ON THE BOUNDEDNESS OF COMMUTATORS OF SINGULAR INTEGRAL OPERATOR AND RIESZ POTENTIAL IN THE GLOBAL MORREY-TYPE SPACES WITH VARIABLE EXPONENTS

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Abstract. We consider the global Morrey-type spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$, $\Omega \subset R^n$ with variable exponents $p(x)$, $\theta(x)$ and general function $w(x,r)$ defining these spaces. In the case of unbounded sets $\Omega \subset \mathbb{R}^n$, we give sufficient conditions for the boundedness of the commutators of singular integral operator and Riesz potential in these spaces.

1 Introduction

In this paper we consider the global Morrey-type spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponents $p(.)$,$\theta(.)$ and a general function $w(x,r)$ defining a Morrey-type norm. The Morrey spaces $M_{p,\lambda}$ introduced in [15] in relation to the study of partial differential equations. The classical result for Calderon-Zygmund operators states that if $1 < p < \infty$ then T is bounded from $L_p(R^n)$ to $L_p(R^n)$, and if $p = 1$ then T is bounded from L_1 to WL_1.(see, for example [3]).

J.Peetre [4] prooved that, if $1 < p < \infty$, $0 < \lambda < n$, then T is bounded from $\mathcal{L}_{p,\lambda}$ to $\mathcal{L}_{p,\lambda}$. Many classical operators of harmonic analysis (for example, maximal,fractional maximal,potential operators) were studied in the Morrey type spaces with constant exponents p,θ ([[9]). The Morrey spaces also attracted attention of researchers in the area of variable exponent analysis ([6]-[7],[11]-[12]). The Morrey spaces $\mathcal{L}_{p(.),\lambda(.)}$ with variable exponent $p(.)$,$\lambda(.)$ were introduced and studied in [1].The general version $M_{p(.),w(.)}(\Omega)$, $\Omega \subset \mathbb{R}^n$ were introduced and studied in [11] in the case of bounded sets $\Omega \subset \mathbb{R}^n$ and in [12] in the case of unbounded sets $\Omega \subset \mathbb{R}^n$. The boundedness of Calderon Zygmund singular integral operator in the generalized Morrey type spaces with a variable exponent were considered in [11] in the case of bounded sets $\Omega \subset \mathbb{R}^n$, in [12] in the case of unbounded sets $\Omega \subset \mathbb{R}^n$.

Let $f \in L_{loc}(\mathbb{R}^n)$.The Calderon-Zygmund singular integral operator defined as

$$Tf(x) = \int_{R^n} K(x,y)f(y)dy, \tag{1.1}$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $(x,y) \in \Omega \times \Omega : x \neq y$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n},$$

for all $x \neq y$,

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^{\delta}}{|x-y|^{n+\delta}}, \delta > 0,$$

if |x-y|>2|y-z|,

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^{\delta}}{|x-y|^{n+\delta}}, \delta > 0,$$

if $|x-y| > 2|x-\xi|$.

The Riesz potential I^{α} with exponent α is defined by :

$$I^{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, 0 < \alpha < n.$$

Here and below, we denote by $B(x,r)$ the ball with center $x \in \mathbb{R}^n$ and radius $r > 0$, $\tilde{B}(x,r) = B(x,r) \cap \Omega$, $\Omega \subset \mathbb{R}^n$.

The space $BMO(\Omega)$ is defined as the space of all integrable functions f with finite norm

$$\|f\|_{BMO} = \|f\|_{*} = \sup_{x\in\Omega, r>0} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy,$$

where $f_{\tilde{B}(x,r)} = |\tilde{B}(x,r)|^{-1} \int_{\tilde{B}(x,r)} f(y)dy$.

Let $b \in BMO(\Omega)$. The commutator of the singular integral operator is defined by

$$[b,T]f = T(bf) - b(Tf) = \int_{R^n} K(x,y)(b(y) - b(x))f(y)dy.$$

The boundedness of commutators of singular integral operator in the generalized weighted Morrey spaces with variable exponent was prooved in [13].

