota—Baxter operator of weight 1. We will use the classification of all Rota—Baxter operators of weight 1 on $gl_2(\mathbb{C})$ obtained in [6]. The description was made up to the conjugation with automorphisms from $\mathrm{Aut}(gl_2(\mathbb{C}))$.

Unfortunately, we can't use the result from [6] directly to describe all solutions of the classical Yang—Baxter equation on $gl_2(\mathbb{C})$. Indeed, let R be a Rota—Baxter operator on $gl_2(\mathbb{C})$, $r = \sum a_i \otimes b_i$ be the corresponding tensor and φ be an automorphism from $\mathrm{Aut}(gl_2(\mathbb{C}))$. Consider $R_1 = \varphi^{-1} \circ R \circ \varphi$. Then the corresponding to R_1 tensor is the following:

$$r_1 = \sum \varphi^*(a_i) \otimes \varphi^{-1}(b_i).$$

In other words, tensors r and r_1 are not necessarily conjugate by an automorphism from $\mathrm{Aut}(gl_2(\mathbb{C}))$. There may be a situation when r is a solution of CYBE while r_1 is not a solution.

Moreover, given a Rota—Baxter operator R of weight 1 satisfying (8) and (9), the conjugate operator $\varphi \circ R \circ \varphi^{-1}$ not necessarily satisfies (8) and (9).

Nevertheless, we have the following

Proposition 1. *If φ is an automorphism of $M_2(\mathbb{C})$ (as an associative algebra), then the dual map φ^* satisfies $\varphi^* = \varphi^{-1}$. Thus, if $R : gl_2(\mathbb{C}) \to gl_2(\mathbb{C})$ is a linear map, φ is an automorphism of $M_2(\mathbb{C})$ and $R_1 = \varphi^{-1} \circ R \circ \varphi$, then corresponding tensors r and r_1 (to R and R_1 respectively) are conjugate:*

$$r_1 = (\varphi^{-1} \otimes \varphi^{-1})r.$$

Proof. Indeed, if $\varphi \in \mathrm{Aut}(M_2(\mathbb{C}))$, then

$$\omega(\varphi(x), \varphi(y)) = \mathrm{tr}(\varphi(x)\varphi(y)) = \mathrm{tr}(\varphi(xy)) = \mathrm{tr}(xy) = \omega(x, y).$$

That means that $\varphi^* = \varphi^{-1}$. □

Definition 6. *For any $\theta \in \mathbb{C}$, $\theta \neq 0$, we can define an automorphism ψ_θ of $gl_2(\mathbb{C})$ as follows:*

$$\psi_\theta(\mathrm{E}) = \theta \mathrm{E},\ \ \psi_\theta(a) = a \tag{10}$$

for any a satisfying $\mathrm{tr}(a) = 0$.

Remark 3. Since $gl_2(\mathbb{C}) = sl_2(\mathbb{C}) \oplus \mathbb{C}\mathrm{E}$ is a split null extension of the algebra $sl_2(\mathbb{C}) = [sl_2(\mathbb{C}), sl_2(\mathbb{C})]$, the group of automorphisms $Aut(gl_2(\mathbb{C}))$ is isomorphisc to the direct product of $Aut(sl_2(\mathbb{C}))$ and the multiplicative group of the field $\mathbb{C}$: $Aut(gl_2(\mathbb{C})) = Aut(sl_2(\mathbb{C})) \times \mathbb{C}^*$. This means that for any $\varphi \in \mathrm{Aut}(gl_2(\mathbb{C}))$, there are $0 \neq \theta \in \mathbb{C}$ and $\phi \in \mathrm{Aut}(M_2(\mathbb{C}))$ such that

$$\varphi = \psi_\theta \circ \phi = \phi \circ \psi_\theta.$$

Let e_{ij} $(i, j = 1, 2)$ be the usual matrix unit, $h = e_{11} - e_{22}$. In what follows, we will take a set $\{\mathrm{E}, h, e_{12}, e_{21}\}$ as a basis of $gl_2(\mathbb{C})$.

