

NUMERICAL RADII INEQUALITIES FOR CERTAIN OPERATOR SUMS

ALI ZAND VAKILI AND ALI FAROKHINIA

ABSTRACT. We give several numerical radius inequalities for the product and the sum of operators in Hilbert space. To do this end, we employ some block matrices method. The advantage of our results is that they extend and refine some well-known inequalities in the literature.

1. INTRODUCTION AND PRELIMINARIES

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathbb{H})$ denotes the C^* -algebra of all bounded linear operators on \mathbb{H} . In the case when $\dim \mathbb{H} = n$, we identify $\mathbb{B}(\mathbb{H})$ with the matrix algebra \mathbb{M}_n of all $n \times n$ matrices with entries in the complex field. For any $T \in \mathbb{B}(\mathbb{H})$, we can write $T = \mathcal{R}T + i\mathcal{I}T$ in which $\mathcal{R}T = \frac{T+T^*}{2}$ and $\mathcal{I}T = \frac{T-T^*}{2i}$ are Hermitian operators. The numerical radius of $T \in \mathbb{B}(\mathbb{H})$ is defined by

$$\omega(T) = \sup_{\substack{x \in \mathbb{H} \\ \|x\|=1}} |\langle Tx, x \rangle|.$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $T \in \mathbb{B}(\mathbb{H})$,

$$\frac{1}{2} \|T\| \leq \omega(T) \leq \|T\|.$$

The inequalities involving numerical radius have been of particular interest (see, e.g., [6, 11, 15]). A principal inequality for $\omega(T)$ is the power inequality stating that $\omega(T^n) \leq \omega^2(T)$, $n = 1, 2, \dots$ [5]. It has been shown in [10], that if $T \in \mathbb{B}(\mathbb{H})$, then

$$\omega^2(T) \leq \frac{1}{2} \left(\| |T|^2 + |T^*|^2 \| \right),$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T .

Let $\mathbb{H}_1, \mathbb{H}_2$ be Hilbert spaces, and consider the direct sum $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$. respecting to this decomposition, every operator $T \in \mathbb{B}(\mathbb{H})$ has a 2×2 operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathbb{B}(\mathbb{H}_j, \mathbb{H}_i)$, the space of all bounded linear operators from \mathbb{H}_j to \mathbb{H}_i

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($1 \leq i, j \leq 2$). Operator matrices provide a useful tool for studying Hilbert space operators, which have been extensively studied in the literature.

In this paper, we establish some generalizations of inequalities that are based on the 2×2 operator matrices. We also show some numerical radii inequalities involving the product of two and three operators.

2. MAIN RESULTS

The starting point of this section is the following general numerical radius inequality.

Theorem 2.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. Then for any $0 \leq v \leq 1$,*

$$\omega^2(AB + CD) \leq \left\| v(|B|^2 + |D|^2)^{\frac{1}{v}} + (1-v)(|A^*|^2 + |C^*|^2)^{\frac{1}{1-v}} \right\|.$$

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathbb{H}_2, \mathbb{H}_1)$. For any unit vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{H}_1 \oplus \mathbb{H}_2$ (as a matter of fact $\|x_1\|^2 + \|x_2\|^2 = 1$), we have

$$\begin{aligned} |\langle TX, X \rangle| &= \left| \left\langle \begin{bmatrix} AB + CD & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\ &= \left| \left\langle \begin{bmatrix} A & C \\ O & O \end{bmatrix} \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\ &= \left| \left\langle \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \\ &\leq \left\| \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \left\| \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \quad (\text{by the Cauchy-Schwarz inequality}) \\ &= \sqrt{\left\langle \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\ &= \sqrt{\left\langle \begin{bmatrix} B^* & D^* \\ O & O \end{bmatrix} \begin{bmatrix} B & O \\ D & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} A & C \\ O & O \end{bmatrix} \begin{bmatrix} A^* & O \\ C^* & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\ &= \sqrt{\left\langle \begin{bmatrix} B^*B + D^*D & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} AA^* + CC^* & O \\ O & O \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle}. \end{aligned}$$

