NUMERICAL RADII INEQUALITIES FOR CERTAIN OPERATOR SUMS

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Abstract. We give several numerical radius inequalities for the product and the sum of operators in Hilbert space. To do this end, we employ some block matrix methods. The advantage of our results is that they extend and refine some well-known inequalities in the literature.

1. Introduction and Preliminaries

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathbb{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathbb{H}$. In the case when $\dim \mathbb{H} = n$, we identify $\mathbb{B}(\mathbb{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. For any $T \in \mathbb{B}(\mathbb{H})$, we can write $T = \mathcal{R}T + i\mathcal{I}T$ in which $\mathcal{R}T = \frac{T+T^*}{2}$ and $\mathcal{I}T = \frac{T-T^*}{2i}$ are Hermitian operators. The numerical radius of $T \in \mathbb{B}(\mathbb{H})$ is defined by

$$\omega(T) = \sup_{\substack{x\in\mathbb{H} \\ \|x\|=1}} |\langle Tx, x\rangle| .$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $T \in \mathbb{B}(\mathbb{H})$,

$$\frac{1}{2}\|T\| \le \omega(T) \le \|T\| .$$

The inequalities involving numerical radius have been particularly interesting (see, e.g., [8, 13, 17]). A principal inequality for $\omega(T)$ is the power inequality stating that $\omega(T^n) \le \omega^2(T)$, $n = 1, 2, \ldots$ [7]. It has been shown in [12], that if $T \in \mathbb{B}(\mathbb{H})$, then

$$\omega^2(T) \le \frac{1}{2}\left\||T|^2 + |T^*|^2\right\| ,$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T.

The direct sum of two copies of $\mathbb{H}$ is denoted by $\mathbb{H} \oplus \mathbb{H}$. If $A, B, C, D \in \mathbb{B}(\mathbb{H})$, then the operator matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$, which is defined

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by $Tx = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$ for every vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{H} \oplus \mathbb{H}$. Operator matrices provide a useful tool for studying Hilbert space operators, which have been extensively studied in the literature.

In this paper, we establish some generalizations of inequalities that are based on the 2×2 operator matrices. We also show some numerical radii inequalities involving the product of two and three operators.

2. Main Results

The starting point of this section is the following general numerical radius inequality.

Theorem 2.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. Then for any $0 \le v \le 1$,*

$$\omega^2 (AB + CD) \le \left\| v\left(|B|^2 + |D|^2\right)^{\frac{1}{v}} + (1-v)\left(|A^*|^2 + |C^*|^2\right)^{\frac{1}{1-v}} \right\|.$$

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathbb{H} \oplus \mathbb{H})$. For any unit vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{H} \oplus \mathbb{H}$ (as a matter of fact $\|x_1\|^2 + \|x_2\|^2 = 1$), we have

$$\begin{aligned}
|\langle TX, X\rangle| &= \left| \left\langle \begin{bmatrix} AB + CD & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\
&= \left| \left\langle \begin{bmatrix} A & C \\ O & O \end{bmatrix} \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\
&= \left| \left\langle \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\
&\le \left\| \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \left\| \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \sqrt{\left\langle \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
&= \sqrt{\left\langle \begin{bmatrix} B^* & D^* \\ O & O \end{bmatrix} \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} A & C \\ O & O \end{bmatrix} \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
&= \sqrt{\left\langle \begin{bmatrix} B^*B + D^*D & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} AA^* + CC^* & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle}.
\end{aligned}$$

