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ON THE PROJECTIVE SPECIAL LINEAR GROUPS $\mathbf{L_p(2)}$

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Abstract: In this paper, we prove that projective special linear groups $\mathbf{L_p(2)}$, where $2^p - 1$, is prime number can be uniquely determined by its order and the largest elements order.

1. **Introduction**

For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the largest element of the set $\pi_e(G)$ of element orders of G is denoted by $k(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where $|G|$ is of even order, we assume that $2 \in \pi_1$.
One of an important problems of finite groups theory is, group characterization by specific property. In the way, the researchers proved that many groups by using methods such as elements order, the set of elements with the same order and graphs ,.... One of the methods is group characterization by using the order of group and the largest elements order. In the other words, we say the group G is characterizable by using the order of group and the largest elements order if there is the group H, so that if $k(G) = k(H)$ and $|G| = |H|$, then $G \cong H$. In the way, for example the authors in([7], [3],[4],[2],[5],[9]) proved the simple groups $L_3(q)$ and $U_3(q)$ where q is some special power of prime, the group $L_2(q)$ where $q = p^n < 125$, the simple K_4 -group of type $L_2(p)$, where p is a prime but not 2^n-1, sporadic groups, $PGL(2, q)$ and suzuki

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group $Sz(q)$, where $q-1$ or $q \pm \sqrt{2q}+1$ is a prime number by using the largest element order and order of group are characterizable. In this paper, we characterize projective special linear groups $\mathbf{L_p(2)}$, where 2^p-1, is prime number can be uniquely determined by their order and the largest elements order.

2. Notation and Preliminaries

Lemma 2.1. [6] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G)=2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(b) *$|H|$ divides $|K|-1$;*
(c) *K is nilpotent.*

Definition 2.2. *A group G is called a 2-Frobenius group if there is a normal series $1 \unlhd H \unlhd K \unlhd G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively.*

Lemma 2.3. [3] *Let G be a 2-Frobenius group of even order. Then*

(a) *$t(G)=2$, $\pi(H)\cup\pi(G/K)=\pi_1$ and $\pi(K/H)=\pi_2$;*
(b) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.4. [12] *Let G be a finite group with $t(G)\geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*
(b) *G is a 2-Frobenius group;*
(c) *G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.*

Lemma 2.5. [13] *Let q,k,l be natural numbers. Then*

(a) $(q^k-1,q^l-1)=q^{(k,l)}-1$.
(b) $(q^k+1,q^l+1)=\begin{cases} q^{(k,l)}+1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,}\\ (2,q+1) & \text{otherwise.}\end{cases}$
(c) $(q^k-1,q^l+1)=\begin{cases} q^{(k,l)}+1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,}\\ (2,q+1) & \text{otherwise.}\end{cases}$

In particular, for every $q\geq 2$ and $k\geq 1$ the inequality $(q^k-1,q^k+1)\leq 2$ holds.

3. Proof of the Main Theorem

In this section, we prove the main theorem. For this purpose, we denote the projective special linear groups $L_p(2)$ by L and also isolated vertex $2^p - 1$ by r. To proof the main theorem, we denote the several lemmas, that by proving this lemmas, in finally the main theorem be proved. For this purpose, we have the main theorem:

Theorem 3.1. *Let G be a group and $L := L_p(2)$ be projective special groups, where $2^p - 1$ is prime number. Then $k(G) = k(L)$ and $|G| = |L|$ if and only if $G \cong L$.*

Lemma 3.2. *p is an isolated vertex of $\Gamma(G)$*

Proof. First, assume that $G \cong L$. Then we can easily prove that $k(G) = k(L)$ and $|G| = |L|$. Now we need prove sufficient condition, that is, if $k(G) = k(L)$ and $|G| = |L|$, then $G \cong L$. By Lemma [8], we have $k(L) = 2^p - 1$. Let p be the number $2^p - 1$. We prove that p is an isolated vertex of $\Gamma(G)$. Assume that it is not. So there is the natural number t belong to $\pi(G)$ such that $t \neq p$ and $tp \in \pi_e(G)$. Thus we deduce that $tr \geq 2r \geq 2(2^p - 1) > (2^{p+1} - 1) > 2^p - 1$ and hence $k(G) > 2^p - 1$ which is impossible. So we conclude that p is an isolated vertex of $\Gamma(G)$ and $t(G) \geq 2$. Now Lemma 2.4 implies that G satisfies one of the following cases.

