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PROJECTIVE CURVATURE TENSOR OF C_9-MANIFOLDS

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Abstract: In this paper, the projective curvature tensor components of C_9-manifold determined. Only fifteen of such components had been non-zero under the projective curvature properties. The projective invariant classes of these non-zero components established and their relationships to Einstein manifolds investigated. Moreover, the case when C_9-manifold has dimension 3 discussed
Keywords: C_9-manifold, projective curvature tensor, Einstein manifold.

1 Introduction

In 1990, D. Chinea and C. Gonzalaz classified the almost contact metric manifolds into many classes [1]. One of these classes is a C_9-manifold where its geometry studied by Rustanov et al. [2]. There are another important classes for instance, manifold of Kenmotsu type and $C_{12}-$manifold that introduced and examined respectively by M. Y. Abass, H. M. Abood [3, 4, 5] and M. Y. Abass, Q. S. A. Al-Zamil [6].

H. M. Abood and N. J. Mohammed focused on studying of the geometric identities of projective curvature tensor of nearly cosymplectic manifold [7] and on the geometry of the pseudo projectively tensor of nearly cosymplectic manifold [8].

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K. De and U. C. De [9] studied the flatness of $\phi-$projectively and $\xi-$projectively for connected $3-$dimensional trans-Sasakian manifolds. They also, studied the semi-symmetric projective property of aforementioned manifolds. A. R. Rustanov et al. [10] investigated the projective tensor and its invariants on almost $C(\lambda)-$manifolds.

T. Raghuwanshi et al. [11] generalized the projective curvature tensor and discussed it on para-Kenmotsu manifold. On the other hand, there are many researches which are related to this article, such as [12, 13, 14]

This article divided into four sections. After the introduction is section 2 that devoted to reviewed the basic related definitions and theorem. In section 3, the components of projective tensor are concluded for C_9-manifolds and their relationship to Einstein manifold decided. The last section has new classification based on projective tensor.

2 Preliminaries

Let M be the $(2n+1)-$dimensional manifold with $n \in \mathbb{Z}^+$, ∇ is Levi-Civita connection, and $X(M)$ be the $C^\infty(M)-$module of smooth vector fields on M.

Definition 1. [1] *A quadruple (η, ξ, Φ, g) of tensor fields on M is called an almost contact metric $(AC-)$ structure on M, if η is a differential $1-$form, ξ is a vector field named the characteristic vector field, Φ is a $(1,1)-$tensor field named the structure endomorphism of the module $X(M)$, and $g = \langle \cdot, \cdot \rangle$ is a Riemannian metric, such that the following satisfied:*

i) $\eta(\xi) = 1$; *ii)* $\eta \circ \Phi = 0$; *iii)* $\Phi(\xi) = 0$; *iv)* $\Phi^2 = -id + \eta \otimes \xi$;
v) $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad \forall\ X, Y \in X(M)$.

Additionally, a manifold M equipped with an $AC-$structure (η, ξ, Φ, g) is called an almost contact metric $(AC-)$manifold.

Definition 2. [2] *If an $AC-$manifold satisfies the following identity:*

$$\nabla_X(\Phi)Y = \eta(Y)\nabla_{\Phi X}\xi - \langle \Phi X, \nabla_Y \xi \rangle \xi, \ for\ all\ X, Y \in X(M),$$

then it is called a C_9-manifold.

Proposition 1. [2] *Let $S = (\eta, \xi, \Phi, g)$ be an $AC-$structure on a manifold M. Hence the next terms are equivalent:*
1) S is an $AC-$structure of class C_9; *2)* $B = C = D_1 = E = F_1 = G = 0$;
3) $F_{ab} = -B_{ab} = -\sqrt{-1}\Phi^0_{a,b}$; $F^{ab} = -B^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}}$; $F_{ab} = \overline{F_{ab}}$;
$F_{ab} = F_{ba}$; $F^{ab} = F^{ba}$.

Proposition 2. [2] *C_9-manifold coincide cosymplectic manifold if and only if $F^{ab} = F_{ab} = 0$ on the space of associated G-structure*

Definition 3. [15] *A Riemannian curvature tensor of type $(3,1)$ on $AC-$manifold is known as:*

$$R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z,$$

for all $X, Y, Z \in X(M)$.

The researchers can be learned about the space of associated $G-$structure ($AG-$structure for short) from the references [10] and [16], such on this space, the tensors Φ and g of any $AC-$manifold are given by:

$$(\Phi^i_j) = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sqrt{-1}\boldsymbol{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\sqrt{-1}\boldsymbol{I}_n \end{pmatrix} \quad (1)$$

$$g_{ij} = \begin{cases} 1 & ; \quad if\ i = j = 0 \\ \delta^a_b & ; \quad if\ i = b\ \&\ j = \hat{a}\ or\ i = \hat{a}\ \&\ j = b \\ 0 & ; \quad if\ i = a\ \&\ j = b\ or\ i = \hat{a}\ \&\ j = \hat{b} \end{cases} \quad (2)$$

where $\boldsymbol{I}_n$ is the identity matrix of rank n, $i, j, = 0, 1, 2, \cdots, 2n$,
$a, b, = 1, 2, \cdots, n$, and $\hat{a} = a + n$.

Theorem 1. [2] *The components of Riemann-Christoffel tensor of type* $(3, 1)$ *are:*

$$(1)\ R^0_{a\hat{b}0} = F_{ac}F^{cb}\ ; \quad (2)\ R^0_{ab0} = -F_{ab0}; \quad (3)\ R^0_{ab\hat{c}} = -F_{ab}{}^{\ c}\ ;$$
$$(4)\ R^a_{bc\hat{d}} = A^{ad}_{bc} + F^{ad}F_{bc}\ ; \qquad (5)\ R^{\hat{a}}_{bcd} = -2F_{a[c}\ F_{|b|d]},$$

and the other components are identical zero. Note that

a) R has the following properties:

1. $R_{ijkl} = R^{\hat{i}}_{jkl}$; 2. $-R_{ijlk} = R_{ijkl} = -R_{jikl}$; 3. $R_{ijkl} = R_{klij}$;
4. $\overline{R_{ijkl}} = R_{\hat{i}\hat{j}\hat{k}\hat{l}}$; 5. $R_{ijkl} + R_{iklj} + R_{iljk} = 0 = R_{ijkl} + R_{kijl} + R_{jkil}$.

b) $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$.

where $\hat{0} = 0$; $i, j, k, l = 0, 1, 2, \cdots, 2n$, $a, b, c, d = 1, 2, \cdots, n$, $\hat{a} = a + n$, *and* $\hat{\hat{a}} = a$.