Consider the description of Rota—Baxter operators of weight 1 on $gl_2(\mathbb{C})$ modulo the action of the group $\mathrm{Aut}(M_2(\mathbb{C}))$. For this, we need to take the representatives R of orbits from [6, Theorem 1], then take $0 \neq \theta \in \mathbb{C}$ and consider the action $\psi_\theta^{-1} \circ R \circ \psi_\theta$. We get the following

Theorem 3. *Let R be a Rota—Baxter operator of weight 1 on $gl_2(\mathbb{C})$. Then there is a map $\psi \in Aut(M_2(\mathbb{C}))$ such that $R = \psi^{-1} \circ R_1 \circ \psi$, where and R_1 is one of the operators below*

$$1.\ R(\mathrm{E}) = \lambda\mathrm{E} + \theta e_{12},\ R(h) = R(e_{12}) = R(e_{21}) = 0;$$

$$2.\ R(\mathrm{E}) = \lambda\mathrm{E} + \theta e_{12},\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21};$$

$$3.\ R(\mathrm{E}) = \lambda\mathrm{E} + \theta h,\ R(h) = 0,\ R(e_{12}) = R(e_{21}) = 0,\ \lambda \in \mathbb{C};$$

$$4.\ R(\mathrm{E}) = \lambda\mathrm{E} + \theta h,\ R(h) = -h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = -e_{21},\ \lambda \in \mathbb{C};$$

$$5.\ R(\mathrm{E}) = \lambda\mathrm{E} + \theta h,\ R(h) = \alpha_1\mathrm{E} + \alpha_2 h,\qquad R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda, \alpha_i \in \mathbb{C};$$

$$6.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = 0,\ R(e_{12}) = -e_{12} + th;\ R(e_{21}) = 0,\ t \in \{0, 1\};$$

$$7.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = R(e_{21} = 0,\ R(e_{12}) = -e_{12} + th + \theta E,\ t \in \{0, 1\};$$

$$8.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = \theta E,\ R(e_{12}) = -e_{12} + h + \alpha\mathrm{E},\ R(e_{21}) = 0,\ \alpha \in \mathbb{C};$$

$$9.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = \theta E,\ R(e_{12}) = -e_{12} + \theta\mathrm{E};\ R(e_{21}) = 0;$$

$$10.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th,\ R(e_{21}) = 0,\ R(e_{12}) = -e_{12},\ t \in \mathbb{C},\ t \neq 0;$$

$$11.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th + \theta\mathrm{E},\ R(e_{21}) = 0,\ R(e_{12}) = -e_{12},\ 0 \neq t \in \mathbb{C};$$

$$12.\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = -h + \alpha\mathrm{E},\ R(e_{21}) = \theta\mathrm{E},\ R(e_{12}) = -e_{12},\ \alpha \in \mathbb{C};$$

$$13.\ R(\mathrm{E}) = \lambda\mathrm{E}, R(h) = th,\ R(e_{12}) = te_{12},\ R(e_{21}) = te_{21},\ t \in \{0, -1\},$$

where $\lambda, \theta \in \mathbb{C}$, $\theta \neq 0$.

Remark 4. Here, different scalars θ not necessarily give us different orbits with respect to $\mathrm{Aut}(M_2(\mathbb{C}))$. For example, if R is a map of type 1, then it is possible to take $\theta = 1$ since in this particular case, the conjugation of R by ψ_θ is equal to the conjugation of R by φ_A, where $\varphi_A(x) = AxA^{-1}$ for every $x \in M_2(\mathbb{C})$ and $A = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$. However, for our purposes, it is enough to consider such a rough description.

Maps that lie in the same orbit (with respect to $\mathrm{Aut}(M_2(\mathbb{C}))$) from Theorem 3 correspond to isomorphic tensors. Note that a map R satisfies (6) or (7) if and only if for any $\varphi \in \mathrm{Aut}(M_2(\mathbb{C}))$, the map $\varphi^{-1} \circ R \circ \varphi$ satisfies the same conditions. Thus, it is enough to consider one representative from every orbit in Theorem 3.

Let $R : gl_2(\mathbb{C}) \to gl_2(\mathbb{C})$ be a Rota—Baxter operator of weight 1 and $r = \sum a_i \otimes b_i \in gl_2(\mathbb{C}) \otimes gl_2(\mathbb{C})$. From Statement 1, it follows that if $r + \tau(r)$

is $gl_2(\mathbb{C})$-invariant, then

$$R(\mathrm{E}) + R^*(\mathrm{E}) = \gamma\mathrm{E} \tag{11}$$

for some $\gamma \in \mathbb{C}$.