Lemma 3.3. *G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1(a), $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since r is an isolated vertex of $\Gamma(G)$, we have (i) $|H| = |G|/r$ and $|K| = r$, or (ii) $|H| = r$ and $|K| = |G|/r$. Assume that $|H| = |G|/r$ and $|K| = r$. Then Lemma 2.1(b) implies that $|G|/r$ divides $r - 1$ and hence $|G|/r \leq r - 1$ which is impossible. So the case $|H| = r$ and $|K| = |G|/r$ can be occure. Lemma 2.1(b) implies that r divides $|G|/r - 1$. Now we show that it is impossible. For this since $r = 2^p - 1$,and also $|L_p(2)| = 2^{p(p-1)/2} \prod_{i=2}^{p} (2^i - 1)$ then $(2^p - 1 \mid \frac{2^{p(p-1)/2} \prod_{i=2}^{p} (2^i - 1)}{2^p - 1} - 1$. So we deduce that $(2^p - 1 \mid 2^{p(p-1)/2} \prod_{i=2}^{p-1} (2^i - 1)$, in conclude must be have $2^p - 1 \mid \prod_{i=2}^{p-1} (2^i - 1)$ which is impossible. □

Lemma 3.4. *G is not a 2-Frobenius group.*

Proof. Let G be 2-Frobenius group. Then by Lemma 2.3, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively, $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$ and $\pi(K/H) = \pi_2$, G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$. Since r is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{r\}$ and $|K/H| = r$. If $r = (2^p - 1)$, then by Lemma 2.5, $(r, r-1) = 1$ so $(2^p - 1, 2^p - 2) = 1$ and since $|G/K| \mid r - 1$, we deduce that $(2^p - 1)$ divides $|H|$. So $K/H \rtimes H_t$ is a Frobenius group with kenel H_t where $t = 2^p - 1$. Hence Lemma 2.1(b) implies that $r \mid (2^p - 2)$. Hence must be $2^p - 1 \mid (2^p - 2)$ which is a contradiction.

□

Lemma 3.5. *G is isomorphic to L.*

Proof. By the third case of Lemma 2.4 then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups K/H is a non-abelian simple group.Since K/H is a non-abelian simple group.The other hand we have $k(G) = p$,where that is only odd order components of G. According to the classification of the finite simple groups we know that the possibilites for K/H are alternating group A_m, $m \geq 5$, 26 sporadic groups, simple groups Lie type. Now K/H is isomorphic one of the following groups :

□

Step 1. Let $K/H \cong A_m$, where $m \geq 5$ and $m = p', p'+1, p'+2$. On the other hand, we know $k(A_m) = m$ and $|A_m| \mid |G|$. So, we consider $2^{p(p-1)/2} \prod_{i=2}^{p}(2^i - 1) = |A_m|$ and $m \geq 2^p - 1$, on the otherhand, we suppose r is number between $2^{p-1} - 1$ and $2^p - 1$, so $r \in \pi(A_m) - \pi(G)$, where this is a contradiction.

Step 2. If K/H be isomorphic sporadic groups, then by [2], k(S) =11,23,15,24,20,30,60,28,40,39,70,119,19,31,67,29,66, where S be a sporadic groups and also this numbers be the largest elements order of sporadic groups. Now we consider $2^p - 1 = 11, 23, 15, 24, 20, 30, 60, 28, 40, 39, 70, 119, 19, 31, 67, 29, 66$, we can see easily this equations has not any solution. Hence this a contradiction

Step 3. In this here, we consider K/H is isomorphic by the groups by Lie-type. For this purpose, we consider the following

Case1. Let $t(K/H) = 2$, then K/H is not isomorphic the following groups:

(1) $C_r(q)$, $r = 2^m \geq 2$; $D_r(q)$,$r \geq 5$, $q = 2, 3, 5$; $D_{r+1}(q)$, $q = 2, 3$;