Then

$$
\begin{aligned}
&|\langle TX, X\rangle| \\
&\le \sqrt{\left\langle \begin{bmatrix} |B|^2+|D|^2 & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} |A^*|^2+|C^*|^2 & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
&= \sqrt{\left\langle \begin{bmatrix} |B|^2+|D|^2 & O \\ O & O \end{bmatrix}^{\frac{v}{v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} |A^*|^2+|C^*|^2 & O \\ O & O \end{bmatrix}^{\frac{1-v}{1-v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
&\le \sqrt{\left\langle \begin{bmatrix} |B|^2+|D|^2 & O \\ O & O \end{bmatrix}^{\frac{1}{v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle^{v} \left\langle \begin{bmatrix} |A^*|^2+|C^*|^2 & O \\ O & O \end{bmatrix}^{\frac{1}{1-v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle^{1-v}} \\
&\quad \text{(by the Hölder-McCarthy inequality [6, Theorem 1.4])} \\
&\le \sqrt{v\left\langle \begin{bmatrix} |B|^2+|D|^2 & O \\ O & O \end{bmatrix}^{\frac{1}{v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle + (1-v)\left\langle \begin{bmatrix} |A^*|^2+|C^*|^2 & O \\ O & O \end{bmatrix}^{\frac{1}{1-v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
&\quad \text{(by the weighted arithmetic-geometric mean inequality)} \\
&= \sqrt{v\left\langle \begin{bmatrix} \left(|B|^2+|D|^2\right)^{\frac{1}{v}} & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle + (1-v)\left\langle \begin{bmatrix} \left(|A^*|^2+|C^*|^2\right)^{\frac{1}{1-v}} & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
&\quad \text{(see, e.g., [6, pp. 3-4])} \\
&= \sqrt{\left\langle \begin{bmatrix} v\left(|B|^2+|D|^2\right)^{\frac{1}{v}} + (1-v)\left(|A^*|^2+|C^*|^2\right)^{\frac{1}{1-v}} & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle}.
\end{aligned}
$$

Therefore,

$$\omega^2(AB+CD) \le \left\| v\left(|B|^2+|D|^2\right)^{\frac{1}{v}} + (1-v)\left(|A^*|^2+|C^*|^2\right)^{\frac{1}{1-v}} \right\|.$$

Hence, we get the desired inequality. □

Theorem 2.1 induces several numerical radius inequalities as follows.

Corollary 2.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. Then,*

$$\omega(AB+CD) \le \sqrt{\left\| |B|^2+|D|^2 \right\| \left\| |A^*|^2+|C^*|^2 \right\|}.$$

Proof. Here, we use the same method as in the proof of [9, (2.6)]. Replacing B, D by tB, tD and A, C by $\frac{1}{t}A$, $\frac{1}{t}C$, with $t > 0$, in Theorem 2.1, then

$$
\begin{aligned}
\omega^2(AB+CD) &\le \frac{1}{2}\left\| t\left(|B|^2+|D|^2\right)^2 + \frac{1}{t}\left(|A^*|^2+|C^*|^2\right)^2 \right\| \\
&\le \frac{1}{2}\left(t\left\| \left(|B|^2+|D|^2\right)^2 \right\| + \frac{1}{t}\left\| \left(|A^*|^2+|C^*|^2\right)^2 \right\| \right).
\end{aligned}
$$

Taking minimum over $t>0$,

$$\begin{aligned}\omega^2\left(AB+CD\right) &\le \sqrt{\left\|\left(|B|^2+|D|^2\right)^2\right\|\left\|\left(|A^*|^2+|C^*|^2\right)^2\right\|}\\ &= \sqrt{\left\||B|^2+|D|^2\right\|^2\left\||A^*|^2+|C^*|^2\right\|^2}\\ &= \left\||B|^2+|D|^2\right\|\left\||A^*|^2+|C^*|^2\right\|.\end{aligned}$$

Hence, we get the desired inequality. □

Corollary 2.2. *Let $T\in\mathbb{B}\left(\mathbb{H}\right)$. Then for any $0\le t,v\le 1$,*

$$\|\mathcal{R}T\|\le\frac{1}{2}\sqrt{\left\||T|^{2t}+|T^*|^{2v}\right\|\left\||T|^{2(1-v)}+|T^*|^{2(1-t)}\right\|}.$$

Proof. We mimic some ideas of [14, Theorem 2.4]. Let $T=U\left|T\right|$ be the polar decomposition of T. Then $T^*=U^*\left|T^*\right|$ is also the polar decomposition of T^* (see [5, p. 59]). Letting $A=U|T|^{1-t}$, $B=|T|^t$, $C=U^*|T^*|^{1-v}$, and $D=|T^*|^v$, with $0\le t,v\le 1$, in Corollary 2.1. Then

$$\|\mathcal{R}T\|\le\frac{1}{2}\sqrt{\left\||T|^{2t}+|T^*|^{2v}\right\|\left\||T|^{2(1-v)}+|T^*|^{2(1-t)}\right\|}.$$

□

We use some ideas of [3, Theorem 2.2] to prove the next result.