Proposition 2. *In Theorem 3, only operators of type 5 (with $\alpha_1 = -\theta$), 6, 10 or 13 satisfy* (11).

Proof. Consider E, $h = e_{11} - e_{22}$, e_{12} and e_{21} as a basis of $gl_2(\mathbb{C})$. To check the condition (11), we need to compute $R^*(\mathrm{E})$. For this, we need to find $v \in gl_2(\mathbb{C})$ such that $\mathrm{tr}(\mathrm{E}R(v)) \neq 0$.

Suppose that R lies in an orbit of type 1. Then $\mathrm{tr}(\mathrm{E}R(v)) \neq 0$ if and only if $v = \gamma\mathrm{E}$ ($\gamma \neq 0$). Therefore, $R^*(\mathrm{E}) = \lambda\mathrm{E}$ and

$$R(\mathrm{E}) + R^*(\mathrm{E}) = 2\lambda\mathrm{E} + \theta e_{12},\ \theta \neq 0.$$

Thus, R doesn't satisfy (11). Using similar arguments, we get that operators of types 2,3,4 do not satisfy (11).

Consider the type 7. In this case, $R^*(\mathrm{E}) = \lambda\mathrm{E} + 2\theta e_{21}$. Thus,

$$R(\mathrm{E}) + R^*(\mathrm{E}) = 2\lambda\mathrm{E} + 2\theta e_{21} \neq \gamma\mathrm{E}.$$

Similar arguments can be used to show that operators that are conjugate to operators of types 8, 9,11,12 do not satisfy (11).

It remains to consider types 5, 6, 10, 13.

Suppose that R is conjugate to an operator of type 5. Then $R^*(\mathrm{E}) = \lambda\mathrm{E} + \alpha_1 h$. Thus, $R(\mathrm{E}) + R^*(\mathrm{E}) = \gamma\mathrm{E}$ if and only if $\alpha_1 = -\theta$.

In 6, 10 and 13 it is easy to see that $R^*(\mathrm{E}) = \lambda\mathrm{E}$. Thus, in this case, R satisfies (11) for any values of parameters. □

From Proposition 2, it follows that it is enough to consider the following operators:

$$(R1).\ R(\mathrm{E}) = \lambda\mathrm{E} + \theta h,\ R(h) = -\theta\mathrm{E} + \alpha_2 h,$$
$$R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda, \alpha_2 \in \mathbb{C},\ \theta \neq 0;$$
$$(R2).\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = R(e_{21}) = 0,\ R(e_{12}) = th - e_{12},\ \lambda \in \mathbb{C},\ t \in \{0, 1\};$$
$$(R3).\ R(\mathrm{E}) = \lambda\mathrm{E},\ R(h) = th,\ R(e_{21}) = 0,\ R(e_{12}) = -e_{12},\ t, \lambda \in \mathbb{C},\ t \neq 0;$$
$$(R4).\ R(\mathrm{E}) = \lambda\mathrm{E}, R(x) = tx,\ x \in sl_2(\mathbb{C}),\ \lambda \in \mathbb{C},\ t \in \{0, -1\}.$$

We will consider operators (R1)-(R4) consequently.

Proposition 3. *Let R be the Rota—Baxter operator of type (R1) or (R3). Then R satisfies* (6) *if and only if $\alpha_2 = -\frac{1}{2}$. In this case, for every $a \in sl_2(\mathbb{C})$ we have $R(a) + R^*(a) + a = 0$. Therefore, if $\alpha_2 = -\frac{1}{2}$, then R also satisfies* (7).

Proof. Direct computations show that $R^*(\mathrm{E}) = \lambda\mathrm{E} - \theta h$, $R(h) = \theta\mathrm{E} + \alpha_2 h$, $R^*(e_{12}) = 0$ and $R^*(e_{21}) = -e_{21}$.

Suppose that R satisfies (6). We have

$$R([h, e_{12}]) + R^*([h, e_{12}]) = 2R(e_{12}) + 2R^*(e_{12}) = -2e_{12}.$$

On the other hand,

$$[R(h), e_{12}] + [R^*(h), e_{12}] = 2\alpha_2 e_{12} + 2\alpha_2 e_{12} = 4\alpha_2 e_{12}.$$

Therefore, $\alpha_2 = -\frac{1}{2}$.