$F_4(q)$, q odd; $G_2(q)$, $q \equiv \pm1(mod3)$;${}^2D_r(3)$,$r \geq 5$, $r \neq 2^n+1$;${}^2D_n(2)$,$n = 2^m+1 \geq 5$;${}^2D_n(3)$,$9 \leq n = 2^m+1 \neq r$; ${}^3D_4(q)$,$C_r(3)$, $B_r(3)$;$B_r(q)$,$r = 2^m \geq 4$, q odd, ${}^2A_3(2)$ and ${}^2F_4(2)$'

1.1. If $K/H \cong C_r(q)$ where $r = 2^v > 2$. Then by [8] $k(C_r(q)) = q^r+1/(2,q-1)$ and also $|C_r(q)| = \frac{1}{(2,q-1)}q^{r^2}\prod_{i=1}^{r}(2^{2i}-1)$. For this purpose, we consider $2^{p(p-1)/2}\prod_{i=2}^{p}(2^i-1) = \frac{1}{(2,q-1)}q^{r^2}\prod_{i=1}^{r}(2^{2i}-1)$ and $2^p-1 = (q^r+1)/2$. Now since $p \mid |C_r(q)|$, then if $p \mid q$, we can see easily that is imposible, in the way since that every p-part of$|G|$ is p, so $p \mid 2^{2t}-1$, where $1 < t < n$. On the other hand, $q^{r^2} \mid 2^t-1$ where $1 < t < n$. As a result, we obtain $q^{r^2} \leq 2^t-1 \leq 2^n-1 \leq (q^r+1)/2 \leq q^r$. Hence $q^{r^2} \leq q^r$, where this is a contradiction. Now if $k(C_r(q)) = q^r+1$, then we consider $2^p-1 = q^r+1$, so we deduce $q^r = 2^p-2$. Now since q^{r^2} must be divided$|G|$ but we can see easily $(2^p-2)^2 \nmid |G|$, where this is a contradiction. For $K/H \cong B_r(q)$, $C_r(q)$ with $r = 2^v \geq 4$, we have a contradiction, similarily

1.2. If $K/H \cong D_r(q)$, where $r \geq 5$, $q = 2,3,5$ and also $|D_r(q)| = q^{r(r-1)}\prod_{i=1}^{r-1}(q^{2i}-1)(q^r-1/(q-1,4)$. For this purpose, first we suppose $q = 2$ so $K/H \cong D_r(2)$. Now by [8], $k(D_r(2)) = 2^r-1$, hence we consider $2^p-1 = 2^r-1$ as a result $n = r$. Since $2^{p(p-1)} \nmid |G|$, hence this is a contradiction. Now if $K/H \cong D_r(3)$, then $k(D_r(3)) = 3^r-1$. So, we consider $2^p-1 = 3^r-1$, in conclution $2^p = 3^r$, where this is a contradiction. For $K/H \cong D_r(5)$ we have a contradiction, similarily.

1.3. If $K/H \cong D_{r+1}(q)$, where $q = 2,3$. First if $K/H \cong D_{r+1}(2)$, then $k(D_{r+1}(2)) = 2^{r+1}-1$. In the way, we consider $2^p-1 = 2^{r+1}-1$, then we obtain $n = r+1$. In finally, since $|D_n(2)| \nmid |L_p(2)|$, where this is a contradiction.The other case is a contradiction, similarily.

1.4. If $K/H \cong F_4(q)$, then by [8], $k(F_4(q)) = (q^3-1)(q+1)$ and also $|F_4(q)| = q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. For this purpose, we consider$2^{n(n-1)/2}\prod_{i=2}^{n}(2^i-1) = q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$ and also $2^n-1 = (q^3-1)(q+1)$. Now since $p \mid |F_4(q)|$, and every p-part of $|G|$ is p hence $p \mid q^t-1$ where $t \in \{12,8,6,2\}$. On the other hand, $q^{24} \mid 2^t-1$ where $2 \leq t \leq n$ so $q^{24} \leq 2^t-1 \leq 2^n-1 \leq (q^3-1)(q+1) \leq q^4$, which is imposible.