Corollary 2.3. *Let $T\in\mathbb{B}\left(\mathbb{H}\right)$. Then for any $0\le t,v\le 1$,*

$$\omega\left(T\right)\le\frac{1}{2}\sqrt{\left\||T|^{2t}+|T^*|^{2v}\right\|\left\||T|^{2(1-v)}+|T^*|^{2(1-t)}\right\|}.$$

In particular,

$$\omega\left(T\right)\le\frac{1}{2}\sqrt{\left\||T|^{2t}+|T^*|^{2t}\right\|\left\||T|^{2(1-t)}+|T^*|^{2(1-t)}\right\|}.$$

Proof. If we replace T by $e^{i\theta}T$, in Corollary 2.2, and then take supremum over $\theta\in\mathbb{R}$ (see [18]), we get

$$\omega\left(T\right)\le\frac{1}{2}\sqrt{\left\||T|^{2t}+|T^*|^{2v}\right\|\left\||T|^{2(1-v)}+|T^*|^{2(1-t)}\right\|}.$$

The case $v=t$ gives the second inequality. □

The case $t=v=1/2$, in Corollary 2.3, recovers the following well-known inequality [10, (8)]

$$\omega\left(T\right)\le\frac{1}{2}\left\||T|+|T^*|\right\|.$$

For positive operators A,B, the following facts will be needed (see [1, Theorem IX.2.1] and [11], respectively).

Lemma 2.1. *If* $0 \le p \le 1$, *then*

$$\|A^p B^p\| \le \|AB\|^p,$$

and

$$\|A+B\| \le \max\left(\|A\|, \|B\|\right) + \left\|A^{\frac{1}{2}} B^{\frac{1}{2}}\right\|.$$

Theorem 2.2. *Let* $T \in \mathbb{B}(\mathbb{H})$. *Then for any* $0 \le v \le 1$,

$$\omega(T) \le \frac{1}{2}\sqrt{\left(\|T\|^{2t} + \|T^2\|^t\right)\left(\|T\|^{2(1-t)} + \|T^2\|^{1-t}\right)}.$$

Proof. By Corollary 2.3 and Lemma 2.1 we have

$$\begin{aligned}
&\omega(T)\\
&\le \frac{1}{2}\sqrt{\left\||T|^{2t} + |T^*|^{2t}\right\| \left\||T|^{2(1-t)} + |T^*|^{2(1-t)}\right\|}\\
&\le \frac{1}{2}\sqrt{\left(\max\left(\left\||T|^{2t}\right\|, \left\||T^*|^{2t}\right\|\right) + \||T|^t|T^*|^t\|\right)\left(\max\left(\left\||T|^{2(1-t)}\right\|, \left\||T^*|^{2(1-t)}\right\|\right) + \left\||T|^{1-t}|T^*|^{1-t}\right\|\right)}\\
&= \frac{1}{2}\sqrt{\left(\|T\|^{2t} + \||T|^t|T^*|^t\|\right)\left(\|T\|^{2(1-t)} + \left\||T|^{1-t}|T^*|^{1-t}\right\|\right)}\\
&\le \frac{1}{2}\sqrt{\left(\|T\|^{2t} + \||T|\,|T^*|\|^t\right)\left(\|T\|^{2(1-t)} + \||T|\,|T^*|\|^{1-t}\right)}\\
&= \frac{1}{2}\sqrt{\left(\|T\|^{2t} + \|T^2\|^t\right)\left(\|T\|^{2(1-t)} + \|T^2\|^{1-t}\right)}.
\end{aligned}$$

Thus,

$$\omega(T) \le \frac{1}{2}\sqrt{\left(\|T\|^{2t} + \|T^2\|^t\right)\left(\|T\|^{2(1-t)} + \|T^2\|^{1-t}\right)},\ 0 \le t \le 1.$$

Hence, we get the desired inequality. □

Remark 2.1. *Notice that, the case* $t = 1/2$, *in Theorem 2.2, implies*

$$\omega(T) \le \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right).$$

In fact, Theorem 2.2 is the weighted version of [10, Theorem 1].