Then

$$\begin{aligned}
& |\langle TX, X \rangle| \\
& \leq \sqrt{\left\langle \left[\begin{array}{cc|c} |B|^2 + |D|^2 & O & \\ O & O & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \left[\begin{array}{cc|c} |A^*|^2 + |C^*|^2 & O & \\ O & O & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
& = \sqrt{\left\langle \left[\begin{array}{cc|c} |B|^2 + |D|^2 & O & \\ O & O & \end{array} \right]^{\frac{v}{v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \left\langle \left[\begin{array}{cc|c} |A^*|^2 + |C^*|^2 & O & \\ O & O & \end{array} \right]^{\frac{1-v}{1-v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
& \leq \sqrt{\left\langle \left[\begin{array}{cc|c} |B|^2 + |D|^2 & O & \\ O & O & \end{array} \right]^{\frac{1}{v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle^v \left\langle \left[\begin{array}{cc|c} |A^*|^2 + |C^*|^2 & O & \\ O & O & \end{array} \right]^{\frac{1}{1-v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle^{1-v}} \\
& \leq \sqrt{v \left\langle \left[\begin{array}{cc|c} |B|^2 + |D|^2 & O & \\ O & O & \end{array} \right]^{\frac{1}{v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle + (1-v) \left\langle \left[\begin{array}{cc|c} |A^*|^2 + |C^*|^2 & O & \\ O & O & \end{array} \right]^{\frac{1}{1-v}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
& = \sqrt{v \left\langle \left[\begin{array}{cc|c} (|B|^2 + |D|^2)^{\frac{1}{v}} & O & \\ O & O & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle + (1-v) \left\langle \left[\begin{array}{cc|c} (|A^*|^2 + |C^*|^2)^{\frac{1}{1-v}} & O & \\ O & O & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle} \\
& = \sqrt{\left\langle \left[\begin{array}{cc|c} v(|B|^2 + |D|^2)^{\frac{1}{v}} + (1-v)(|A^*|^2 + |C^*|^2)^{\frac{1}{1-v}} & O & \\ O & O & \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle}.
\end{aligned}$$

Therefore,

$$\omega^2(AB + CD) \leq \left\| v(|B|^2 + |D|^2)^{\frac{1}{v}} + (1-v)(|A^*|^2 + |C^*|^2)^{\frac{1}{1-v}} \right\|.$$

Hence we get the desired inequality. \square

Theorem 2.1 induces several numerical radius inequalities as follows.

Corollary 2.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. Then,*

$$\omega(AB + CD) \leq \sqrt{\| |B|^2 + |D|^2 \| \| |A^*|^2 + |C^*|^2 \|}.$$

Proof. Here we use the same method as in the proof of [8, (2.6)]. Replacing B, D by tB, tD and A, C by $\frac{1}{t}A, \frac{1}{t}C$, with $t > 0$, in Theorem 2.1, then

$$\begin{aligned}
\omega^2(AB + CD) & \leq \frac{1}{2} \left\| t(|B|^2 + |D|^2)^2 + \frac{1}{t}(|A^*|^2 + |C^*|^2)^2 \right\| \\
& \leq \frac{1}{2} \left(t \left\| (|B|^2 + |D|^2)^2 \right\| + \frac{1}{t} \left\| (|A^*|^2 + |C^*|^2)^2 \right\| \right).
\end{aligned}$$

Taking minimum over $t > 0$,

$$\begin{aligned}\omega^2(AB + CD) &\leq \sqrt{\|(|B|^2 + |D|^2)^2\| \|(|A^*|^2 + |C^*|^2)^2\|} \\ &= \sqrt{\| |B|^2 + |D|^2 \|^2 \| |A^*|^2 + |C^*|^2 \|^2} \\ &= \| |B|^2 + |D|^2 \| \| |A^*|^2 + |C^*|^2 \|.\end{aligned}$$

Hence we get the desired inequality. \square

Corollary 2.2. *Let $T \in \mathbb{B}(\mathbb{H})$. Then for any $0 \leq t, v \leq 1$,*

$$\|\mathcal{R}T\| \leq \frac{1}{2} \sqrt{\| |T|^{2t} + |T^*|^{2v} \| \| |T|^{2(1-v)} + |T^*|^{2(1-t)} \|}.$$

Proof. We mimic some ideas of [12, Theorem 2.4]. Let $T = U|T|$ be the polar decomposition of T . Then $T^* = U^*|T^*|$ is also the polar decomposition of T^* (see [4, p. 59]). Letting $A = U|T|^{1-t}$, $B = |T|^t$, $C = U^*|T^*|^{1-v}$, and $D = |T^*|^v$, with $0 \leq t, v \leq 1$, in Theorem 2.1. Then

$$\|\mathcal{R}T\| \leq \frac{1}{2} \sqrt{\| |T|^{2t} + |T^*|^{2v} \| \| |T|^{2(1-v)} + |T^*|^{2(1-t)} \|}.$$

\square

We use some ideas of [3, Theorem 2.2] to prove the next result.