1.5. If $K/H \cong^3 D_4(q)$, then by [8], $k({}^3D_4(q)) = (q^3-1)(q+1)$. On the other hand, we have $|{}^3D_4(q)| = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. Now we consider $2^{n(n-1)/2}\prod_{i=2}^{n}(2^i-1) = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$ and also $2^p-1 = (q^3-1)(q+1)$. Now we have $p \mid (q^8+q^4+1)$ or $p \mid q^t-1$

where $t \in \{2,6\}$, on the other hand $q^{12} \mid 2^t - 1$, $2 \le t \le n$, in conclution $q^{12} \le 2^t - 1 \le 2^n - 1 \le (q^3-1)(q+1) \le q^4$, which is a contradiction.
1.6. If $K/H \cong^2 A_3(2)$, ${}^2F_4(2)'$, then by [8], $k({}^2A_3(2)) = 5$ and $k({}^2F_4(2)') = 13$. Now we consider $2^p - 1 = 5$, $2^p - 1 = 13$, we can see easily these equation has not any solution, where this is a contradiction.
(2) In the way K/H is not isomorphic $C_r(2)$ and ${}^2D_r(q)$, where $r = 2^m \ge 4$, hence we have the following cases:
2.1. If $K/H \cong C_r(2)$, then by [8], $k(C_r(2)) = (2^r - 1)$. So we consider $(2^p - 1) = 2^r - 1$, in conclude we obtain $p = r$, now since $|C_r(2) \nmid |G|$, which is a contradiction.
2.2. If $K/H \cong {}^2D_r(q)$, where $r = 2^m \ge 4$, then by [8], $k({}^2D_r(q)) = (q^r + 1)/(2, q+1)$. First we suppose q be even, for this purpose we consider $2^p - 1 = q^r + 1$, so we obtain $q^r = 2^p - 2$. Now we have q^{r^2} must be divided $|G|$, hence $(2^n - 2)^2 \mid |G|$, where this is a contradiction. Now if $2^p - 1 = q^r + 1/2$, then $q^r = 2^{p+1} - 3$, so by last proof since $(2^{p+1} - 3)^2 \nmid |G|$ hence this is a contradiction.
(3). In this here, we prove that K/H is not isomorphic by the following groups:
$A_{r-1}(q) \cong L_r(q)$, $(r,q) \ne (3,2)$,$(3,4)$; $A_r(q) \cong L_{r+1}(q)$,$q-1 \mid r+1$; ${}^2A_{r-1}(q)$ and ${}^2A_r(q)$, $q+1 \mid r+1$,$(r,q) \ne (3,3)$,$(5,2)$, where r is an odd prime.
3.1. If $K/H \cong L_r(q)$, wher $(r,q) \ne (3,2),(3,4)$. On the other hand by [8], $k(L_r(q)) = (q^r - 1)/(r, q-1)(q-1)$. So we consider $2^p - 1 = (q^r-1)/(r,q-1)(q-1)$, now we assume $q = t^s$, $s > 1$, $t = 3$ then we have $2^p - 1 = (q^r-1)/(r,q-1)(q-1) = 3^{sr}-1/(r,3^s-1)(3^s-1) \le 3^{sr}-1$, now since $3^{sr} - 1 \nmid |G|$. If $t = 2$ then $q = 2^s$, again we have $2^n - 1 \le 2^{sr} - 1$, now since$2^{sr} - 1 \nmid |G|$, in conclude we obtain a contradiction. Now if $t > 3$ then we have a contradiction, similarily.
3.2. If $K/H \cong L_{r+1}(q)$, wher $q - 1 \mid r - 1$, then by [8], $k(L_{r+1}(q)) = (q^r - 1)/(q-1)$. So we consider $2^p - 1 = (q^r-1)/(q-1)$, then by proof of last case we have a contradiction. For $K/H \cong^2 A_{r-1}(q)$ and ${}^2A_r(q)$, $q+1 \mid r+1$,$(r,q) \ne (3,3)$,$(5,2)$, where r is an odd prime,we have a contradiction, similarily.