3. Further Inequalities

The following upper bound for the numerical radius of the product of two operators has been given in [16, Theorem 2.10]. Let $A, B \in \mathbb{B}(\mathbb{H})$. Then

$$\omega(A^*B) \le \frac{1}{4}\|AA^* + BB^*\| + \frac{1}{2}\omega(BA^*). \tag{3.1}$$

We extend this inequality to the product of three operators. We note that, in the proof of the following theorem, we used $r(T)$ to denote the spectral radius of the operator T.

Theorem 3.1. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that X is positive. Then*

$$\omega\left(A^{*}XB\right) \leq \frac{1}{4}\left\|AA^{*}X + XBB^{*}\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right).$$

Proof. We have

$$\begin{aligned}
4\omega\left(A^{*}XB\right) &= 4\omega\left(A^{*}X^{\frac{1}{2}}X^{\frac{1}{2}}B\right) \\
&= 4\omega\left(\left(X^{\frac{1}{2}}A\right)^{*}\left(X^{\frac{1}{2}}B\right)\right) \\
&\leq \left\|X^{\frac{1}{2}}\left(AA^{*} + BB^{*}\right)X^{\frac{1}{2}}\right\| + 2\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \quad \text{(by (3.1))} \\
&= r\left(X^{\frac{1}{2}}\left(AA^{*} + BB^{*}\right)X^{\frac{1}{2}}\right) + 2\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&= r\left((AA^{*} + BB^{*})^{\frac{1}{2}}X(AA^{*} + BB^{*})^{\frac{1}{2}}\right) + 2\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&= \left\|(AA^{*} + BB^{*})^{\frac{1}{2}}X(AA^{*} + BB^{*})^{\frac{1}{2}}\right\| + 2\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right).
\end{aligned}$$

Thus,

$$\omega\left(A^{*}XB\right) \leq \frac{1}{4}\left\|(AA^{*} + BB^{*})^{\frac{1}{2}}X(AA^{*} + BB^{*})^{\frac{1}{2}}\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right). \tag{3.2}$$

On the other hand, we know that if ST is Hermitian, then $\|ST\| \leq \|\mathfrak{R}TS\|$ (see [1, Proposition IX.1.2]). Using this fact, we have,

$$\begin{aligned}
\omega\left(A^{*}XB\right) &\leq \frac{1}{4}\left\|(AA^{*} + BB^{*})^{\frac{1}{2}}X(AA^{*} + BB^{*})^{\frac{1}{2}}\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&\leq \frac{1}{4}\left\|\mathfrak{R}\left(\left(AA^{*} + BB^{*}\right)X\right)\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&= \frac{1}{8}\left\|\left(AA^{*} + BB^{*}\right)X + X\left(AA^{*} + BB^{*}\right)\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&= \frac{1}{8}\left\|\left(AA^{*}X + XBB^{*}\right) + \left(XAA^{*} + BB^{*}X\right)\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&\leq \frac{1}{8}\left(\left\|AA^{*}X + XBB^{*}\right\| + \left\|XAA^{*} + BB^{*}X\right\|\right) + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right) \\
&\qquad \text{(by the triangle inequality for the usual operator norm)} \\
&= \frac{1}{4}\left\|AA^{*}X + XBB^{*}\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right).
\end{aligned}$$

Namely,

$$\omega\left(A^{*}XB\right) \leq \frac{1}{4}\left\|AA^{*}X + XBB^{*}\right\| + \frac{1}{2}\omega\left(X^{\frac{1}{2}}BA^{*}X^{\frac{1}{2}}\right),$$

as desired. □

In the remaining part of this section, we prepare two different upper bounds for the numerical radii of the product of two operators. To this end, the following lemma is required.