Corollary 2.3. *Let $T \in \mathbb{B}(\mathbb{H})$. Then for any $0 \leq t, v \leq 1$,*

$$\omega(T) \leq \frac{1}{2} \sqrt{\| |T|^{2t} + |T^*|^{2v} \| \| |T|^{2(1-v)} + |T^*|^{2(1-t)} \|}.$$

In particular,

$$\omega(T) \leq \frac{1}{2} \sqrt{\| |T|^{2t} + |T^*|^{2t} \| \| |T|^{2(1-t)} + |T^*|^{2(1-t)} \|}.$$

Proof. If we replace T by $e^{i\theta}T$, in Corollary 2.2, and then take supremum over $\theta \in \mathbb{R}$ (see [16]), we get

$$\omega(T) \leq \frac{1}{2} \sqrt{\| |T|^{2t} + |T^*|^{2v} \| \| |T|^{2(1-v)} + |T^*|^{2(1-t)} \|}.$$

The case $v = t$ gives the second inequality. \square

The case $t = v = 1/2$, in Corollary 2.3, recovers the following well-known inequality [9, (8)]

$$\omega(T) \leq \frac{1}{2} \| |T| + |T^*| \|.$$

For positive operators A, B , the following fact will be needed.

Lemma 2.1. [1, Theorem IX.2.1] *If $0 \leq p \leq 1$, then*

$$\|A^p B^p\| \leq \|AB\|^p,$$

and

$$\|A + B\| \leq \max(\|A\|, \|B\|) + \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|.$$

Theorem 2.2. *Let $T \in \mathbb{B}(\mathbb{H})$. Then for any $0 \leq v \leq 1$,*

$$\omega(T) \leq \frac{1}{2} \sqrt{(\|T\|^{2t} + \|T^2\|^t) (\|T\|^{2(1-t)} + \|T^2\|^{1-t})}.$$

Proof. By Corollary 2.2,

$$\begin{aligned} \omega(T) &\leq \frac{1}{2} \sqrt{\| |T|^{2t} + |T^*|^{2t} \| \| |T|^{2(1-t)} + |T^*|^{2(1-t)} \|} \\ &\leq \frac{1}{2} \sqrt{\left(\max(\| |T|^{2t} \|, \| |T^*|^{2t} \|) + \| |T|^t |T^*|^t \| \right) \left(\max(\| |T|^{2(1-t)} \|, \| |T^*|^{2(1-t)} \|) + \| |T|^{1-t} |T^*|^{1-t} \| \right)} \\ &= \frac{1}{2} \sqrt{\left(\|T\|^{2t} + \| |T|^t |T^*|^t \| \right) \left(\|T\|^{2(1-t)} + \| |T|^{1-t} |T^*|^{1-t} \| \right)} \\ &\leq \frac{1}{2} \sqrt{\left(\|T\|^{2t} + \| |T| |T^*| \| \right) \left(\|T\|^{2(1-t)} + \| |T| |T^*| \| \right)} \\ &= \frac{1}{2} \sqrt{\left(\|T\|^{2t} + \|T^2\|^t \right) \left(\|T\|^{2(1-t)} + \|T^2\|^{1-t} \right)}. \end{aligned}$$

Thus,

$$\omega(T) \leq \frac{1}{2} \sqrt{\left(\|T\|^{2t} + \|T^2\|^t \right) \left(\|T\|^{2(1-t)} + \|T^2\|^{1-t} \right)}, \quad 0 \leq t \leq 1.$$

Hence we get the desired inequality. \square

Remark 2.1. *Notice that, the case $t = 1/2$, in Theorem 2.2, implies*

$$\omega(T) \leq \frac{1}{2} \left(\|T\| + \|T^2\|^{\frac{1}{2}} \right).$$

In fact, Theorem 2.2 is the weighted version of [9, Theorem 1].

The following result provides an alternative proof for [7, Theorem 2.4].