On the space of $AG-$structure if $r_{ij} = -R^k_{ijk}$, then r is called Ricci tensor of a Riemannian manifold. On C_9-manifold, the components of Ricci tensor are [2]:

$$r_{00} = -2F_{ab}F^{ba}; \quad r_{a0} = -F_{ba}{}^{\ b}; \quad r_{ab} = F_{ab0}; \quad r_{a\hat{b}} = A^{bc}_{ac}. \quad (3)$$

Definition 4. [10] *The projective curvature tensor of type* $(3, 1)$ *on AC-manifold M is given by the relation*

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}\{r(Y, Z)X - r(X, Z)Y\}, \qquad \forall X, Y, Z \in X(M)$$

Proposition 3. [10] *On an $AC-$manifold, the projective curvature tensor of type* $(4, 0)$ *has the following properties:*

1) $P_{ijkl} = -P_{ijlk}$; 2) $P_{ijkl} + P_{iklj} + P_{iljk} = 0$; 3) $P_{ijkl} = \overline{P_{\hat{i}\hat{j}\hat{k}\hat{l}}}$,

Definition 5. [2] *An $AC-$manifold is called an Einstein manifold if it satisfies the identity* $r_{ij} = \lambda\ g_{ij}$, *where* λ *is a smooth function.*

3 Geometry of projective curvature tensor

Theorem 2. *The components of projective curvature tensor on the space of AG−structure for C_9-manifold are given by:*

(1) $P_{000d} = -\frac{1}{2n}\ r_{0d}$;
(2) $P_{0b0d} = \frac{2n-1}{2n} r_{bd}$;
(3) $P_{0b0\hat{d}} = -F_{bc}\ F^{cd} - \frac{1}{2n} r_{b\hat{d}}$;
(4) $P_{0bc\hat{d}} = -F_{bc}{}^{d}$;
(5) $P_{a00d} = -r_{ad}$;
(6) $P_{a00\hat{d}} = F_{ac}F^{cd} + \frac{1}{2n}\delta_a^d\ r_{00}$;
(7) $P_{a0c\hat{d}} = F_{ac}{}^{d} + \frac{1}{2n}\delta_a^d\ r_{0c}$;
(8) $P_{a0\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta_a^c\ r_{0\hat{d}} - \delta_a^d\ r_{0\hat{c}}\}$;
(9) $P_{ab0\hat{d}} = \frac{1}{2n}\delta_a^d\ r_{b0}$;
(10) $P_{abcd} = -2F_{a[c}\ F_{|b|d]}$;
(11) $P_{abc\hat{d}} = \frac{1}{2n}\delta_a^d\ r_{bc}$;
(12) $P_{ab\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta_a^c\ r_{b\hat{d}} - \delta_a^d\ r_{b\hat{c}}\}$;
(13) $P_{a\hat{b}0\hat{d}} = F^{bd}{}_{a} + \frac{1}{2n}\delta_a^d\ r_{\hat{b}0}$;
(14) $P_{a\hat{b}c\hat{d}} = -(A_{ac}^{bd} + F^{bd}F_{ac}) + \frac{1}{2n}\delta_a^d\ r_{\hat{b}c}$;
(15) $P_{a\hat{b}\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta_a^c\ r_{\hat{b}\hat{d}} - \delta_a^d\ r_{\hat{b}\hat{c}}\}$,

and the remaining components are identical to zero or can determined by the Proposition 3.

Proof. The components of projective curvature tensor P of type (4,0) on AC-manifold is given by the following formula:

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}\{g_{ik}\ r_{jl} - g_{il}\ r_{jk}\}. \tag{4}$$

This formula give us 81 components.These 81 components are divided into the following collections:

(i) The non-zero 15 components are:

$$\{P_{000d}, P_{0b0d}, P_{0b0\hat{d}}, P_{0bc\hat{d}}, P_{a00d}, P_{a00\hat{d}}, P_{a0c\hat{d}}, P_{a0\hat{c}\hat{d}},$$
$$P_{ab0\hat{d}}, P_{abcd}, P_{abc\hat{d}}, P_{ab\hat{c}\hat{d}}, P_{a\hat{b}0\hat{d}}, P_{a\hat{b}c\hat{d}}, P_{a\hat{b}\hat{c}\hat{d}}\},$$

and their conjugates:

$$\{P_{000\hat{d}}, P_{0\hat{b}0\hat{d}}, P_{0\hat{b}0d}, P_{0\hat{b}\hat{c}d}, P_{\hat{a}00\hat{d}}, P_{\hat{a}00d}, P_{\hat{a}0\hat{c}d},$$
$$P_{\hat{a}0cd}, P_{\hat{a}\hat{b}0d}, P_{\hat{a}\hat{b}\hat{c}\hat{d}}, P_{\hat{a}\hat{b}\hat{c}d}, P_{\hat{a}\hat{b}cd}, P_{\hat{a}b0d}, P_{\hat{a}b\hat{c}d}, P_{\hat{a}bcd}\},$$

are 15 components, also.

(ii) The zero 13 components are:

$$\{P_{00cd}, P_{00c\hat{d}}, P_{0b00}, P_{0bcd}, P_{0b\hat{c}\hat{d}}, P_{a000}, P_{a0cd},$$
$$P_{ab00}, P_{ab0d}, P_{abc0}, P_{a\hat{b}00}, P_{a\hat{b}cd}\} \bigcup \{P_{0000}\},$$

and their conjugates are:

$$\{P_{00\hat{c}\hat{d}}, P_{00\hat{c}d}, P_{0\hat{b}00}, P_{0\hat{b}\hat{c}\hat{d}}, P_{0\hat{b}cd}, P_{\hat{a}000}, P_{\hat{a}0\hat{c}\hat{d}}, P_{\hat{a}\hat{b}00}, P_{\hat{a}\hat{b}0\hat{d}}, P_{\hat{a}\hat{b}\hat{c}0}, P_{\hat{a}b00}, P_{\hat{a}b\hat{c}\hat{d}}\},$$

just, 12 components.

(iii) The component $P_{a\hat{b}0d}$ and its conjugate $P_{\hat{a}b0\hat{d}}$ can find them by using Proposition 3; items 2 and 3, respectively.

(iv) The following 12 components determined by Proposition 3; item 1:

$$\{P_{00c0}, P_{0bc0}, P_{0b\hat{c}0}, P_{0b\hat{c}d}, P_{a0c0}, P_{a0\hat{c}0},$$
$$P_{a0\hat{c}d}, P_{ab\hat{c}0}, P_{ab\hat{c}d}, P_{a\hat{b}c0}, P_{a\hat{b}\hat{c}0}, P_{a\hat{b}\hat{c}d}\},$$

and their conjugates are:

$$\{P_{00\hat{c}0}, P_{0\hat{b}\hat{c}0}, P_{0\hat{b}c0}, P_{0\hat{b}c\hat{d}}, P_{\hat{a}0\hat{c}0}, P_{\hat{a}0c0},$$
$$P_{\hat{a}0c\hat{d}}, P_{\hat{a}\hat{b}c0}, P_{\hat{a}\hat{b}c\hat{d}}, P_{\hat{a}b\hat{c}0}, P_{\hat{a}bc0}, P_{\hat{a}bc\hat{d}}\},$$

are also 12 components.