Conversely, let $\alpha_2 = -\frac{1}{2}$. It is easy to see that in this case, for all $a \in sl_2(\mathbb{C})$ we have

$$R(a) + R^*(a) + a = 0.$$

This means that equations (6) and (7) are true for all $a, b \in sl_2(\mathbb{C})$. Since $R(\mathrm{E}) + R^*(\mathrm{E}) \in Z(gl_2(\mathbb{C}))$, it follows that equations (6) and (7) are true for all $a, b \in gl_2(\mathbb{C})$. □

Proposition 4. *Let R be the Rota—Baxter operator of type (R2). Then R does not satisfy (6) for any t and λ.*

Proof. For R we have:

$$R^*(\mathrm{E}) = \lambda\mathrm{E},\ R^*(h) = te_{21},\ R^*(e_{12}) = 0,\ R^*(e_{21}) = -e_{21},\ t \in \{0, 1\}.$$

Then,

$$R([h, e_{12}]) + R^*([h, e_{12}]) = -2e_{12}.$$

On the other hand,

$$[R(h), e_{12}] + [R^*(h), e_{12}] = -th.$$

Thus, R does not satisfy (6). □

Proposition 5. *Let R be the Rota—Baxter operator of type (R4). Then for any $t \in \{0, -1\}$, R satisfies (6). Moreover, R satisfies (7) if and only if $t = 0$.*

Proof. The first statement is obvious since the restriction of R on $sl_2(\mathbb{C})$ is equal to $t \cdot \mathrm{id}$.

If $t = 0$, then $R(sl_2(\mathbb{C})) = 0$. Since $[gl_2(\mathbb{C}), gl_2(\mathbb{C})] = sl_2(\mathbb{C})$, R satisfies (7).

If $t = -1$, then direct computations show that

$$R(R([h, e_{12}] + R^*([h, e_{12}]) + [h, e_{12}]) = -2R(e_{12}) \neq 0.$$

Thus, if $t = -1$, then R does not satisfy (7). □

Now we are ready to prove the main result of the section:

Theorem 4. *Let $r \in gl_2(\mathbb{C}) \otimes gl_2(\mathbb{C})$ be a solution of CYBE such that $r + \tau(r)$ is $gl_2(\mathbb{C})$-invariant and the corresponding map is a Rota—Baxter operator of weight 1. Then, up to the action of $Aut(gl_2(\mathbb{C}))$, r is equal to the one of the following:*

$$r = \lambda\mathrm{E} \otimes \mathrm{E} + \mathrm{E} \otimes h - h \otimes \mathrm{E} - \frac{1}{4} h \otimes h - e_{21} \otimes e_{12},\ \lambda \in \mathbb{C}; \quad (12)$$

$$r = \lambda\mathrm{E} \otimes \mathrm{E} - \frac{1}{4} h \otimes h - e_{21} \otimes e_{12},\ \lambda \in \{0, 1\}, \quad (13)$$

$$r = \lambda\mathrm{E} \otimes \mathrm{E}, \lambda \in \{0, 1\}. \quad (14)$$

Proof. Let $r \in gl_2(\mathbb{C}) \otimes gl_2(\mathbb{C})$ be a solution of CYBE such that $r + \tau(r)$ is $gl_2(\mathbb{C})$-invariant. Suppose that the corresponding map R defined as (3) is a Rota—Baxter operator of weight 1. Thus, R satisfies (6) and (7) (as a Rota-Baxter operator of weight 1). From Propositions 2-5 it follows that up to a conjugation with automorphisms from $\mathrm{Aut}(M_2(\mathbb{C}))$, R is one of the following:

$$R(\mathrm{E}) = \lambda\mathrm{E} + \theta h,\ R(h) = -\theta\mathrm{E} - \frac{1}{2}h,\ R(e_{12}) = -e_{12},\ R(e_{21}) = 0,\ \lambda, \theta \in \mathbb{C};$$
$$R(\mathrm{E}) = \lambda\mathrm{E}, R(h) = R(e_{12}) = R(e_{21}) = 0\ \lambda \in \mathbb{C}.$$

The kernel of the second operator contains $sl_2(\mathbb{C}))$ while the dimension of the kernel of the first operator can't exceed 2. This means that they can't be conjugated. A simple check shows that maps of type 1 with different scalars λ are not conjugated by elements of $\mathrm{Aut}(M_2(\mathbb{C}))$.