Proposition 2.1. *Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(AB \pm CD) \leq 2\omega \left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix} \right).$$

Proof. By [7, (4.11)] in

$$|\langle (A + B)x, x \rangle| = |\langle Ax, x \rangle + \langle Bx, x \rangle| \leq 2\|x\|^2 \omega \left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right),$$

for all $x \in \mathbb{H}$. Therefore,

$$\omega(AB + CD) \leq 2\omega \left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix} \right).$$

By [7, Lemma 2.1 (b)], we also have

$$\omega(AB - CD) \leq 2\omega \left(\begin{bmatrix} O & AB \\ -CD & O \end{bmatrix} \right) = 2\omega \left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix} \right).$$

So,

$$\omega(AB \pm CD) \leq 2\omega \left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix} \right).$$

□

The following upper bound for the numerical radius of the product of two operators has been given in [14, Theorem 2.10]. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$(2.1) \quad \omega(A^*B) \leq \frac{1}{4} \|AA^* + BB^*\| + \frac{1}{2} \omega(BA^*).$$

We extend this inequality to the product of three operators.

Theorem 2.3. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ such that X is positive. Then*

$$\omega(A^*XB) \leq \frac{1}{4} \|AA^*X + XBB^*\| + \frac{1}{2} \omega(X^{\frac{1}{2}}BA^*X^{\frac{1}{2}}).$$

Proof. We have

$$\begin{aligned} 4\omega(A^*XB) &= 4\omega\left(A^*X^{\frac{1}{2}}X^{\frac{1}{2}}B\right) \\ &= 4\omega\left(\left(X^{\frac{1}{2}}A\right)^*\left(X^{\frac{1}{2}}B\right)\right) \\ &\leq \left\|X^{\frac{1}{2}}(AA^* + BB^*)X^{\frac{1}{2}}\right\| + 2\omega\left(X^{\frac{1}{2}}BA^*X^{\frac{1}{2}}\right) \quad (\text{by (2.1)}) \\ &= r\left(X^{\frac{1}{2}}(AA^* + BB^*)X^{\frac{1}{2}}\right) + 2\omega\left(X^{\frac{1}{2}}BA^*X^{\frac{1}{2}}\right) \\ &= r\left(\left(AA^* + BB^*\right)^{\frac{1}{2}}X\left(AA^* + BB^*\right)^{\frac{1}{2}}\right) + 2\omega\left(X^{\frac{1}{2}}BA^*X^{\frac{1}{2}}\right) \\ &= \left\|\left(AA^* + BB^*\right)^{\frac{1}{2}}X\left(AA^* + BB^*\right)^{\frac{1}{2}}\right\| + 2\omega\left(X^{\frac{1}{2}}BA^*X^{\frac{1}{2}}\right). \end{aligned}$$

Thus,

$$(2.2) \quad \omega(A^*XB) \leq \frac{1}{4} \left\|\left(AA^* + BB^*\right)^{\frac{1}{2}}X\left(AA^* + BB^*\right)^{\frac{1}{2}}\right\| + \frac{1}{2} \omega\left(X^{\frac{1}{2}}BA^*X^{\frac{1}{2}}\right).$$

On the other hand, we know that if ST is Hermitian, then $\|ST\| \leq \|\Re TS\|$ (see [1, Proposition IX.1.2]). Using this fact, we have,

$$\begin{aligned}
\omega(A^*XB) &\leq \frac{1}{4} \left\| (AA^* + BB^*)^{\frac{1}{2}} X (AA^* + BB^*)^{\frac{1}{2}} \right\| + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right) \\
&\leq \frac{1}{4} \|\Re((AA^* + BB^*)X)\| + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right) \\
&= \frac{1}{8} \|(AA^* + BB^*)X + X(AA^* + BB^*)\| + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right) \\
&= \frac{1}{8} \|(AA^*X + XBB^*) + (XAA^* + BB^*X)\| + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right) \\
&\leq \frac{1}{8} (\|AA^*X + XBB^*\| + \|XAA^* + BB^*X\|) + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right) \\
&\quad \text{(by the triangle inequality for the usual operator norm)} \\
&= \frac{1}{4} \|AA^*X + XBB^*\| + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right).
\end{aligned}$$

Namely,

$$\omega(A^*XB) \leq \frac{1}{4} \|AA^*X + XBB^*\| + \frac{1}{2} \omega \left(X^{\frac{1}{2}} B A^* X^{\frac{1}{2}} \right),$$

as desired. □

In the remaining part of this section, we prepare two different upper bounds for the numerical radii of the product of two operators. To this end, the following lemma is required.

Lemma 2.2. (*Buzano's inequality* [2]) *Let $x, y, e \in \mathbb{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e \rangle \langle y, e \rangle| \leq \frac{1}{2} (|\langle x, y \rangle| + \|x\| \|y\|).$$

For a recent paper devoted to various inequalities of Buzano types, see [13].