Hence, the demonstration of them, for P defined on C_9-manifold, can be done by the substitutions of the values R_{ijkl} from Theorem 1 and g_{ij} from equation (2) in the equation (4) give the desired as in the following chosen components:

$$\begin{aligned}
P_{0b0d} =& R_{0b0d} - \frac{1}{2n}\ \{g_{00}\ r_{bd} - g_{0d}\ r_{b0}\},\\
=& - R_{0bd0} - \frac{1}{2n} r_{bd},\\
=& - R^0_{bd0} - \frac{1}{2n} r_{bd},\\
=& F_{bd0} - \frac{1}{2n} r_{bd} = \frac{2n-1}{2n} r_{bd}. \qquad \text{from equation (3)}.\\
P_{a00\hat{d}} =& R_{a00\hat{d}} - \frac{1}{2n}\ \{g_{a0}\ r_{0\hat{d}} - g_{a\hat{d}}\ r_{00}\},\\
=& R_{0a\hat{d}0} + \frac{1}{2n}\delta^d_a\ r_{00},\\
=& R^0_{a\hat{d}0} + \frac{1}{2n}\delta^d_a\ r_{00},\\
=& F_{ac}F^{cd} + \frac{1}{2n}\delta^d_a\ r_{00}.\\
P_{a\hat{b}c\hat{d}} =& R_{a\hat{b}c\hat{d}} - \frac{1}{2n}\ \{g_{ac}\ r_{\hat{b}\hat{d}} - g_{a\hat{d}}\ r_{\hat{b}c}\},\\
=& - R_{\hat{b}ac\hat{d}} + \frac{1}{2n}\delta^d_a\ r_{\hat{b}c},\\
=& - R^b_{ac\hat{d}} + \frac{1}{2n}\delta^d_a\ r_{\hat{b}c},\\
=& - (A^{bd}_{ac} + F^{bd}F_{ac}) + \frac{1}{2n}\delta^d_a\ r_{\hat{b}c}.\\
P_{0000} =& R_{0000} - \frac{1}{2n}\{g_{00}\ r_{00} - g_{00}\ r_{00}\},\\
=& - \frac{1}{2n}\{r_{00} - \ r_{00}\} = 0.
\end{aligned}$$

The component $P_{a\hat{b}0d}$ will be proved as follow: from Proposition 3, $P_{a\hat{b}0d} + P_{a0d\hat{b}} + P_{ad\hat{b}0} = 0$. So, $P_{a\hat{b}0d} = -P_{a0d\hat{b}} - P_{ad\hat{b}0}$. From the non-zero components, we obtain: $P_{a0d\hat{b}} = F_{ad}{}^{b} + \frac{1}{2n}\delta_a^b\ r_{0d}$, and from Proposition 3, we get: $P_{ad\hat{b}0} = -P_{ad0\hat{b}} = -\frac{1}{2n}\delta_a^b\ r_{d0}$, then

$$\begin{aligned} P_{a\hat{b}0d} &= -\ (F_{ad}{}^{b} + \frac{1}{2n}\delta_a^b\ r_{0d}) + \frac{1}{2n}\delta_a^b\ r_{d0} \\ &= -\ F_{ad}{}^{b} = -F_{da}{}^{b}. \end{aligned}$$

At last we will calculate the component P_{00c0} as follow: by Proposition 3,

$$P_{00c0} = -P_{000c} = -(R_{000c} - \tfrac{1}{2n}\{g_{00}\ r_{0c} - g_{0c}\ r_{00}\}) = \tfrac{1}{2n}\ r_{0c}. \qquad \square$$

Corollary 1. *When M in Theoerm 2 of dimension 3, then the non-zero components are given by:*

1) $P_{0112} = -2P_{0001} = -2P_{1012} = 2P_{1102} = r_{01}$;
2) $P_{1001} = -2P_{1112} = -2P_{0101} = -r_{11}$;
3) $P_{0102} = P_{1212} = \frac{1}{2}(r_{00} - r_{12})$;
4) $P_{1202} = -\frac{1}{2}r_{20}$.

Proof. Since M is a C_9-manifold of dimension 3, then $n = 1$ and hence $a = b = c = d = 1$ and $\hat{a} = \hat{b} = \hat{c} = \hat{d} = 2$.

The results be fulfilled from Theorem 2 and the following facts:

$$F^{ab} = F^{11}; \quad F_{ab} = F_{11};\ F_{11}{}^{1} = -r_{01}; \quad F^{11}{}_{1} = -r_{02};$$
$$A^{11}_{11} = r_{12}; \quad F_{11}F^{11} = -\tfrac{1}{2}r_{00}; \quad \delta_b^a = 1,$$

and the facts attained from equation (3). $\square$

Theorem 3. *If a C_9-manifold M of dimension greater than 3 has flat projective curvature tensor, then M has flat Ricci tensor.*

Proof. From Theorem 2, we have the following components:

$$P_{000d} = -\frac{1}{2n}\ r_{0d} \tag{5}$$

$$P_{0b0\hat{d}} = -F_{bc}\ F^{cd} - \frac{1}{2n}\{r_{b\hat{d}}\} \tag{6}$$

$$P_{a00d} = -r_{ad} \tag{7}$$

$$P_{a00\hat{d}} = F_{ac}F^{cd} + \frac{1}{2n}\{\delta_a^d\ r_{00}\} \tag{8}$$

$$P_{ab\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta_a^c\ r_{b\hat{d}} - \delta_a^d\ r_{b\hat{c}}\} \tag{9}$$

since P is flat, then $P_{ijkl} = 0$. So, from (5), we get $r_{0d} = 0$. The equation (7) gives: $r_{ad} = 0$.

Now, from equation (9), we obtain: $\delta_a^c\ r_{b\hat{d}} - \delta_a^d\ r_{b\hat{c}} = 0$, and by contracting (a,c), we get:

$$\begin{aligned} \delta_a^a\ r_{b\hat{d}} - \delta_a^d\ r_{b\hat{a}} &= 0, \\ (n-1)\ r_{b\hat{d}} &= 0, \\ n = 1 \text{ or } r_{b\hat{d}} &= 0. \end{aligned}$$

If $n = 1$, then $2n + 1 = 3$.This contradict with hypothesis, which M of dimension greater than 3. Then $r_{b\hat{d}} = 0$. Hence from equation (6), we have:

$$F_{bc}\ F^{cd} = 0. \tag{10}$$

From equations (8) and (10), we obtain: $r_{00} = 0$. Then M has flat Ricci tensor. □

Corollary 2. *If C_9-manifold M of dimension greater than 3 has flat projective curvature tensor, then M will be cosymplectic manifold.*

Proof. Let M has $P_{ijkl} = 0$, then

$$F_{bc}F^{cd} = 0, \qquad \text{from equation (10)}$$

$$F_{bc}F^{cb} = 0, \qquad \text{by contracting (b,d)}$$

$$\sum_{c,b} |F^{cb}|^2 = 0.$$

So, $F^{cb} = 0$ *and* $F_{bc} = 0$. Then the tensor F must be zero and this gives the result, from Proposition 2 □

Corollary 3. *If C_9-manifold M of dimension greater than or equal to 5 has flat projective curvature tensor, then $A^{ba}_{ac} = 0$.*