Therefore, up to the action of $\mathrm{Aut}(M_2(\mathbb{C}))$ and multiplication by a nonzero scalar, r is one of the following:

$$\frac{1}{2}\mathrm{E} \otimes (\lambda\mathrm{E} + \theta h) + \frac{1}{2}h \otimes (-\theta\mathrm{E} - \frac{1}{2}h) - e_{21} \otimes e_{12},\ \lambda, \theta \in \mathbb{C}; \tag{15}$$
$$\lambda\mathrm{E} \otimes \mathrm{E}, \lambda \in \mathbb{C}. \tag{16}$$

By Remark 3, it remains to consider the action of an automorphism ψ_β defined in (10) for $\beta \in \mathbb{C}$, $\beta \neq 0$.

If in (15) $\theta \neq 0$, then the action of $\psi_{2\theta^{-1}}$ gives us tensors of type (12).

Suppose that $\theta = 0$. If $\lambda = 0$, then we obtain the solution (13) with $\lambda = 0$. If $\lambda \neq 0$, then after the action of ψ_β with $\beta = \frac{\sqrt{2}}{\sqrt{\lambda}}$, we obtain (13) with $\lambda = 1$.

Similar arguments show that in the case $r = \lambda\mathrm{E} \otimes \mathrm{E}$, λ is equal to 0 or 1 up to the action of automorphisms of type ψ_β. □

5 Classification of solutions of CYBE with an ad-invariant symmetric part on $gl_2(\mathbb{C})$, the case of weight 0.

In this case, we first need to classify all Rota—Baxter operators R of weight 0 on $gl_2(\mathbb{C})$ such that

$$R(x) + R^*(x) = 0 \quad x \in sl_2(\mathbb{C}), \tag{17}$$
$$R(\mathrm{E}) + R^*(\mathrm{E}) = \alpha\mathrm{E}, \quad \alpha \in \mathbb{C}. \tag{18}$$

As was mentioned in the previous section, we need the classification up to the action of the group of automorphisms of $M_2(\mathbb{C})$, that is, by a conjugation with an invertible matrix.

Let R be a Rota—Baxter operator of weight 0 on $gl_2(\mathbb{C})$ satisfying (17) and (18). Define a map $R_1 : sl_2(\mathbb{C}) \to sl_2(\mathbb{C})$ as follows: $R(x) = R_1(x) + \alpha(x)\mathrm{E}$, where $\alpha : sl_2(\mathbb{C}) \to \mathbb{C}$ is a linear functional on $sl_2(\mathbb{C})$. Since $sl_2(\mathbb{C})$ is a quadratic Lie algebra with the form given by $\omega(x, y) = \mathrm{tr}(xy)$, there is $t \in sl_2(\mathbb{C})$ such that $\alpha(x) = \omega(t, x)$ for all $t \in sl_2(\mathbb{C})$. Moreover, since R is a Rota—Baxter operator of weight 0 on $gl_2(\mathbb{C})$, E belongs to the center of

$gl_2(\mathbb{C})$ and $gl_2(\mathbb{C}) = sl_2(\mathbb{C}) \dotplus \mathrm{E}$, R_1 is a Rota—Baxter operator of weight 0 on $sl_2(\mathbb{C})$. In [21], the classification of Rota—Baxter operators of weight 0 on $sl_2(\mathbb{C})$, up to the action of the group of automorphisms of $sl_2(\mathbb{C})$, was given.