Let $b \in BMO(\Omega)$. The commutator of the Riesz potential is defined by

$$[b, I^{\alpha}]f = I^{\alpha}(bf) - b(I^{\alpha}f) = \int_{R^n} \frac{(b(y) - b(x))}{|x-y|^{n-\alpha}} f(y)dy, 0 < \alpha < n.$$

The boundedness of commutators of Riesz potential in the generalized weighted Morrey spaces with variable exponent was prooved in [14].

2 Preliminaries.Variable Exponent Lebesgue Spaces $L_{p(.)}$. Generalized variable exponent Morrey spaces $M_{p(.),w(.)}$

Let $p(x)$ be a measurable function on an open set $\Omega \subset \mathbb{R}^n$ with values $(1,\infty)$. Let

$$1 < p_{-} \leq p(x) \leq p_{+} < \infty \tag{2.1}$$

where $p_- = p_-(\Omega) = esinf_{x\in\Omega}\, p(x)$, $p_+ = p_+(\Omega) = essup_{x\in\Omega}\, p\,(x)$. We denote by $L_{p(.)}(\Omega)$ the space of all measurable functions $f(x)$ on Ω such that

$$J_{p(.)}(f) = \int_{\Omega} [f(x)]^{p(x)}\, dx < \infty,$$

where the norm is defined as follows

$$||f||_{p(.)} = \inf\left\{\eta > 0, J_{p(.)}\left(\frac{f}{\eta}\right) \leq 1\,\right\}.$$

This is a Banach space. The conjugate exponent $p^{'}$ is defined by the formula

$$p^{'}(x) = \frac{p(x)}{p(x)-1}$$

Holder inequality for the variable exponents $p(.)$, $p^{'}(.)$

$$\int_{\Omega} f(x)g(x)dx \leq C(p)\|f\|_{L_{p(.)}(\Omega)}\|g\|_{L_{p^{'}(.)}(\Omega)},$$

where $C(p) = \frac{1}{p_-} + \frac{1}{p^{'}_-}$.

$\mathcal{P}(\Omega)$ is the set of measurable functions $p : \Omega \to [1, \infty)$, $\mathcal{P}^{\log}(\Omega)$ is the set of measurable functions $p(x)$ satisfying the local log -condition

$$|p(x) - p(y)| \leq \frac{A_p}{-ln\,|x-y|}, : |x-y| \leq \frac{1}{2}, x, y \in \Omega$$

where A_p is independent of x и y. $\mathbb{P}^{\log}(\Omega)$ is the set of measurable functions $p(x)$ satisfying 2.1 and the log-condition. In case Ω is a unbounded set, we denote by $\mathbb{P}^{log}_{\infty}(\Omega)$ the set of exponents which is a subset of the set of $\mathbb{P}^{\log}(\Omega)$ and satisfying the decay condition

$$|p(x) - p(\infty)| \leq A_\infty ln(2+|x|), x \in \mathbb{R}^n. \tag{2.2}$$

$\mathbb{A}^{log}(\Omega)$ is the set of bounded exponents $\alpha : \Omega \to \mathbb{R}$ satisfying the log-condition.

Let Ω be a open bounded set, $p \in \mathbb{P}^{\log}(\Omega)$ and $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}_{p(.),\lambda(.)}(\Omega)$ was introduced in [1] with the norm

$$\|f\|_{\mathcal{L}_{p(.),\lambda(.)}(\Omega)} = \sup_{x\in\Omega, t>0} t^{-\frac{\lambda(x)}{p(x)}}\|f\|_{L_{p(.)}(\tilde{B}(x,t))}.$$

Let $w(x, r)$ - nonnegative measurable function on Ω, where $\Omega \subset \mathbb{R}^n$ is a open bounded set .The generalized Morrey type space $M_{p(.),w(.)}(\Omega)$ with variable exponent was defined in [11] with the norm

$$||f||_{M_{p(.),w(.)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{w(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))}.$$

Let $w(x,r)$ - nonnegative measurable function on Ω, where $\Omega \subset \mathbb{R}^n$ is a open unbounded set .The generalized Morrey type space $M_{p(.),w(.)}(\Omega)$ with variable exponent was defined in [12] with the norm

$$||f||_{M_{p(.),w(.)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{||f||_{L_{p(.)}(\tilde{B}(x,r))}}{w(x,r)}.$$