Proof. Let $P_{ijkl} = 0$, and M of dimension greater than or equal to 5, then from Theorem 3, we get that $r_{a\hat{b}} = 0$. Thus, from equation (3), we have $A^{ba}_{ca} = A^{ba}_{ac} = 0$ □

Theorem 4. *Let M be a C_9-manifold of dimension 3. M has flat projective curvature tensor if and only if M is Einstein manifold with $\lambda = -2|F_{11}|^2 = A^{11}_{11}$*

Proof. $P_{ijkl} = 0$ since M has flat projective curvature tensor. Then from Corollary 1, we obtain $r_{00} = r_{12}$ and $r_{10} = r_{11} = 0$. By using equation (3), we get:

$$r_{00} = r_{12} = A^{11}_{11} = -2F_{11}\ F^{11} = -2|F_{11}|^2.$$

So, M is Einstein manifold with $\lambda = -2|F_{11}|^2 = A^{11}_{11}$

conversely, let M is Einstein manifold with $\lambda = -2|F_{11}|^2 = A^{11}_{11}$. By using the equation (3), we obtain: $r_{00} = r_{12}$ and $r_{01} = r_{11} = 0$. So, Corollary 1, gives $P_{ijkl} = 0$. □

Theorem 5. *If a Riemannian manifold M of dimension $2n+1$ satisfies the following property:*

$$P_{ijkl} + P_{kijl} + P_{jkil} = 0,$$

then M is an Einstein manifold.

Proof. Let $P_{ijkl}+P_{kijl}+P_{jkil} = 0$, then from the fact $R_{ijkl}+R_{kijl}+R_{jkil} = 0$ and equation (4), we obtain:

$$-\frac{1}{2n}\{g_{ik}\ r_{jl} - g_{il}\ r_{jk} + g_{kj}\ r_{il} - g_{kl}\ r_{ij} + g_{ji}\ r_{kl} - g_{jl}\ r_{ki}\} = 0.$$

Thus,

$$\{g_{ik}\ r_{jl} - g_{il}\ r_{jk} + g_{kj}\ r_{il} - g_{kl}\ r_{ij} + g_{ji}\ r_{kl} - g_{jl}\ r_{ki}\}g^{ik} = 0,$$

$$g_{ik}\ g^{ik}\ r_{jl} = (2n+1)\ r_{jl}; \quad g_{kj}\ g^{ik}\ r_{il} = \delta^i_j\ r_{il}; \quad g_{ji}\ g^{ik}\ r_{kl} = \delta^k_j\ r_{kl}$$
$$g_{il}\ g^{ik}\ r_{jk} = \delta^k_l\ r_{jk}; \quad g_{kl}\ g^{ik}\ r_{ij} = \delta^i_l\ r_{ij}; \quad g_{jl}\ g^{ik}\ r_{ki} = \gamma\ g_{jl},$$

where γ is a scalar curvature of Riemannian manifold M. Then

$$(2n+1)\ r_{jl} - \gamma\ g_{jl} = 0 \Rightarrow r_{jl} = \frac{\gamma}{2n+1}\ g_{jl}. \quad \square$$

Theorem 6. *If M is C_9-manifold with the property above in Theorem 5, then $A^{ac}_{ac} = -2nF_{ac}F^{ca}$.*

Proof. From Theorem 5, M is an Einstein manifold with dimension 2n+1. From Theorem 5, $\gamma\ g_{ij} = (2n+1)\ r_{ij}$.
when $i = j = 0$ and from equation (3),

$$\gamma = -2(2n+1)F_{ab}F^{ba}, \tag{11}$$

when $i = a$ & $j = \hat{b}$ and from equation (3), $\gamma\delta^b_a = (2n+1)A^{bc}_{ac}$,

$$\gamma = \frac{2n+1}{n}A^{ac}_{ac}. \tag{12}$$

From equations (11) and (12), we get on $A^{ac}_{ac} = -2nF_{ac}F^{ca}$. $\square$

4 Projective invariant classes

Consider the following collections of functions: $P_1 = \{P_{000d}, P_{000\hat{d}}\}$; $P_2 = \{P_{0b0d}, P_{0\hat{b}0\hat{d}}\}$; $P_3 = \{P_{0b0\hat{d}}, P_{0\hat{b}0d}\}$; $P_4 = \{P_{0bc\hat{d}}, P_{0\hat{b}\hat{c}d}\}$; $P_5 = \{P_{a00d}, P_{\hat{a}00\hat{d}}\}$; $P_6 = \{P_{a00\hat{d}}, P_{\hat{a}00d}\}$; $P_7 = \{P_{a0c\hat{d}}, P_{\hat{a}0\hat{c}d}\}$; $P_8 = \{P_{a0\hat{c}\hat{d}}, P_{\hat{a}0cd}\}$; $P_9 = \{P_{ab0\hat{d}}, P_{\hat{a}\hat{b}0d}\}$; $P_{10} = \{P_{abcd}, P_{\hat{a}\hat{b}\hat{c}\hat{d}}\}$; $P_{11} = \{P_{abc\hat{d}}, P_{\hat{a}\hat{b}\hat{c}d}\}$; $P_{12} = \{P_{ab\hat{c}\hat{d}}, P_{\hat{a}\hat{b}cd}\}$; $P_{13} = \{P_{a\hat{b}0\hat{d}}, P_{\hat{a}b0d}\}$; $P_{14} = \{P_{a\hat{b}c\hat{d}}, P_{\hat{a}b\hat{c}d}\}$; $P_{15} = \{P_{a\hat{b}\hat{c}\hat{d}}, P_{\hat{a}bcd}\}$, which define tensors on the manifold M^{2n+1}. Such tensors are called the *basic projective invariants* of an C_9-manifold.

Definition 6. *A C_9-manifold is named C_9-manifold of class P_α, $\alpha = 1, 2, \cdots, 15$, if $P_\alpha = 0$*

Lemma 1. *Let $\Pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$ and $\overline{\Pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$, then $\{\Pi(X)\}^i = X^a$ and $\{\overline{\Pi}(X)\}^i = X^{\hat{a}}$, $\forall\ X \in X(M)$.*

Proof.

$$\begin{aligned}\{\Pi(X)\}^i &= -\frac{1}{2}\{(\Phi^2 + \sqrt{-1}\Phi)X\}^i = -\frac{1}{2}\{\Phi^i_j\Phi^j_k + \sqrt{-1}\Phi^i_k\}X^k \\ &= -\frac{1}{2}\{(\sqrt{-1})^2 + (\sqrt{-1})^2\}X^a - \frac{1}{2}\{(-\sqrt{-1})^2 - (\sqrt{-1})^2\}X^{\hat{a}} \\ &= X^a\end{aligned}$$

In the same way we can prove that, $\{\overline{\Pi}(X)\}^i = X^{\hat{a}}$ $\square$