Lemma 3.1. *(Buzano's inequality* [2]*) Let $x, y, e \in \mathbb{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle \langle y, e\rangle| \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\| \|y\|\right).$$

For a recent paper devoted to various inequalities of Buzano types, see [15].

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(AB) \leq \frac{1}{4}\left(\left\||B|^2 + |A^*|^2\right\| + \sqrt{2}\sqrt{\omega\left(|A^*|^2|B|^2\right) + \|A\|^2\|B\|^2}\right).$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Then

$$\begin{aligned}
|\langle ABx, x\rangle| &= |\langle Bx, A^*x\rangle| \\
&\leq \|Bx\| \|A^*x\| \quad \text{(by the Cauchy-Schwarz inequality)} \\
&\leq \frac{1}{4}(\|Bx\| + \|A^*x\|)^2 \quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{4}\left(\left\langle |B|^2x, x\right\rangle^{\frac{1}{2}} + \left\langle |A^*|^2x, x\right\rangle^{\frac{1}{2}}\right)^2 \\
&= \frac{1}{4}\left(\left\langle \left(|B|^2 + |A^*|^2\right)x, x\right\rangle + 2\sqrt{\left\langle |B|^2x, x\right\rangle \left\langle |A^*|^2x, x\right\rangle}\right) \\
&\leq \frac{1}{4}\left(\left\langle \left(|B|^2 + |A^*|^2\right)x, x\right\rangle + \sqrt{2}\sqrt{\left|\left\langle |A^*|^2|B|^2x, x\right\rangle\right| + \left\||B|^2x\right\| \left\||A^*|^2x\right\|}\right) \\
&\qquad \text{(by Lemma 3.1)} \\
&\leq \frac{1}{4}\left(\left\||B|^2 + |A^*|^2\right\| + \sqrt{2}\sqrt{\omega\left(|A^*|^2|B|^2\right) + \|A\|^2\|B\|^2}\right).
\end{aligned}$$

This implies

$$\omega(AB) \leq \frac{1}{4}\left(\left\||B|^2 + |A^*|^2\right\| + \sqrt{2}\sqrt{\omega\left(|A^*|^2|B|^2\right) + \|A\|^2\|B\|^2}\right)$$

as desired. □

Theorem 3.3. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(AB) \leq \frac{1}{4}\left(\left\||B|^2 + |A^*|^2\right\| + \sqrt{2}\sqrt{\omega\left(|A^*|^2|B|^2\right) + \frac{1}{2}\left\||B|^4 + |A^*|^4\right\|}\right).$$

Proof. It has been shown in the proof of Theorem 3.2 that

$$\begin{aligned}
&|\langle ABx, x\rangle| \\
&\leq \frac{1}{4}\left(\left\langle \left(|B|^2 + |A^*|^2\right)x, x\right\rangle + \sqrt{2}\sqrt{\left|\left\langle |A^*|^2|B|^2x, x\right\rangle\right| + \left\||B|^2x\right\| \left\||A^*|^2x\right\|}\right).
\end{aligned}$$

On the other hand, by the arithmetic-geometric mean inequality, we have

$$\begin{aligned}\left\||B|^2x\right\| \left\||A^*|^2x\right\| &\le \frac{1}{2}\left(\left\||B|^2x\right\|^2 + \left\||A^*|^2x\right\|^2\right)\\ &= \frac{1}{2}\left(\left\langle |B|^4x, x\right\rangle + \left\langle |A^*|^4x, x\right\rangle\right)\\ &= \frac{1}{2}\left\langle \left(|B|^4 + |A^*|^4\right)x, x\right\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}&|\langle ABx, x\rangle|\\ &\le \frac{1}{4}\left(\left\langle \left(|B|^2 + |A^*|^2\right)x, x\right\rangle + \sqrt{2}\sqrt{\left|\left\langle |A^*|^2|B|^2x, x\right\rangle\right| + \frac{1}{2}\left\langle \left(|B|^4 + |A^*|^4\right)x, x\right\rangle}\right)\\ &\le \frac{1}{4}\left(\left\||B|^2 + |A^*|^2\right\| + \sqrt{2}\sqrt{\omega\left(|A^*|^2|B|^2\right) + \frac{1}{2}\left\||B|^4 + |A^*|^4\right\|}\right)\end{aligned}$$

i.e.,

$$|\langle ABx, x\rangle| \le \frac{1}{4}\left(\left\||B|^2 + |A^*|^2\right\| + \sqrt{2}\sqrt{\omega\left(|A^*|^2|B|^2\right) + \frac{1}{2}\left\||B|^4 + |A^*|^4\right\|}\right).$$