Theorem 5. [21] *Up to conjugation with an automorphism of $sl_2(\mathbb{C})$ and up to a scalar multiple, we have that a Rota—Baxter operator R_1 of weight 0 on $sl_2(\mathbb{C})$ is one of the following:*

$$1.\ R_1 = 0,$$
$$2.\ R_1(e_{12}) = 0,\ R_1(e_{21}) = te_{12} - h,\ R_1(h) = 2e_{12},$$
$$3.\ R_1(e_{12}) = R_1(e_{21}) = 0,\ R_1(h) = h,$$
$$4.\ R_1(e_{12}) = 0,\ R_1(e_{21}) = h,\ R_1(h) = 0,$$
$$5.\ R_1(e_{12}) = 0,\ R_1(e_{21}) = e_{12},\ R_1(h) = 0.$$

Since R is a skew-symmetric map, so is R_1. Obviously, $R_1 = 0$ is skew-symmetric. Consider an operator of type (2) from Theorem 5. Direct computation shows that $R_1^*(e_{12}) = 0$, $R_1^*(e_{21}) = te_{12} + h$, $R_1^*(h) = -2e_{12}$. Thus, R_1 is skew-symmetric if and only if $t = 0$.

Similarly, one can compute that operators 3)-5) from Theorem 5 are not skew-symmetric.

Thus, we need to consider two cases: $R_1 = 0$ and R_1 is of type 2 from Theorem 5 with $t = 0$.

Proposition 6. *If $R_1 = 0$, then there are $x \in sl_2(\mathbb{C})$ and $\theta \in \mathbb{C}$ such that*

$$R(s) = \omega(x, s)\mathrm{E},\ R(\mathrm{E}) = -2x + \theta\mathrm{E}. \tag{19}$$

Proof. Since $R_1 = 0$, we have that $R(s) = \alpha(s)\mathrm{E}$ for all $s \in sl_2(\mathbb{C})$. Suppose that $\alpha(s) = \omega(x, s)$ for all $s \in sl_2(\mathbb{C})$.

Let $R(E) = p + \theta\mathrm{E}$ for some $p \in sl_2(\mathbb{C})$ and $\theta \in \mathbb{C}$. For all $s \in sl_2(\mathbb{C})$ we have

$$\omega(R^*(\mathrm{E}), s) = \omega(\mathrm{E}, R(s)) = \omega(\mathrm{E}, \mathrm{E})\omega(x, s) = \omega(2x, s).$$

Therefore, $R^*(\mathrm{E}) = 2x + \alpha\mathrm{E}$. From (18) we deduce that $p + 2x = 0$.

Since the image of R is an abelian subalgebra in $gl_2(\mathbb{C})$, R is a Rota—Baxter operator of weight 0 for any $x \in sl_2(\mathbb{C})$, $\theta \in \mathbb{C}$. □

Proposition 7. *Suppose that $R_1(e_{12}) = 0$, $R_1(e_{21}) = -h$, $R_1(h) = 2e_{12}$. Then there are $\beta, \theta \in \mathbb{C}$:*

$$R(e_{12}) = 0,$$
$$R(e_{21}) = -h + \beta\mathrm{E},$$
$$R(h) = 2e_{12}$$
$$R(E) = -2\beta e_{12} + \alpha\mathrm{E}.$$

Proof. Using similar arguments as above, we have that:

$$\begin{gathered} R(e_{12}) = \omega(x, e_{12})\mathrm{E}, \\ R(e_{21}) = -h + \omega(x, e_{21})\mathrm{E}, \\ R(h) = 2e_{12} + \omega(x, h)\mathrm{E}, \\ R(E) = -2x + \theta\mathrm{E},\ x \in sl_2(\mathbb{C}),\ \theta \in \mathbb{C}. \end{gathered}$$

Since R is a Rota—Baxter operator, we have that

$$0 = [R(\mathrm{E}), R(e_{12})] = R([\mathrm{E}, R(e_{12})] + [R(\mathrm{E}), e_{12}]) = -2R([x, e_{12}]).$$

Note that if $(\alpha_1, \alpha_2) \neq (0, 0)$, then $R(\alpha_1 e_{21} + \alpha_2 h) \neq 0$. Therefore, $x = \beta e_{12} + \gamma h$ for some $\beta, \gamma \in \mathbb{C}$. Similarly,

$$\begin{aligned} -4\beta e_{12} = [-2\beta e_{12} - 2\gamma h, -h] = [R(\mathrm{E}), R(e_{21})] \\ = R([R(\mathrm{E}), e_{21}]) = R(-2\beta h + 4\gamma e_{21}) = -4\beta e_{12} - 4\gamma h. \end{aligned}$$