Theorem 7. *The invariants* P_α, *where* $\alpha = 1, 2, \cdots, 15$, *of* C_9-*manifold* M *are determined by the following formulas, for all* $X, Y, Z \in X(M)$:

$$\begin{aligned}
P_1(X) &= P(\xi, X)\xi + \Phi^2 \circ P(\xi, X)\xi;\\
P_2(X,Y) &= P(\xi,\Pi(X))\Pi(Y) + \Phi^2 \circ P(\xi,\Pi(X))\Pi(Y) + P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y)\\
&\quad + \Phi^2 \circ P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y);\\
P_3(X,Y) &= P(\xi,\overline{\Pi}(X))\Pi(Y) + \Phi^2 \circ P(\xi,\overline{\Pi}(X))\Pi(Y) + P(\xi,\Pi(X))\overline{\Pi}(Y)\\
&\quad + \Phi^2 \circ P(\xi,\Pi(X))\overline{\Pi}(Y);\\
P_4(X,Y)Z &= P(\Pi(X),\overline{\Pi}(Y))\Pi(Z) + \Phi^2 \circ P(\Pi(X),\overline{\Pi}(Y))\Pi(Z)\\
&\quad + P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z) + \Phi^2 \circ P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z);\\
P_5(X) &= \tfrac{1}{2}\{\Phi \circ P(\xi,\Phi X)\xi - \Phi^2 \circ P(\xi, X)\xi\};\\
P_6(X) &= \tfrac{1}{2}\{\Phi^2 \circ P(X,\xi)\xi + \Phi \circ P(\Phi X,\xi)\xi\};\\
P_7(X,Y) &= \overline{\Pi} \circ P(\Pi(X),\overline{\Pi}(Y))\xi + \Pi \circ P(\overline{\Pi}(X),\Pi(Y))\xi;\\
P_8(X,Y) &= \overline{\Pi} \circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\xi + \Pi \circ P(\Pi(X),\Pi(Y))\xi;\\
P_9(X,Y) &= \overline{\Pi} \circ P(\xi,\overline{\Pi}(Y))\Pi(X) + \Pi \circ P(\xi,\Pi(Y))\overline{\Pi}(X);\\
P_{10}(X,Y)Z &= \overline{\Pi} \circ P(\Pi(X),\Pi(Y))\Pi(Z) + \Pi \circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z);\\
P_{11}(X,Y)Z &= \overline{\Pi} \circ P(\Pi(X),\overline{\Pi}(Y))\Pi(Z) + \Pi \circ P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z);\\
P_{12}(X,Y)Z &= \overline{\Pi} \circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\Pi(Z) + \Pi \circ P(\Pi(X),\Pi(Y))\overline{\Pi}(Z);\\
P_{13}(X,Y) &= \overline{\Pi} \circ P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y) + \Pi \circ P(\xi,\Pi(X))\Pi(Y);\\
P_{14}(X,Y)Z &= \overline{\Pi} \circ P(\Pi(X),\overline{\Pi}(Y))\overline{\Pi}(Z) + \Pi \circ P(\overline{\Pi}(X),\Pi(Y))\Pi(Z);\\
P_{15}(X,Y)Z &= \overline{\Pi} \circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z) + \Pi \circ P(\Pi(X),\Pi(Y))\Pi(Z).
\end{aligned}$$

Proof. We will prove some of P_α and the other should be in the same way. We need to know in the A-frame $\xi^a = 0 = \xi^{\hat{a}}, \xi^0 = 1$ and use the equation (1). So, to prove P_1 do the following:

$$\begin{aligned}
\{P(\xi, X)\xi\}^i &= P^i_{jkl}\xi^j\xi^k X^l = P_{\hat{i}jkl}\xi^j\xi^k X^l,\\
&= P_{000d}X^d + P_{000\hat{d}}X^{\hat{d}} + P_{a00d}X^d + P_{\hat{a}00\hat{d}}X^{\hat{d}} + P_{a00\hat{d}}X^{\hat{d}}\\
&\quad + P_{\hat{a}00d}X^d.\\
\{\Phi^2 \circ P(\xi, X)\xi\}^i &= \Phi^i_t\Phi^t_s P^s_{jkl}\xi^j\xi^k X^l = \Phi^i_t\Phi^t_s P_{\hat{s}00l}\, X^l, \qquad s,t = 0,1,2,\cdots,2n\\
&= -\{P_{a00d}X^d + P_{\hat{a}00\hat{d}}X^{\hat{d}} + P_{a00\hat{d}}X^{\hat{d}} + P_{\hat{a}00d}X^d\}.
\end{aligned}$$

Then, $\{P(\xi, X)\xi + \Phi^2 \circ P(\xi, X)\xi\}^i = P_{000d}X^d + P_{000\hat{d}}X^{\hat{d}}$.
Hence the functions $\{P_{000d}, P_{000\hat{d}}\}$ are components of the following tensor: $\{P(\xi, X)\xi + \Phi^2 \circ P(\xi, X)\xi\}$, and therefore this tensor coincides with the tensor $P_1(X)$. The first equality is satisfied.

To prove P_3 do the following: by use Lemma 1 and equation (1):

$$\begin{aligned}
\{P(\xi,\overline{\Pi}(X))\Pi(Y)\}^i &= P^i_{jkl}\{\Pi(Y)\}^j\xi^k\{\overline{\Pi}(X)\}^l,\\
&= P_{0b0\hat{d}}Y^b X^{\hat{d}} + P_{ab0\hat{d}}Y^b X^{\hat{d}} + P_{\hat{a}b0\hat{d}}Y^b X^{\hat{d}}.\\
\{\Phi^2 \circ P(\xi,\overline{\Pi}(X))\Pi(Y)\}^i &= \Phi^i_t\Phi^t_s P^s_{jkl}\{\Pi(Y)\}^j\xi^k\{\overline{\Pi}(X)\}^l,\\
&= -\{P_{ab0\hat{d}}Y^b X^{\hat{d}} + P_{\hat{a}b0\hat{d}}Y^b X^{\hat{d}}\}.\\
\{P(\xi,\Pi(X))\overline{\Pi}(Y)\}^i &= P^i_{jkl}\{\overline{\Pi}(Y)\}^j\xi^k\{\Pi(X)\}^l,\\
&= P_{0\hat{b}0d}Y^{\hat{b}}X^d + P_{a\hat{b}0d}Y^{\hat{b}}X^d + P_{\hat{a}\hat{b}0d}Y^{\hat{b}}X^d.\\
\{\Phi^2 \circ P(\xi,\Pi(X))\overline{\Pi}(Y)\}^i &= \Phi^i_t\Phi^t_s P^s_{jkl}\{\overline{\Pi}(Y)\}^j\xi^k\{\Pi(X)\}^l,
\end{aligned}$$

$$= -\{P_{a\hat{b}0d}Y^{\hat{b}}X^d + P_{\hat{a}\hat{b}0d}Y^{\hat{b}}X^d\}.$$

Then,

$$P_{0b0\hat{d}}Y^bX^{\hat{d}}+P_{0\hat{b}0d}Y^{\hat{b}}X^d = \{P(\Pi(X),\overline{\Pi}(Y))\Pi(Z)+\Phi^2\circ P(\Pi(X),\overline{\Pi}(Y))\Pi(Z) +P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z)+\Phi^2\circ P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z)\}^i.$$

Hence, the functions $\{P_{0b0\hat{d}}, P_{0\hat{b}0d}\}$ are the components of the following tensor: $P(\Pi(X),\overline{\Pi}(Y))\Pi(Z) + \Phi^2 \circ P(\Pi(X),\overline{\Pi}(Y))\Pi(Z) + P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z) + \Phi^2\circ P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z)$, and therefore, this tensor coincides with the tensor $P_3(X,Y)$ and the third equality is satisfied.