Let

$$\eta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & \text{if } r \leq 1; \\ \frac{n}{p(\infty)}, & \text{if } r > 1. \end{cases}$$

Definition 1. Let $p \in P^{log}(\Omega)$, $w(x,r)$ is a positive function on $\Omega \times [0,\infty]$, where $\Omega \in R^n$. The Global Morrey type spaces with variable exponents $GM_{p(.),\theta(.),w(.)}(\Omega)$ is defined as the set of functions $f \in L^{loc}_{p(.)}(\Omega)$ with finite norm

$$||f||_{GM_{p(.),\theta(.),w(.)}(\Omega)} = \sup_{x\in\Omega} ||w(x,r) r^{-\eta_p(x,r)} ||f||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\infty)}. \tag{2.3}$$

We assume that the positive measurable function $w(x,r)$ satisfies the condition

$$\sup_{x\in\Omega} ||w(x,r)||_{L_{\theta(.)}(0,\infty)} < \infty.$$

Then the space contains bounded functions and thereby is nonempty. In case $w(x,r) = r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}$the corresponding space is denoted by $GM^{\lambda(.)}_{p(.),\theta(.)}$:

$$GM^{\lambda(.)}_{p(.),\theta(.)}(\Omega) = GM_{p(.),w(.),\theta}\big|_{w(x,r)=r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}},$$

$$||f||_{GM^{\lambda(.)}_{p(.),\theta(.)}(\Omega)} = \sup_{x\in\Omega} ||r^{-\frac{\lambda(x)}{p(x)}} ||f||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\infty)}.$$

In case $\theta = \infty$ the space $GM_{p(.),\infty,w(.)}(\Omega)$ concides the generalized Morrey space with variable exponent $M_{p(.),w(.)}(\Omega)$ with finite quasi-norm

$$||f||_{M_{p(.),w(.)}(\Omega)} = \sup_{x\in\Omega} w(x,r) r^{-\eta_p(x,r)} ||f||_{L_{p(.)}(\tilde{B}(x,r))}.$$

If $p(.) = p = const$, $\theta(x) = \theta = const$ the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ coincides with the ordinary global Morrey space $GM_{p,\theta,w}(\Omega)$, which was considered in the works of V. I. Burenkov, V. Guliev and others [9].

3 The auxiliary theorems

The next theorem was proved in [13]

Theorem 3.1. *Let $\Omega \subset R^n$ be an unbounded domain $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $b \in BMO(\Omega)$. Then*

$$||[b,T]f]||_{L_{p(.)}(\tilde{B}(x,t))} \leq C\|b\|_* t^{\eta_p(x,t)} \int_t^\infty (1+ln\frac{r}{t}) r^{-\eta_p(x,r)-1} ||f||_{L_{p(.)}(\tilde{B}(x,r))} dr, \tag{3.1}$$

where C does not depend on $x \in \Omega$ and $t > 0$.

The next theorem was proved in [14]

Theorem 3.2. *Let $\Omega \subset R^n$ be an unbounded domain, $0<\alpha<n$, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$, $b \in BMO(\Omega)$.Then*

$$||[b, I^{\alpha} f]||_{L_{q(.)}(\tilde{B}(x,t))} \leq C\|b\|_* t^{\eta_q(x,t)} \int_t^{\infty} (1+ln\frac{r}{t}) r^{-\eta_q(x,r)-1} ||f||_{L_{p(.)}(\tilde{B}(x,r))} dr, \qquad (3.2)$$

where C does not depend on $x \in \Omega$ and $t>0$.

Let u and v be a positive measurable functions.The dual Hardy operator is defined by the identity

$$\tilde{H}_{v,u} f(x) = v(x) \int_x^{\infty} f(t)u(t)dt, x \in R^n \qquad (3.3)$$

Suppose that a is a positive fixed number. Let $\theta_{1,a}(x) = essinf_{y\in[x,a)}\, \theta_1(y)$,

$$\tilde{\theta}_1(x) = \begin{cases} \theta_{1,a}(x) & \text{if } x \in [0,a]; \\ \overline{\theta}_1 = const & \text{if } x \in [a,\infty); \end{cases}, \theta_1 = \underset{x\in R_+}{essinf}\, \theta_1(x), \Theta_2 = \underset{x\in R_+}{essup}\, \theta_2(x)$$

The next theorem was proved in [10].