(4) The groups K/H is not isomorphic with $E_6(q)$ and ${}^2E_6(q)$, $q > 2$
4.1. If $K/H \cong^2 E_6(q)$, then by [8] $k({}^2E_6(q))) = (q+1)(q^2+1)(q^3-1)/(3, q+1)$ .Now we consider $2^p - 1 = (q+1)(q^2+1)(q^3-1)/(3,q+1)$, so $2^n - 1 = q^6 + q^5 + q^4 - q^2 - q - 1$ so we obtain$2^n = q^6 + q^5 + q^4 - q^2 - q$,

where we can see easily this equation has not any solution, so this a contradiction, now if $K/H \cong E_6(q)$ then we have a contradiction,similarily.
Case 2. If t(K/H)=3, then K/H is not isomorphic the following groups:
2.1. If $K/H \cong L_2(q)$, where $4 \mid q \pm 1$ then by [8], $k(L_2(q)) = q$. So we consider $(2^p - 1) = q$, in conclude $q^2 = (2^n - 1)^2$ so $q^2 - 1/2 = (2^n - 1)^2 - 1)/2 \nmid |G|$, where this is a contradiction.
2.2. If $K/H \cong L_2(q)$, where $q > 2$ then by [8] $k(L_2(q)) = q, q+1$. So we consider $(2^p - 1) = q$, in conclude $q^2 = (2^n - 1)^2$ so $q^2 - 1/2 = (2^n - 1)^2 - 1)/2 \nmid |G|$, where this is a contradiction. The other case if $2^p - 1 = q + 1$ then $2^p - 2 = q$, ,now since $|L_2(q)| \nmid |G|$ where this is imposible.
2.3. If $K/H \cong G_2(q)$, where $3 \mid q$,then by [8], $k(G_2(q)) = q^2 + q + 1$. So we consider $(2^p - 1) = q^2 + q + 1$, in conclude $q(q+1) = 2(2^n - 1))$. Since$(q, q-1) = 1$ so $q = 2$, where this is a contrary by $3 \mid q$ also if $q + 1 = 2^{p-1} - 1$ then $n = 3$, now since $|G_2(2)| \nmid |G|$, where this is a contradiction.
2.4. If $K/H \cong^2 G_2(3^{2m+1})$, where $m \geq 1$then by [8], $k(^2G_2(3^{2m+1}))) = 3^{2m+1} + 3^{m+1} + 1$ .Now we consider $(2^p - 1)/2 = 3^{2m+1} + 3^{m+1} + 1$, so we obtain $2^p - 2 = (3^{m+1}(3^m + 1)$.In conclude $2(2^{n-1} = 3^{m+1}(3^m + 1)$. Now since $(3^{m+1}, 3^m + 1) = 1$ so we obtain $2 = 3^m + 1$ and $2^{p-1} - 1 = 3^{m+1}$. Now if $2 = 3^m + 1$ then $m = 0$ so $q = 3$ and also since $2^{n-1} - 1 = 3^{m+1}$ then $n = 3$ in conclude $|^2G_2(3| \nmid |L_3(2)|$, where this a contradiction.
2.5. If $K/H \cong^2 D_r(3)$, where $r = 2^n + 1 \geq 3$ then by [8], $k(^2D_r(3)) = 3^r + 1/2$.Now we consider $2^p - 1 = 3^r + 1/2$ in conclude $3^r + 3 = 2^{p+1}$, in the way $3(3^{r-1} + 1) = 2(2^p)$ then $3 = 2^p$ and $2^p = 3^{r-1} + 1$, that is a contradiction.
2.6. If $K/H \cong F_4(q)$, where $2 \mid q$ then by [8], $k(F_4(q)) = 2(q+1)(q^2 + 1)$,also we know $|F_4(q)| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. For this purpose, we consider$2^{n(n-1)/2} \prod_{i=2}^{n}(2^i - 1) = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$ and also $2^p - 1 = 2q^3 + 2q^2 + 2q + 2$. Now since $p \mid |F_4(q)|$, on the other hand, since the p-part of $|G|$ is p hence $p \mid q^t - 1$ where $t \in \{12, 8, 6, 2\}$ on the other hand $q^{24} \mid 2^t - 1$ where $2 \leq t \leq n$ so $q^{24} \leq 2^t - 1 \leq 2^n - 1 \leq 2q^3 + 2q^2 + 2q + 2$, where this is imposible.
2.7. If $K/H \cong^2 F_4(q)$, where $q = 2^{2m+1} > 2$ then by [8], $k(^2F_4(q)) = 2^{(2m+1)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$. So we consider $2^p - 1 = 2^{(2m+1)} + 2^{3m+2} + 2^{2m+1} + 2^{m+1} + 1$ in conclude $2^{m+1}(2^m + 2^{2m+1} + 1) = 2(2^{n-1} - 1)$. Now since $(2, 2^{n-1} - 1) = 1$ so $2^{m+1} = 2$ in conclude $m = 0$ then $q = 2$ now since $|^2F_4(q)| \nmid |G|$,where this is a contradiction.If $2^m + 2^{2m+1} + 1 = 2^{n-1} - 1$ then $5 = 2^{n-1}$, where this is a contradiction.