To prove P_5 do the following:

$$\begin{aligned}\{\Phi\circ P(\xi,\Phi X)\xi\}^i &= \Phi^i_t P^t_{jkl}\xi^j\xi^k\Phi^l_s X^s,\\ &= P_{a00d}X^d + P_{\hat{a}00\hat{d}}X^{\hat{d}} - P_{a00\hat{d}}X^{\hat{d}} - P_{\hat{a}00d}X^d.\\ \{\Phi^2\circ P(\xi,X)\xi\}^i &= \Phi^i_t\Phi^t_s P^s_{jkl}\xi^j\xi^k X^l,\\ &= -P_{a00d}X^d - P_{\hat{a}00\hat{d}}X^{\hat{d}} - P_{a00\hat{d}}X^{\hat{d}} - P_{\hat{a}00d}X^d.\end{aligned}$$

Then, $\frac{1}{2}\{\Phi\circ P(\xi,\Phi X)\xi - \Phi^2\circ P(\xi,X)\xi\}^i = P_{a00d}X^d + P_{\hat{a}00\hat{d}}X^{\hat{d}}$. Hence, the functions $\{P_{a00d}, P_{\hat{a}00\hat{d}}\}$ are the components of the following tensor: $\frac{1}{2}\{\Phi\circ P(\xi,\Phi X)\xi - \Phi^2\circ P(\xi,X)\xi\}$, and therefore, this tensor coincides with the tensor $P_5(X,Y)$ and the fifth equality is satisfied.

To prove P_{10} do the following:

$$\begin{aligned}\{\overline{\Pi}\circ P(\Pi(X),\Pi(Y))\Pi(Z)\}^i &= \{\overline{\Pi}(P(\Pi(X),\Pi(Y))\Pi(Z))\}^i\\ &= \{P(\Pi(X),\Pi(Y))\Pi(Z)\}^{\hat{a}}\\ &= P^{\hat{a}}_{jkl}\{\Pi(X)\}^k\{\Pi(Y)\}^l\{\Pi(Z)\}^j\\ &= P_{abcd}X^cY^dZ^b,\\ \{\Pi\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^i &= \{\Pi(P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z))\}^i\\ &= \{P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^a\\ &= P^a_{jkl}\{\overline{\Pi}(X)\}^k\{\overline{\Pi}(Y)\}^l\{\overline{\Pi}(Z)\}^j\\ &= P_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{c}}Y^{\hat{d}}Z^{\hat{b}}.\end{aligned}$$

Then,

$$\{\overline{\Pi}\circ P(\Pi(X),\Pi(Y))\Pi(Z) + \Pi\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^i = P_{abcd}X^cY^dZ^b + P_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{c}}Y^{\hat{d}}Z^{\hat{b}}.$$

Hence, the functions $\{P_{abcd}, P_{\hat{a}\hat{b}\hat{c}\hat{d}}\}$ are the components of the following tensor: $\overline{\Pi}\circ P(\Pi(X),\Pi(Y))\Pi(Z)+\Pi\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)$, and therefore, this tensor coincides with the tensor $P_{10}(X,Y)Z$ and the tenth equality is satisfied.

To prove P_{14} do the following:

$$\{\overline{\Pi}\circ P(\Pi(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^i = P_{a\hat{b}c\hat{d}}X^cY^{\hat{d}}Z^{\hat{b}}.$$

$$\{\Pi\circ P(\overline{\Pi}(X),\Pi(Y))\Pi(Z)\}^i = P_{\hat{a}b\hat{c}d}X^{\hat{c}}Y^dZ^b.$$

Then, $\{\overline{\Pi}\circ P(\Pi(X),\overline{\Pi}(Y))\overline{\Pi}(Z) + \Pi\circ P(\overline{\Pi}(X),\Pi(Y))\Pi(Z)\}^i$

$$= P_{a\hat{b}c\hat{d}}X^cY^{\hat{d}}Z^{\hat{b}} + P_{\hat{a}b\hat{c}d}X^{\hat{c}}Y^dZ^b.$$

Hence, the functions $\{P_{a\hat{b}c\hat{d}}, P_{\hat{a}b\hat{c}d}\}$ are the components of the following tensor: $\overline{\Pi}\circ P(\Pi(X),\overline{\Pi}(Y))\overline{\Pi}(Z)+\Pi\circ P(\overline{\Pi}(X),\Pi(Y))\Pi(Z)$, and therefore, this tensor coincides with the tensor $P_{14}(X,Y)Z$ and the fourteenth equality is satisfied.

The rest of the equalities can satisfy by the same way. □

Corollary 4. *Some of the invariants in Theorem 7 can be written as follow:*

$$\begin{aligned}
P_2(X,Y) &= \tfrac{1}{2}\Phi^2 \circ \{P(\xi,\Phi^2X)\Phi^2Y - P(\xi,\Phi X)\Phi Y\}\\
&\quad +\tfrac{1}{2}\{P(\xi,\Phi^2X)\Phi^2Y - P(\xi,\Phi X)\Phi Y\};\\
P_3(X,Y) &= \tfrac{1}{2}\Phi^2 \circ \{P(\xi,\Phi^2X)\Phi^2Y + P(\xi,\Phi X)\Phi Y\}\\
&\quad +\tfrac{1}{2}\{P(\xi,\Phi^2X)\Phi^2Y + P(\xi,\Phi X)\Phi Y\};\\
P_4(X,Y)Z &= \tfrac{-1}{4}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\Phi^2Z + P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi Z + P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad -\tfrac{1}{4}\{P(\Phi^2X,\Phi^2Y)\Phi^2Z + P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi Z + P(\Phi X,\Phi Y)\Phi^2Z\};\\
P_7(X,Y) &= \tfrac{-1}{4}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\xi + P(\Phi X,\Phi Y)\xi\}\\
&\quad +\tfrac{1}{4}\Phi \circ \{P(\Phi^2X,\Phi Y)\xi - P(\Phi X,\Phi^2Y)\xi\};\\
P_8(X,Y) &= \tfrac{-1}{4}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\xi - P(\Phi X,\Phi Y)\xi\}\\
&\quad +\tfrac{1}{4}\Phi \circ \{P(\Phi^2X,\Phi Y)\xi + P(\Phi X,\Phi^2Y)\xi\};\\
P_9(X,Y) &= \tfrac{-1}{4}\Phi^2 \circ \{P(\xi,\Phi^2Y)\Phi^2X + P(\xi,\Phi Y)\Phi X\}\\
&\quad +\tfrac{1}{4}\Phi \circ \{P(\xi,\Phi Y)\Phi^2X - P(\xi,\Phi^2Y)\Phi X\};\\
P_{10}(X,Y)Z &= \tfrac{1}{8}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi Z - P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad +\tfrac{1}{8}\Phi \circ \{P(\Phi^2X,\Phi^2Y)\Phi Z + P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad +P(\Phi X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi Y)\Phi Z\};\\
P_{11}(X,Y)Z &= \tfrac{1}{8}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\Phi^2Z + P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi Z + P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad +\tfrac{1}{8}\Phi \circ \{P(\Phi^2X,\Phi^2Y)\Phi Z - P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad +P(\Phi X,\Phi^2Y)\Phi^2Z + P(\Phi X,\Phi Y)\Phi Z\};\\
P_{12}(X,Y)Z &= \tfrac{1}{8}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\Phi^2Z + P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad +P(\Phi X,\Phi^2Y)\Phi Z - P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad +\tfrac{1}{8}\Phi \circ \{P(\Phi^2X,\Phi^2Y)\Phi Z - P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi Y)\Phi Z\};\\
P_{13}(X,Y) &= \tfrac{-1}{4}\Phi^2 \circ \{P(\xi,\Phi^2X)\Phi^2Y - P(\xi,\Phi X)\Phi Y\}\\
&\quad +\tfrac{1}{4}\Phi \circ \{P(\xi,\Phi^2X)\Phi Y + P(\xi,\Phi X)\Phi^2Y\};\\
P_{14}(X,Y)Z &= \tfrac{1}{8}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad +P(\Phi X,\Phi^2Y)\Phi Z + P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad +\tfrac{1}{8}\Phi \circ \{-P(\Phi^2X,\Phi^2Y)\Phi Z - P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad +P(\Phi X,\Phi^2Y)\Phi^2Z - P(\Phi X,\Phi Y)\Phi Z\};\\
P_{15}(X,Y)Z &= \tfrac{1}{8}\Phi^2 \circ \{P(\Phi^2X,\Phi^2Y)\Phi^2Z - P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi Z - P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad +\tfrac{1}{8}\Phi \circ \{-P(\Phi^2X,\Phi^2Y)\Phi Z - P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi^2Z + P(\Phi X,\Phi Y)\Phi Z\}.
\end{aligned}$$