Taking supremum over $x \in \mathbb{H}$ with $\|x\| = 1$, we get the desired bound, which completes the proof. □

Applying a similar argument as in the proof of Corollary 2.2, one can obtain from Theorems 3.2 and 3.3:

Corollary 3.1. *Let $T \in \mathbb{B}(\mathbb{H})$. Then for any $0 \le t \le 1$,*

$$\omega(T) \le \frac{1}{4}\left(\left\||T|^{2t} + |T^*|^{2(1-t)}\right\| + \sqrt{2}\sqrt{\omega\left(|T^*|^{2(1-t)}|T|^{2t}\right) + \|T\|^2}\right),$$

and

$$\omega(T) \le \frac{1}{4}\left(\left\||T|^{2t} + |T^*|^{2(1-t)}\right\| + \sqrt{2}\sqrt{\omega\left(|T^*|^{2(1-t)}|T|^{2t}\right) + \frac{1}{2}\left\||T|^{4t} + |T^*|^{4(1-t)}\right\|}\right).$$

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References

[1] R. Bhatia, *Matrix analysis*, Springer-Verlag, New York, USA, (1997).

[2] M. L. Buzano, *Generalizzazione della diseguaglianza di Cauchy–Schwarz'*, Rend. Sem. Mat. Univ. Politech. Torino., **31** (1971/73), 405–409; (1974) (in Italian).

[3] C. Conde, M. Sababheh, and H. R. Moradi, *Some weighted numerical radius inequalities*, arXiv:2204.07620 [math.FA]

[4] S. S. Dragomir, *Kato's Type Inequalities for Bounded Linear Operators in Hilbert Spaces*. Springer Briefs in Mathematics, 2019.

[5] T. Furuta, *Invitation to linear operators*, Taylor Francis, London, (2001).

[6] T. Furuta, J. Mićić Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić method in operator inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[7] K. E. Gustafson, D. K. M. Rao, *Numerical range, the field of values of linear operators and matrices*, Springer, New York, 1997.

[8] M. Hassani, M. E. Omindvar, and H. R. Moradi, *New estimates on numerical radius and operator norm of Hilbert space operators*, Tokyo J. Math., 44(2) (2021), 439–449.

[9] O. Hirzallah, F. Kittaneh, and K. Shebrawi, *Numerical radius inequalities for commutators of Hilbert space operators*, Numer. Funct. Anal. Optim., 32 (2011), 739–749.

[10] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the F robenius companion matrix*, Studia Math., 158(1) (2003), 11–17.

[11] F. Kittaneh, *Norm inequalities for certain operator sums*, J. Funct. Anal, 143 (1997), 337–348.

[12] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math., 168(1) (2005), 73–80.

[13] F. Kittaneh, H. R. Moradi, *Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl., 23(3) (2020), 1117–1125.

[14] M. Sababheh, C. Conde, and H. R. Moradi, *Further numerical radius inequalities*, arXiv:2204.12045 [math.FA]

[15] M. Sababheh, H. R. Moradi, and Z. Heydarbeygi, *Buzano, Kreǐn and Cauchy-Schwarz inequalities*, Oper. Matrices., 16(1) (2022), 239–250.

[16] M. Sattari, M. S. Moslehian, and T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl., 470 (2015), 216–227.

[17] S. Sheybani, M. Sababheh, and H. R. Moradi, *Weighted inequalities for the numerical radius*, Vietnam J. Math., (2021). https://doi.org/10.1007/s10013-021-00533-4

[18] T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math., 178 (2007), 83–89.

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