Theorem 2.4. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(AB) \leq \frac{1}{4} \left(\| |B|^2 + |A^*|^2 \| + \sqrt{2} \sqrt{\omega(|A^*|^2 |B|^2) + \|A\|^2 \|B\|^2} \right).$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Then

$$\begin{aligned}
|\langle ABx, x \rangle| &= |\langle Bx, A^*x \rangle| \\
&\leq \|Bx\| \|A^*x\| \quad (\text{by the Cauchy-Schwarz inequality}) \\
&\leq \frac{1}{4}(\|Bx\| + \|A^*x\|)^2 \quad (\text{by the arithmetic-geometric mean inequality}) \\
&= \frac{1}{4} \left(\langle |B|^2 x, x \rangle^{\frac{1}{2}} + \langle |A^*|^2 x, x \rangle^{\frac{1}{2}} \right)^2 \\
&= \frac{1}{4} \left(\langle (|B|^2 + |A^*|^2) x, x \rangle + 2\sqrt{\langle |B|^2 x, x \rangle \langle |A^*|^2 x, x \rangle} \right) \\
&\leq \frac{1}{4} \left(\langle (|B|^2 + |A^*|^2) x, x \rangle + \sqrt{2} \sqrt{|\langle |A^*|^2 |B|^2 x, x \rangle| + \| |B|^2 x \| \| |A^*|^2 x \|} \right) \\
&\quad (\text{by Lemma 2.2}) \\
&\leq \frac{1}{4} \left(\| |B|^2 + |A^*|^2 \| + \sqrt{2} \sqrt{\omega(|A^*|^2 |B|^2) + \|A\|^2 \|B\|^2} \right).
\end{aligned}$$

This implies

$$\omega(AB) \leq \frac{1}{4} \left(\| |B|^2 + |A^*|^2 \| + \sqrt{2} \sqrt{\omega(|A^*|^2 |B|^2) + \|A\|^2 \|B\|^2} \right)$$

as desired. \square

Theorem 2.5. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(AB) \leq \frac{1}{4} \left(\| |B|^2 + |A^*|^2 \| + 2\sqrt{\omega(|A^*|^2 |B|^2) + \frac{1}{2} \| |B|^4 + |A^*|^4 \|} \right).$$

Proof. Employ the arithmetic-geometric mean inequality instead of Lemma 2.2, in the proof of Theorem 2.4, one get

$$\begin{aligned}
|\langle ABx, x \rangle| &\leq \frac{1}{4} \left(\langle (|B|^2 + |A^*|^2) x, x \rangle + 2\sqrt{|\langle |A^*|^2 |B|^2 x, x \rangle| + \frac{1}{2} (\| |B|^2 x \|^2 + \| |A^*|^2 x \|^2)} \right) \\
&= \frac{1}{4} \left(\langle (|B|^2 + |A^*|^2) x, x \rangle + 2\sqrt{|\langle |A^*|^2 |B|^2 x, x \rangle| + \frac{1}{2} \langle (|B|^4 + |A^*|^4) x, x \rangle} \right) \\
&\leq \frac{1}{4} \left(\| |B|^2 + |A^*|^2 \| + 2\sqrt{\omega(|A^*|^2 |B|^2) + \frac{1}{2} \| |B|^4 + |A^*|^4 \|} \right).
\end{aligned}$$

Taking supremum over $x \in \mathbb{H}$ with $\|x\| = 1$, we get the desired bound, which completes the proof. \square

Applying a similar argument as in the proof of Corollary 2.2, one can obtain from Theorems 2.4 and 2.5:

Corollary 2.4. *Let $T \in \mathbb{B}(\mathbb{H})$. Then for any $0 \leq t \leq 1$,*

$$\omega(T) \leq \frac{1}{4} \left(\left\| |T|^{2t} + |T^*|^{2(1-t)} \right\| + \sqrt{2} \sqrt{\omega \left(|T^*|^{2(1-t)} |T|^{2t} \right) + \|T\|^2} \right),$$

and

$$\omega(T) \leq \frac{1}{4} \left(\left\| |T|^{2t} + |T^*|^{2(1-t)} \right\| + 2 \sqrt{\omega \left(|T^*|^{2(1-t)} |T|^{2t} \right) + \frac{1}{2} \left\| |T|^{4t} + |T^*|^{4(1-t)} \right\|} \right).$$

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(A Zand Vakili) Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran

E-mail address: alizvakili982@gmail.com

(A Farokhinia) Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran

E-mail address: farokhinia@iaushiraz.ac.ir