Proof. We will explain P_2 and P_{15} and the rest will be in the same way.

According to the definitions of Π and $\overline{\Pi}$ in Lemma 1, P_2 appears as

$$\begin{aligned}
P(\xi,\Pi(X))\Pi(Y) &= P(\xi,\tfrac{-1}{2}\Phi^2X + \tfrac{-\sqrt{-1}}{2}\Phi X)(\tfrac{-1}{2}\Phi^2Y + \tfrac{-\sqrt{-1}}{2}\Phi Y),\\
&= \tfrac{1}{4}\{P(\xi,\Phi^2X)\Phi^2Y + \sqrt{-1}P(\xi,\Phi^2X)\Phi Y\\
&\quad +\sqrt{-1}P(\xi,\Phi X)\Phi^2Y - P(\xi,\Phi X)\Phi Y\}.
\end{aligned}$$

$$\begin{aligned}
\Phi^2\circ P(\xi,\Pi(X))\Pi(Y) &= \tfrac{1}{4}\Phi^2\circ\{P(\xi,\Phi^2X)\Phi^2Y+\sqrt{-1}P(\xi,\Phi^2X)\Phi Y\\
&\quad +\sqrt{-1}P(\xi,\Phi X)\Phi^2Y-P(\xi,\Phi X)\Phi Y\}.\\
P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y) &= P(\xi,\tfrac{-1}{2}\Phi^2X+\tfrac{\sqrt{-1}}{2}\Phi X)(\tfrac{-1}{2}\Phi^2Y+\tfrac{\sqrt{-1}}{2}\Phi Y),\\
&= \tfrac{1}{4}\{P(\xi,\Phi^2X)\Phi^2Y-\sqrt{-1}P(\xi,\Phi^2X)\Phi Y\\
&\quad -\sqrt{-1}P(\xi,\Phi X)\Phi^2Y-P(\xi,\Phi X)\Phi Y\}.\\
\Phi^2\circ P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y) &= \Phi^2\circ P(\xi,\tfrac{-1}{2}\Phi^2X+\tfrac{1}{2}\Phi X)(\tfrac{-1}{2}\Phi^2Y+\tfrac{1}{2}\Phi Y),\\
&= \tfrac{1}{4}\Phi^2\circ\{P(\xi,\Phi^2X)\Phi^2Y-\sqrt{-1}P(\xi,\Phi^2X)\Phi Y\\
&\quad -\sqrt{-1}P(\xi,\Phi X)\Phi^2Y-P(\xi,\Phi X)\Phi Y\}.
\end{aligned}$$

So,

$$\begin{aligned}
P_2(X,Y) &= P(\xi,\Pi(X))\Pi(Y)+\Phi^2\circ P(\xi,\Pi(X))\Pi(Y)+P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y)\\
&\quad +\Phi^2\circ P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y),\\
&= \tfrac{1}{2}\Phi^2\circ\{P(\xi,\Phi^2X)\Phi^2Y-P(\xi,\Phi X)\Phi Y\}+\tfrac{1}{2}\{P(\xi,\Phi^2X)\Phi^2Y\\
&\quad -P(\xi,\Phi X)\Phi Y\}.
\end{aligned}$$

Continued in the same manner, then P_{15} appears as:

