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BINARY WEAKLY ORDERED MINIMAL THEORIES AND 1-CONSERVATIVE EXTENSIONS

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Abstract. The present paper concerns application of approach introduced by David Marker and Charles I. Steinhorn for o-minimal theories [2] into binary weakly o-minimal theories. We study non-definable types using criterion of non-definability for weakly o-minimal theories [5]. It is proved that in 1-conservative, non-conservative pair of models of binary weakly o-minimal theory there exists an irrational 1-type over small model which is not weakly orthogonal to some tuple from big model. As corollary of this result, we give an analogy of Marker-Steihnorn theorem for quite or almost o-minimal theories.

Keywords: definable types, binary theory, weakly o-minimal theory, orthogonality of types, conservative extension.

A linearly ordered structure $\mathcal{M} = \langle M; =, <, \ldots\rangle$ is called *weakly o-minimal* if any definable (with parameters) subset of $\mathcal{M}$ is a union of finitely many convex sets in $\mathcal{M}$. A subset A of a linearly ordered structure $\mathcal{M}$ is called *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. We say that theory T is weakly o-minimal if all its models are weakly o-minimal. Theory T is called *binary* if every formula $\phi(x_1, \ldots, x_n)$ is equivalent to Boolean combination of formulas in at most two free variables. We call such formulas *binary*.

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Suppose that $\mathcal{M} = \langle M, \Sigma\rangle$ and $\mathcal{N} = \langle N, \Sigma\rangle$ are models of a complete theory T such that $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$ ($\mathcal{M} \prec \mathcal{N}$). For any set $B \subset N$ we fix following notations:

$$B^+ = \{z \in N \mid \forall b \in B, z > b\},$$
$$B^- = \{z \in N \mid \forall b \in B, z < b\}.$$

Obviously, for a definable set B sets B^+ and B^- are also definable. The following are easily derived from notations:

$$(B^+)^- = \{w \in N \mid \forall b \in B^+, w < b\},$$
$$(B^-)^+ = \{w \in N \mid \forall b \in B^-, w > b\}.$$

Notice that these notations apply for definable sets of formulas as well.

Let $\bar{y}$ be a tuple $\langle y_1, \ldots, y_n\rangle$. Denote the length of tuple $\bar{y}$ by $l(\bar{y})$ ($l(\bar{y}) = n$).

Let $A \subseteq M$, $n < \omega$, $p \in S_n(A)$. The type p is called *definable* if for every $\phi(\bar{x}_n, \bar{v}) \in L(x_n)$ there exists a formula $R_\phi(\bar{v}) \in L(A)$ (A-formula) such that for any $\bar{a} \in A$ holds $\phi(\bar{x}_n, \bar{a}) \in p$ if and only if $\mathcal{M} \models R_\phi(\bar{a})$. We say [4] that $\mathcal{N}$ is *n-conservative extension* of $\mathcal{M}$ ($\mathcal{M} \prec_{n,c} \mathcal{N}$), if for any $\bar{\alpha} \in N \setminus M$ ($l(\bar{\alpha}) = n$) its n-type is definable. $\mathcal{N}$ is *conservative extension* of $\mathcal{M}$ ($\mathcal{M} \prec_c \mathcal{N}$), if it is n-conservative extension of $\mathcal{M}$ for any $n < \omega$.

Van den Dries [1] proved that in an o-minimal structure $\langle \mathbb{R}, <, \ldots\rangle$ any type $p \in S(\mathbb{R})$ is definable. David Marker and Charles I. Steinhorn [2] generalized this result and proved that in an o-minimal theory if a structure $\mathcal{M}$ is 1-conservative in $\mathcal{N}$, then $\mathcal{M}$ is n-conservative in $\mathcal{N}$. This theorem does not apply for weakly o-minimal theories. B.S.Baizhanov constructed [4] an example of weakly o-minimal theory where $\mathcal{N}$ is 1-conservative extension of $\mathcal{M}$ but $\mathcal{N}$ is not 2-conservative extension of $\mathcal{M}$.

Let p be a type from $S_1(A)$ with $A \subset M$. Denote

$L(p) := \{G(x) | G(x)$ is an A-definable formula such that $G(M) < p(M)\}$ and

$R(p) := \{D(x) | D(x)$ is an A-definable formula such that $p(M) < D(M)\}$.