2.8 If $K/H \cong^2 A_5(2)$, or $E_7(3)$, then by [8] $k(^2A_5(2)) = 11$ and $k(E_7(3) = 1093$. Hence we have $2^p - 1 = 11$ and also $2^p - 1 = 1093$, we can see easily these equation has not any solution.

Case 3 If t(K/H)=4,5 then, K/H is not isomorphic the following groups:
3.1. If $K/H \cong A_2(4)$, $^2E_6(2)$, then by [8], $k(A_2(4)) = 7$,$k(^2E_6(2)) = 35$ so we have $2^p - 1 = 7$,where we obtain $n = 3$,now since $|A_2(4)| \nmid |G|$ where is a contradiction.Also if $2^p - 1 = 35$, we can see easily the equation $2^n = 36$ has not any solve.
3.2. If $K/H \cong^2 B_2(2^{2m+1})$, where $m \geq 1$ then by [8] $k(^2B_2(2^{2m+1}))) = 2^{2m+1} + 2^{m+1} + 1$ and also$|^2B_2(2^{2m+1})| = q^2(q^2 + 1)(q - 1)$. For this purpose, we consider$2^{n(n-1)/2} \prod_{i=2}^{n}(2^i - 1) = q^2(q^2 + 1)(q - 1)$ and also $(2^n - 1)/2 = 2^{2m+1} + 2^{m+1} + 1$, so we obtain $2^p - 2 = (2^{m+1}(2^m + 1)$. In conclude $2(2^{p-1}) = 2^{m+1}(2^m + 1)$. Now since $(2^{m+1}, 2^m + 1) = 1$ so we obtain $2 = 2^m + 1$ and $2^{p-1} - 1 = 2^{m+1}$. Now if $2 = 2^m + 1$, then $m = 0$ where this is contrary by$m \geq 1$, so $q = 2$ and also since $2^{n-1} - 1 = 2^{m+1}$ then $3 = 2^{n-1}$ in conclude this is a contradiction.
3.3. If $K/H \cong E_8(q)$, then by [8], $k(E_8(q) = (q + 1)(q^2 + q + 1)(q^5 - 1)$, also we have $|E_8(q)| = q^{120}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^8 - 1)(q^2 - 1)$. So we consider $2^{n(n-1)/2} \prod_{i=2}^{n}(2^i - 1) = q^{120}(q^{30} - 1)(q^{24} - 1)(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^8 - 1)(q^2 - 1)$ and also $2^p - 1 = (q + 1)(q^2 + q + 1)(q^5 - 1)$. Now since $p \mid |E_8(q)|$ and on the other hand since that every p-part of $|G|$ is p so $p \mid q^t - 1$ where $t \in \{30, 24, 20, 18, 14, 12, 8, 2\}$, and on the other hand, $q^{120} \mid 2^t - 1$ where $2 \leq t \leq n$ in conclude $q^{120} \leq 2^t - 1 \leq 2^n - 1 \leq (q+1)(q^2+q+1)(q^5-1) \leq q^8$, where this is a contradiction.
Hence $K/H \cong L$, in conclude $|K/H| = |L|$. We know that $H \trianglelefteq K \trianglelefteq G$, hence since p is an isolated vertex of $\Gamma(G)$.Thus we deduce that $p \mid |K/H|$. Hence $2^p - 1 = 2^{p'} - 1$ and the other hand $k(K/H) \mid k(L)$ so $2^{p'} - 1 \mid 2^p - 1$, in result $p = p'$. Now since $|K/H| = |L|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong L$.

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