$$\begin{aligned}
\overline{\Pi}\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z) &= \{\tfrac{-1}{2}\Phi^2+\tfrac{\sqrt{-1}}{2}\Phi\}\circ P(\tfrac{-1}{2}\Phi^2X+\tfrac{\sqrt{-1}}{2}\Phi X,\tfrac{-1}{2}\Phi^2Y\\
&\quad +\tfrac{\sqrt{-1}}{2}\Phi Y)(\tfrac{-1}{2}\Phi^2Z+\tfrac{\sqrt{-1}}{2}\Phi Z),\\
&= \tfrac{1}{16}\Phi^2\circ P(\Phi^2X,\Phi^2Y)\Phi^2Z-\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2X,\Phi^2Y)\Phi^2Z\\
&\quad -\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2X,\Phi^2Y)\Phi Z-\tfrac{1}{16}\Phi\circ P(\Phi^2X,\Phi^2Y)\Phi Z\\
&\quad -\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2X,\Phi Y)\Phi^2Z-\tfrac{1}{16}\Phi\circ P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad -\tfrac{1}{16}\Phi^2\circ P(\Phi^2X,\Phi Y)\Phi Z+\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi X,\Phi^2Y)\Phi^2Z-\tfrac{1}{16}\Phi\circ P(\Phi X,\Phi^2Y)\Phi^2Z\\
&\quad -\tfrac{1}{16}\Phi^2\circ P(\Phi X,\Phi^2Y)\Phi Z+\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi X,\Phi^2Y)\Phi Z\\
&\quad -\tfrac{1}{16}\Phi^2\circ P(\Phi X,\Phi Y)\Phi^2Z+\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi X,\Phi Y)\Phi^2Z\\
&\quad +\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi X,\Phi Y)\Phi Z+\tfrac{1}{16}\Phi\circ P(\Phi X,\Phi Y)\Phi Z\\
\Pi\circ P(\Pi(X),\Pi(Y))\Pi(Z) &= \{\tfrac{-1}{2}\Phi^2+\tfrac{-\sqrt{-1}}{2}\Phi\}\circ P(\tfrac{-1}{2}\Phi^2X+\tfrac{-\sqrt{-1}}{2}\Phi X,\tfrac{-1}{2}\Phi^2Y\\
&\quad +\tfrac{-\sqrt{-1}}{2}\Phi Y)(\tfrac{-1}{2}\Phi^2Z+\tfrac{-\sqrt{-1}}{2}\Phi Z)\\
&= \tfrac{1}{16}\Phi^2\circ P(\Phi^2X,\Phi^2Y)\Phi^2Z+\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2X,\Phi^2Y)\Phi^2Z\\
&\quad +\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2X,\Phi^2Y)\Phi Z-\tfrac{1}{16}\Phi\circ P(\Phi^2X,\Phi^2Y)\Phi Z\\
&\quad +\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2X,\Phi Y)\Phi^2Z-\tfrac{1}{16}\Phi\circ P(\Phi^2X,\Phi Y)\Phi^2Z\\
&\quad -\tfrac{1}{16}\Phi^2\circ P(\Phi^2X,\Phi Y)\Phi Z-\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad +\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi X,\Phi^2Y)\Phi^2Z-\tfrac{1}{16}\Phi\circ P(\Phi X,\Phi^2Y)\Phi^2Z\\
&\quad -\tfrac{1}{16}\Phi^2\circ P(\Phi X,\Phi^2Y)\Phi Z-\tfrac{-\sqrt{1}}{16}\Phi\circ P(\Phi X,\Phi^2Y)\Phi Z\\
&\quad -\tfrac{1}{16}\Phi^2\circ P(\Phi X,\Phi Y)\Phi^2Z-\tfrac{\sqrt{-1}}{16}\Phi\circ P(\Phi X,\Phi Y)\Phi^2Z\\
&\quad -\tfrac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi X,\Phi Y)\Phi Z+\tfrac{1}{16}\Phi\circ P(\Phi X,\Phi Y)\Phi Z.
\end{aligned}$$

So, $P_{15}(X,Y)Z=\overline{\Pi}\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)+\Pi\circ P(\Pi(X),\Pi(Y))\Pi(Z),$

$$\begin{aligned}
&= \tfrac{1}{8}\Phi^2\circ\{P(\Phi^2X,\Phi^2Y)\Phi^2Z-P(\Phi^2X,\Phi Y)\Phi Z\\
&\quad -P(\Phi X,\Phi^2Y)\Phi Z-P(\Phi X,\Phi Y)\Phi^2Z\}\\
&\quad +\tfrac{1}{8}\Phi\circ\{-P(\Phi^2X,\Phi^2Y)\Phi Z-P(\Phi^2X,\Phi Y)\Phi^2Z
\end{aligned}$$

$$-P(\Phi X, \Phi^2 Y)\Phi^2 Z + P(\Phi X, \Phi Y)\Phi Z\}. \qquad \square$$

Corollary 5. *The invariants P_α, with $\alpha = 4, 7, 8, 10, 11, 12, 14, 15$ has the identities $P_\alpha(X,Y)Z = -P_\alpha(Y,X)Z$, when $\alpha = 4, 10, 11, 12, 14, 15$ and $P_\alpha(X,Y) = -P_\alpha(Y,X)$, when $\alpha = 7, 8$ $\forall X, Y, Z \in X(M)$.*

Proof.
$$\begin{aligned} P_8(X,Y) &= \overline{\Pi} \circ P(\overline{\Pi}(X), \overline{\Pi}(Y))\xi + \Pi \circ P(\Pi(X), \Pi(Y))\xi, \\ &= -\overline{\Pi} \circ P(\overline{\Pi}(Y), \overline{\Pi}(X))\xi - \Pi \circ P(\Pi(Y), \Pi(X))\xi, \\ &= -\{\overline{\Pi} \circ P(\overline{\Pi}(Y), \overline{\Pi}(X))\xi + \Pi \circ P(\Pi(Y), \Pi(X))\xi\}, \\ &= -P_8(Y,X). \end{aligned}$$

In the same way, we can prove the identity for the rest of the invariants. $\square$

Theorem 8. *Let M be a C_9-manifold. If M of classes P_1 and P_{11}, then M is an Einstein manifold with $\lambda = A^{ac}_{ac} = -2F_{ac}F^{ac}$.*

Proof. Suppose that, M of classes P_1 & P_{11} together. Definition 6 and Theorem 2, give $r_{0a} = 0$ & $r_{ab} = 0$. Moreover, Definition 5 and equation (3) produced $\lambda = A^{ac}_{ac} = -2F_{ac}F^{ac}$. Then attains the result. $\square$

Theorem 9. *The C_9-manifold belongs to class P_1 iff it belongs to class P_9*

Proof. Suppose that, M is a C_9-manifold of class P_1. From Theorem 2, we have that $-\frac{1}{2n} r_{0d} = 0$. Hence $r_{0d} = 0$. Since $r_{0d} = r_{d0}$ and $b = d$ then $r_{b0} = 0$. Also, from Theorem 2, we have that $P_{ab0\hat{d}} = \frac{1}{2n}\{\delta^d_a r_{b0}\}$. Hence $P_{ab0\hat{d}} = 0 = P_{\hat{a}\hat{b}0d}$, then $P_9 = 0$, so that M of class P_9.

Conversely, by the same way we can show that if M of class P_9, then M of class P_1 $\square$

Theorem 10. *On C_9-manifold of dimension equal or greater than 5, then*

(1) the following statements are equivalent:
- *(a) C_9-manifold of class P_1;*
- *(b) C_9-manifold of class P_8;*
- *(c) C_9-manifold of class P_9.*

(2) the following statements are equivalent:
- *(a) C_9-manifold of class P_2;*
- *(b) C_9-manifold of class P_5;*
- *(c) C_9-manifold of class P_{11};*
- *(d) C_9-manifold of class P_{15}.*

Proof. Let M^{2n+1} be a C_9-manifold of dimension equal or greater than 5 $(n \geq 2)$. We will prove $1.(b) \Rightarrow (c)$ and $2.(a) \Rightarrow (d)$. The rest cases will be in the same way. Let M^{2n+1} of class P_8, then from Definition 6 and Theorem 2, we get $\delta^c_a r_{0\hat{d}} - \delta^d_a r_{0\hat{c}} = 0$. So, by contracting (a,c), then either $n = 1$, but this contradict with $n \geq 2$ or $r_{0\hat{d}} = 0$. This result gives us $r_{b0} = 0$, so that $P_{ab0\hat{d}} = 0$. So, by Definition 6 M^{2n+1} of class P_9. Now, let M^{2n+1} of class P_2, then by Definition 6 and Theorem 2, we get $r_{bd} = 0$. So, from Theorem 2, we get $P_{a\hat{b}\hat{c}\hat{d}} = 0$. Hence, by Definition 6, M^{2n+1} of class P_{15}. $\square$

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