Theorem 3.3. *Let $\theta_1(x)$ and $\theta_2(x)$ measurable functions on R_+.Suppose that there exists a positive number a for all $x>a$ holds $\theta_1(x) = \overline{\theta}_1 = const$, $\theta_2(x) = \overline{\theta_2} = const$ and $1<\theta_1 \leq \tilde{\theta}_1(x) \leq \theta_2(x) \leq \Theta_2 < \infty$ for a.a. If*

$$G = \sup_{t>0} \int_0^t [v(x)]^{\theta_2(x)} (\int_t^{\infty} u^{\tilde{\theta}_1'(x)}(\tau)d\tau)^{\frac{\theta_2(x)}{(\theta_1)'(x)}} dx < \infty \qquad (3.4)$$

then the operator $\tilde{H}_{v,u}$ is bounded from $L_{\theta_1(.)}(R^+)$ to $L_{\theta_2(.)}(R^+)$.

4 The Main Results

Theorem 4.1. *Let $p(.) \in \mathbb{P}^{log}_{\infty}(\Omega)$ and $\theta_1(r)$ and $\theta_2(r)$ measurable functions on R_+. Suppose that there exists a positive number a such that, for all $r>a$ holds $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ and $1<\theta_1 \leq \tilde{\theta}_1(r) \leq \theta_2(r) \leq \Theta_2 < \infty$ for a.a., the positive measurable functions w_1 u w_2 satisfies the condition*

$$A = \sup_{x\in\Omega, t>0} \int_0^t (\frac{w_2(x,r)}{r})^{\theta_2(r)} (\int_t^{\infty} (\frac{1}{w_1(x,s)})^{[\tilde{\theta}_1(r)]'} ds)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \qquad (4.1)$$

Then the commutator $[b,T]$ is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$.

Corollary 4.1. *Let $p(.) \in \mathbb{P}^{log}_{\infty}(\Omega)$, $w_1(x,r) = w_2(x,r) = r^{\beta(x)}$.If*

$$\inf_{x\in\Omega, r>0} (\beta(x))[\tilde{\theta}_1(r)]' > 1, \qquad (4.2)$$

$$\sup_{x\in\Omega,t>0}\int_0^t r^{\theta_2(r)[\beta(x)-1]}\frac{t^{[-(\beta(x))[\tilde{\theta}_1(r)]'+1]\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}{[(\beta(x))[\tilde{\theta}_1(r)]-1]^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}dr<\infty. \tag{4.3}$$

Then the the commutator $[b,T]$ *is bounded from* $GM_{p(.),\theta_1(.),r^{\beta(.)}}(\Omega)$ *to* $GM_{p(.),\theta_2(.),r^{\beta(.)}}(\Omega)$.

Proof. Proof of the theorem 3.1. According to the Theorem 3.1, we have

$$||[b,T]f||_{GM_{p(.),\theta_2(.),w_2(.)}(\Omega)}=\sup_{x\in\Omega}\left\|w_2(x,r)r^{-\eta_{p(x,r)}}\|[b,T]f\|_{L_{p(.)}(B(x,r))}\right\|_{L_{\theta_2(.)}(0,\infty)}\le$$

$$\le C\sup_{x\in\Omega}||\frac{w_2(x,r)}{r}\int_r^\infty t^{-\eta_{p(x,t)}}||f||_{L_{p(.)}(B(x,t))}dt||_{L_{\theta_2(.)}(0,\infty)},$$

here we use the inequality $1+ln\frac{t}{r}<\frac{t}{r}$ for $t>r>0$. We denote

$$\tilde{H}_{v,u}f(r)=v(r)\int_r^\infty g(t)u(t)dt,$$

where

$$v(r)=\frac{w_2(x,r)}{r},$$

$$g(t)=\frac{w_1(x,t)}{t^{\eta_{p(x,t)}}}||f||_{L_{p(.)}(B(x,t))},$$

$$u(t)=\frac{1}{w_1(x,t)},$$

for every fixed $x\in\Omega$. Then the condition (3.4) takes the form (4.1), from which, according to the Theorem 3.3, it follows that the operator $\tilde{H}_{v,u}f(r)$ is bounded from $L_{\theta_1(.)}(0,\infty)$ to $L_{\theta_2(.)}(0,\infty)$. Finally, we have