Thus, $\gamma = 0$ and $x = \beta e_{12}$. Note that in this case, the last condition

$$0 = [-2\alpha e_{12}, 2e_{12}] = [R(\mathrm{E}), R(h)] = R([R(\mathrm{E}), h]) = R(4\alpha e_{12}) = 0$$

holds automatically. □

Theorem 6. *Let $r \in gl_2(\mathbb{C}) \otimes gl_2(\mathbb{C})$ be a solution of CYBE such that $r + \tau(r)$ is $gl_2(\mathbb{C})$-invariant and the corresponding map is a Rota—Baxter operator of weight 0. Then, up to the action of $Aut(gl_2(\mathbb{C}))$ and multiplication by a nonzero scalar, r is equal to the one of the following:*

$$r = x \otimes \mathrm{E} - \mathrm{E} \otimes x + \alpha \mathrm{E} \otimes \mathrm{E}, \quad x \in \{0, e_{12}, h\}, \alpha \in \mathbb{C}. \tag{20}$$

$$r = h \otimes e_{12} - e_{12} \otimes h + \alpha \mathrm{E} \otimes \mathrm{E},\ \alpha \in 0, 1 \tag{21}$$

$$r = h \otimes e_{12} - e_{12} \otimes h + e_{12} \otimes \mathrm{E} - \mathrm{E} \otimes e_{12} + \alpha \mathrm{E} \otimes \mathrm{E},\ \alpha \in \mathbb{C}. \tag{22}$$

Proof. If R is a map from Proposition 6, then the corresponding tensor

$$r = x \otimes \mathrm{E} - \mathrm{E} \otimes x + \frac{\alpha}{2}\mathrm{E} \otimes \mathrm{E},\ x \in sl_2(\mathbb{C}),\ \alpha \in \mathbb{C}$$

obviously satisfies CYBE. Finally, since the Jordan form of an element $x \in sl_2(\mathbb{C})$ is either $0, e_{12}$ or αh for some $\alpha \in \mathbb{C}$, we have that up to the action of $\mathrm{Aut}(gl_2(\mathbb{C}))$, r is equal to an element of type (20).

If R is a map from Proposition 7, then the corresponding tensor has a form

$$r = h \otimes e_{12} - e_{12} \otimes h + \beta(e_{12} \otimes \mathrm{E} - \mathrm{E} \otimes e_{12}) + \alpha \mathrm{E} \otimes \mathrm{E},$$

where $\alpha, \beta \in \mathbb{C}$. Now we need to consider two cases: $\beta = 0$ and $\beta \neq 0$. In the first case, the conjugation with ψ_γ, where $\gamma = \alpha^{-\frac{1}{2}}$, gives us (21). Similarly, if $\beta \neq 0$, we obtain elements of type (22). □

Remark 5. The tensor (20) form Theorem 6 with $x = 0$ coincides with the tensor (14) in Theorem 4 since in this case, the corresponding map is a Rota—Baxter operator of any weight.

Theorem 7. *Let $r \in gl_2(\mathbb{C})\otimes gl_2(\mathbb{C})$ be a solution of CYBE such that $r+\tau(r)$ is $gl_2(\mathbb{C})$-invariant. Then, up to the action of $Aut(gl_2(\mathbb{C}))$ and multiplication by a nonzero scalar, r is equal to the one of the following:*

$$1.\ r = \mathrm{E}\otimes(\lambda\mathrm{E}+\theta h) - h\otimes(\theta\mathrm{E}+\frac{1}{4}h) - e_{21}\otimes e_{12},\ \ \lambda\in\{0,1\},\ \ \theta\in\mathbb{C};$$
$$2.\ r = x\otimes\mathrm{E} - \mathrm{E}\otimes x + \alpha\mathrm{E}\otimes\mathrm{E},\quad x\in\{0,e_{12},h\},\alpha\in\mathbb{C}.$$
$$3.\ r = h\otimes e_{12} - e_{12}\otimes h + \alpha\mathrm{E}\otimes\mathrm{E},\ \ \alpha\in 0,1$$
$$4.\ r = h\otimes e_{12} - e_{12}\otimes h + e_{12}\otimes\mathrm{E} - \mathrm{E}\otimes e_{12} + \alpha\mathrm{E}\otimes\mathrm{E},\ \ \alpha\in\mathbb{C}.$$

As a corollary of Theorem 7, we obtain a well known description of solutions r of CYBE on $sl_2(\mathbb{C})$ such that $r+\tau(r)$ is $sl_2(\mathbb{C})$-invariant.