$$\|[b,T]f\|_{GM_{p(.),\theta_2(.),w_2(.)}(\Omega)}\le CA\cdot\sup_{x\in\Omega}\left\|w_1(x,t)t^{-\eta_{p(x,t)}}\|f\|_{L_{p(.)}(B(x,t))}\right\|_{L_{\theta_1(.)}(0,\infty)}=$$

$$=CA\cdot\|f\|_{GM_{p(.),\theta_1(.),w_1(.)}(\Omega)},$$

which means that the commutator $[b,T]$ is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$. The theorem 4.1 is proved. □

Proof. Proof of the corollary 4.1. The condition (4.1) takes the form

$$\sup_{x\in\Omega,t>0}\int_0^t r^{\theta_2(r)[\beta(x)-1]}\left(\int_t^\infty s^{-[\beta(x)][\tilde{\theta}_1(r)]'}ds\right)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}dr<\infty.$$

By the convergence of the inner integral, we take the conditions (4.2) and (4.3). □

Theorem 4.2. *Let $p(.) \in \mathbb{P}^{log}_{\infty}(\Omega)$ and $\theta_1(r)$ and $\theta_2(r)$ measurable functions on R_+, such that $1 < \theta_{1-} \leq \theta_1(r) \leq \theta_{1+} < \infty$, $1 < \theta_{2-} \leq \theta_2(r) \leq \theta_{2+} < \infty$. Suppose that, the positive measurable functions $w_1(x,t)$ and $w_2(x,t)$ satisfy the condition*

$$B = \sup_{x\in\Omega} \| \frac{w_2(x,r)}{r} \| \frac{1}{w_1(x,t)} \|_{L_{\theta_1'(.)}(r,\infty)} \|_{L_{\theta_2(.)}(0,\infty)} < \infty. \tag{4.4}$$

Then the commutator $[b,T]$ is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$.

Proof. Using the Holder inequality with the exponents $\theta_1(.)$, $\theta_1'(.)$, we have

$$||[b,T]f||_{GM_{p(.),\theta_2(.),w_2(.)}(\Omega)} = \sup_{x\in\Omega} \|w_2(x,r) r^{-\eta_{p(x,r)}} \|[b,T]f\|_{L_{p(.)}(B(x,r))}\|_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq C \sup_{x\in\Omega} || \frac{w_2(x,r)}{r} \int_r^\infty t^{-\eta_{p(x,t)}} ||f||_{L_{p(.)}(B(x,t))} dt ||_{L_{\theta_2(.)}(0,\infty)} =$$

$$= C \sup_{x\in\Omega} || \frac{w_2(x,r)}{r} \int_r^\infty w_1(x,t) t^{-\eta_{p(x,t)}} ||f||_{L_{p(.)}(B(x,t))} \cdot \frac{1}{w_1(x,t)} dt ||_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq C \sup_{x\in\Omega} \| \frac{w_2(x,r)}{r} \| \frac{1}{w_1(x,t)} \|_{L_{\theta_1'(.)}(r,\infty)} \|w_1(x,t) t^{-\eta_p(x,t)} ||f||_{L_{p(.)}(B(x,t))}\|_{L_{\theta_1(.)}(r,\infty)} \|_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq C \sup_{x\in\Omega} \| \frac{w_2(x,r)}{r} \| \frac{1}{w_1(x,t)} \|_{L_{\theta_1'(.)}(r,\infty)} \|_{L_{\theta_2(.)}(0,\infty)} \sup_{x\in\Omega} \|w_1(x,t) t^{-\eta_p(x,t)} ||f||_{L_{p(.)}(B(x,t))}\|_{L_{\theta_1(.)}(0,\infty)} =$$