Corollary 1. *Up to a multiplication to a nonzero scalar and the action of $Aut(sl_2(\mathbb{C}))$, there are only two solutions:*

$$r_1 = \frac{1}{4}h\otimes h + e_{12}\otimes e_{21}.$$

$$r_2 = h\otimes e_{12} - e_{12}\otimes h.$$

The following result was obtained in [4] using another technique.

Corollary 2. *Up to the action of $Aut(gl_2(\mathbb{C}))$ and multiplication by a nonzero scalar, there are two nontrivial quasitriangular Lie bialgebra structures δ_λ ($\lambda = 0,1$) on $gl_2(\mathbb{C})$ given by*

$$\delta_\lambda(\mathrm{E}) = 0,\quad \delta_\lambda(h) = 0,\quad \delta_\lambda(e_{12}) = (\lambda\mathrm{E}+\frac{1}{2}h)\otimes e_{12} - e_{12}\otimes(\lambda\mathrm{E}+\frac{1}{2}h)$$
$$\delta_\lambda(e_{21}) = e_{21}\otimes(\lambda\mathrm{E}-\frac{1}{2}h) - (\lambda\mathrm{E}-\frac{1}{2}h)\otimes e_{21}\quad \lambda\in\{0,1\}.$$

Proof. A quasitriangular Lie bialgebra structure is given by a non-skew-symmetric solution of the CYBE. In Theorem 6, the symmetric part of the tensors may be omitted since it gives a zero comultiplication. Thus, we need to consider tensors (12) from Theorem 4.

If r_1 and r_2 are conjugated by an automorphism $\varphi\in\mathrm{Aut}(gl_2(\mathbb{C}))$, i.e. $r_1 = (\varphi\otimes\varphi)(r_2)$, then corresponding comultiplications satisfy $\delta_{r_1}\circ\varphi^{-1} = (\varphi^{-1}\otimes\varphi^{-1})\circ\delta_{r_2}$ (note that the converse is false: if δ_{r_1} and δ_{r_2} are conjugate, then r_1 and r_2 are not necessarily conjugate). Therefore, up to multiplication by a scalar and the action of $\mathrm{Aut}(gl_2(\mathbb{C}))$, we have the following class of Lie bialgebra structures on $gl_2(\mathbb{C})$ depending on a parameter $\theta\in\mathbb{C}$:

$$\delta_\theta(\mathrm{E}) = 0,\quad \delta_\theta(h) = 0,\quad \delta_\theta(e_{12}) = (\theta\mathrm{E}+\frac{1}{2}h)\otimes e_{12} - e_{12}\otimes(\theta\mathrm{E}+\frac{1}{2}h)$$
$$\delta_\theta(e_{21}) = e_{21}\otimes(\theta\mathrm{E}-\frac{1}{2}h) - (\theta\mathrm{E}-\frac{1}{2}h)\otimes e_{21}\quad \theta\in\mathbb{C}.$$

If $\theta = 0$, we get δ_0. If $\theta\neq 0$, a conjugation with $\psi_{\theta^{-1}}$ will give us δ_1. □

Corollary 3. *Up to the action of Aut($gl_2(\mathbb{C})$) and multiplication by a nonzero scalar, there are two nontrivial triangular Lie bialgebra structures δ_λ ($\lambda = 0, 1$) on $gl_2(\mathbb{C})$ given by*

1. $\delta(\mathrm{E}) = 0,\ \delta_x(y) = [x, y] \otimes \mathrm{E} - \mathrm{E} \otimes [x, y] \quad x, y \in sl_2(\mathbb{C}),$
2. $\delta(\mathrm{E}) = 0,\ \delta(e_{12}) = 0,\ \delta(h) = e_{12} \wedge h,\ \delta(e_{21}) = e_{12} \wedge e_{21},$
3. $\delta(\mathrm{E}) = \delta(e_{12}) = 0,\ \delta(h) = e_{12} \wedge h + \mathrm{E} \wedge e_{12},\ \delta(e_{21}) = e_{12} \wedge e_{21} + \dfrac{1}{2} h \wedge \mathrm{E}.$

Proof. Similar to corollary 2. □

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