$$= CB \cdot \|f\|_{GM_{p(.),\theta_1(.),w_1(.)}(\Omega)}.$$

The theorem 4.2 is proved. □

Theorem 4.3. *Let $p_1(.) \in \mathbb{P}^{log}_{\infty}(\Omega)$ and a constant number α satisfy the conditions $\alpha > 0$, $(\alpha p_1(.))_+ = sup_{x\in\Omega}\, \alpha p_1(x) < n$ the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, $\theta_1(r)$ and $\theta_2(r)$ are positive measurable functions on R_+ and there exists a positive number a such that, $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, the inequality $1 < \theta_1 \leq \tilde{\theta}_1(r) \leq \theta_2(r) \leq \Theta_2 < \infty$ are holds almost everywhere. Suppose that the positive measurable functions w_1 and w_2 satisfy the condition*

$$D = \sup_{x\in\Omega, t>0} \int_0^t (\frac{w_2(x,r)}{r})^{\theta_2(r)} (\int_t^\infty (\frac{s^\alpha}{w_1(x,s)})^{[\tilde{\theta}_1(r)]'} ds)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \tag{4.5}$$

Then the commutator $[b, I^\alpha]$ is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Corollary 4.2. *Let $p_1(.) \in \mathbb{P}^{log}_{\infty}(\Omega)$ and a constant number α satisfy the conditions $\alpha > 0$, $(\alpha p_1(.))_+ = sup_{x\in\Omega}\, \alpha p_1(x) < n$ the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, $w_1(x,r) = w_2(x,r) = r^{\beta(x)}$.If*

$$\inf_{x\in\Omega, r>0} (\beta(x) - \alpha)[\tilde{\theta}_1(r)]' > 1, \tag{4.6}$$

$$\sup_{x\in\Omega,t>0}\int_0^t r^{\theta_2(r)[\beta(x)-1]}\frac{t^{[-(\beta(x)+\alpha)[\tilde{\theta}_1(r)]'+1]\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}{[(\beta(x)-\alpha)[\tilde{\theta}_1(r)]-1]^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}dr<\infty. \tag{4.7}$$

Then the the commutator $[b,I^\alpha]$ *is bounded from* $GM_{p_1(.),\theta_1(.),r^{\beta(.)}}(\Omega)$ *to* $GM_{p_2(.),\theta_2(.),r^{\beta(.)}}(\Omega)$.

Proof. Proof of the theorem 4.3. According to the Theorem 3.2, we have

$$||[b,I^\alpha]f||_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)}=\sup_{x\in\Omega}\left\|w_2(x,r)r^{-\eta_{p_2}(x,r)}\|[b,I_\alpha]f\|_{L_{p_2(.)}(B(x,r))}\right\|_{L_{\theta_2(.)}(0,\infty)}\leq$$

$$\leq C\sup_{x\in\Omega}||\frac{w_2(x,r)}{r}\int_r^\infty t^{-\eta_{p_2}(x,t)}||f||_{L_{p_1(.)}(B(x,t))}dt||_{L_{\theta_2(.)}(0,\infty)},$$

here we use the inequality $1+ln\frac{t}{r}<\frac{t}{r}$ for $t>r>0$. Denote

$$\tilde{H}_{v,u}f(r)=v(r)\int_r^\infty g(t)u(t)dt,$$

where

$$v(r)=\frac{w_2(x,r)}{r},$$

$$g(t)=\frac{w_1(x,t)}{t^{\eta_{p_1}(x,t)}}||f||_{L_{p_1(.)}(B(x,t))},$$

$$u(t)=\frac{t^\alpha}{w_1(x,t)},$$

for every fixed $x\in\Omega$. Then the condition (3.4) takes the form (4.5), from which, according to the Theorem 3.3, it follows that the operator $\tilde{H}_{v,u}f(r)$ is bounded from $L_{\theta_1(.)}(0,\infty)$ to $L_{\theta_2(.)}(0,\infty)$. Finally, we have

$$\|[b,I^\alpha]f\|_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)}\leq CD\cdot\sup_{x\in\Omega}\left\|w_1(x,t)t^{-\eta_{p_1}(x,t)}\|f\|_{L_{p_1(.)}(B(x,t))}\right\|_{L_{\theta_1(.)}(0,\infty)}=$$

$$=CD\cdot\|f\|_{GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)},$$

which means that the commutator $[b,I^\alpha]$ is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$. The theorem 4.3 is proved. □

Proof. Proof of the corollary 4.2. The condition (4.5) takes the form

$$\sup_{x\in\Omega,t>0}\int_0^t r^{\theta_2(r)[\beta(x)-1]}(\int_t^\infty s^{-[\beta(x)+\alpha][\tilde{\theta}_1(r)]'}ds)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}dr<\infty.$$

By the convergence of the inner integral, we take the conditions (4.6) and (4.7). □

Theorem 4.4. *Let* $p_1(.) \in \mathbb{P}_\infty^{log}(\Omega)$ *and a constant number* α *satisfy the conditions* $\alpha > 0$, $(\alpha p_1(.))_+ = sup_{x\in\Omega}\, \alpha p_1(x) < n$ *the functions* $p_1(x)$ *and* $p_2(x)$ *satisfy the equality* $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, $\theta_1(r)$ *and* $\theta_2(r)$ *measurable functions on* R_+, *such that* $1 < \theta_{1-} \le \theta_1(r) \le \theta_{1+} < \infty$, $1 < \theta_{2-} \le \theta_2(r) \le \theta_{2+} < \infty$. *Suppose that, the positive measurable functions* $w_1(x,t)$ *and* $w_2(x,t)$ *satisfy the condition*

$$E = \sup_{x\in\Omega} \Big\| \frac{w_2(x,r)}{r} \Big\| \frac{t^\alpha}{w_1(x,t)} \Big\|_{L_{\theta_1'(.)}(r,\infty)} \Big\|_{L_{\theta_2(.)}(0,\infty)} < \infty. \tag{4.8}$$

Then the the commutator $[b, I^\alpha]$ *is bounded from* $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ *to* $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Proof. Proof of the theorem 4.4. Using the Holder inequality with the exponents $\theta_1(.)$, $\theta_1'(.)$, we have

$$||[b, I^\alpha]f||_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} = \sup_{x\in\Omega} \Big\| w_2(x,r) r^{-\eta_{p_2}(x,r)} \|[b, I_\alpha]f\|_{L_{p_2(.)}(B(x,r))} \Big\|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le C \sup_{x\in\Omega} \Big|\Big| \frac{w_2(x,r)}{r} \int_r^\infty t^{-\eta_{p_2}(x,t)} ||f||_{L_{p_1(.)}(B(x,t))} dt \Big|\Big|_{L_{\theta_2(.)}(0,\infty)} =$$

$$= C \sup_{x\in\Omega} \Big|\Big| \frac{w_2(x,r)}{r} \int_r^\infty w_1(x,t) t^{-\eta_{p_1}(x,t)} ||f||_{L_{p_1(.)}(B(x,t))} \cdot \frac{t^\alpha}{w_1(x,t)} dt \Big|\Big|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le C \sup_{x\in\Omega} \Big\| \frac{w_2(x,r)}{r} \Big\| \frac{t^\alpha}{w_1(x,t)} \Big\|_{L_{\theta_1'(.)}(r,\infty)} \Big\| w_1(x,t) t^{-\eta_{p_1}(x,t)} ||f||_{L_{p_1(.)}(B(x,t))} \Big\|_{L_{\theta_1(.)}(r,\infty)} \Big\|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le C \sup_{x\in\Omega} \Big\| \frac{w_2(x,r)}{r} \Big\| \frac{t^\alpha}{w_1(x,t)} \Big\|_{L_{\theta_1'(.)}(r,\infty)} \Big\|_{L_{\theta_2(.)}(0,\infty)} \sup_{x\in\Omega} \Big\| w_1(x,t) t^{-\eta_{p_1}(x,t)} ||f||_{L_{p_1(.)}(B(x,t))} \Big\|_{L_{\theta_1(.)}(0,\infty)} =$$

$$= CE \cdot \|f\|_{GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)}.$$

The theorem 4.4 is proved. □

References

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The bibliography should be placed in alphabetical order. Template for the bibliography see below.

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