Definition 1. [5] *Let $\theta(\bar{z})$ and $H(y, \bar{z})$ be an A-definable formulas with $X := \theta(M) \cap A$. We say that the* **condition of left convergence** *of a formula $H(y, \bar{z})$ on a set X or of $\theta(\bar{z})$ to the type p is satisfied and denote by*

$$LC\big(H(y, \bar{z}), X, p\big) \text{ or } LC\big(H(y, \bar{z}), \theta(\bar{z}), p\big)$$

if for any $G(x) \in L(p)$ there exists $\bar{a} \in X$ such that

$\mathcal{M} \models \exists x(G(M) < x < H(M, \bar{a})^+)$, $H(M, \bar{a}) < p(M)$.

We say that the **condition of right convergence** *of a formula $H(y, \bar{z})$ on a set X or of $\theta(\bar{z})$ to the type p is satisfied and denote by*

$$RC\big(H(y, \bar{z}), X, p\big) \text{ or } RC\big(H(y, \bar{z}), \theta(\bar{z}), p\big)$$

if for any $D(x) \in R(p)$ there exists $\bar{a} \in X$ such that

$\mathcal{M} \models \exists x(H(M, \bar{a}) < x < D(M))$, $p(M) < H(M, \bar{a})^+$.

We also say that the **condition of two-side convergence** *of a formula $H(y, \bar{z})$ on a set X or of $\theta(\bar{z})$ to the type p is satisfied and denote by*

$$C\big(H(y,\bar z),X,p\big) \textit{ or } C\big(H(y,\bar z),\theta(\bar z),p\big)$$
if $LC(H,X,p)$ and $RC(H,X,p)$ hold simultaneously.

Let $p \in S_1(A)$ where $A \subseteq M \models T$. We say [8] that the type p is *right quasirational* if there exists convex A-definable formula $U_p(x) \in p$ such that for every sufficiently saturated model $\mathcal{N} \succ \mathcal{M}$ we have $U_p(N)^+ = p(N)^+$. We say that the type p is *left quasirational* if there exists convex A-definable formula $U_p(x) \in p$ such that for every sufficiently saturated model $\mathcal{N} \succ \mathcal{M}$ we have $U_p(N)^- = p(N)^-$. If a type is either right or left quasirational, it is called *quasirational*. If a type is both right and left quasirational, it is called isolated. If a type is neither quasirational nor isolated, we say that it is *irrational*.

Proposition 1. [8] *Let p be a 1-type over a model of a weakly o-minimal theory. The type p is definable if and only if p is quasirational. The type p is non-definable if and only if p is irrational.*

Theorem 1. [5] *Let $A \subset M$ and $p \in S_1(A)$ be an irrational type. Then the following are equivalent:*

(i) p is non-definable;

(ii) exists A-definable formula $H(y,\bar z)$ such that for any A-definable formula $\theta(\bar z)$ holds:

$$C\big(H(y,\bar z),\theta(\bar z),p\big) \bigvee C\big(H(y,\bar z),\neg\theta(\bar z),p\big).$$

We say that the formula $\phi(\bar x,\bar b)$ where $\bar b \in N$ *divides* $C \subset N^l$ (l is the length of tuple $\bar x$) if $\phi(N^l,\bar b)\cap C \neq \emptyset$ and $\neg\phi(N^l,\bar b)\cap C \neq \emptyset$.

Definition 2. [7] *Let $A \subset M$. We say that type $p \in S(A)$ is weakly orthogonal to type $q \in S(A)$ ($p \perp^w q$) if $p(\bar x)\cup q(\bar y)$ define complete $(l(\bar x)+l(\bar y))$-type.*

Type p is *not weakly orthogonal* to type q ($p \not\perp^w q$) if there exists A-definable formula $\phi(\bar x,\bar y)$, $\alpha \in p(M)$ and β_1, $\beta_2 \in q(M)$ such that $\beta_1 \in \phi(N,\alpha)$ and $\beta_2 \notin \phi(N,\alpha)$. Equivalently, we say that type $p \in S(A)$ is *not weakly orthogonal* to type $q \in S(A)$ if there exist $\bar\alpha \in p(M)$ and A-definable formula $\phi(\bar x,\bar y)$ such that $\phi(\bar x,\bar\alpha)$ divides $q(M)$.

At this point, we should recall [5] that in weakly o-minimal theories non weakly orthogonality preserves the property of definability of 1-types, that is when $p \not\perp^w q$, the type p is definable if and only if the type q is definable.

Theorem 2. *Let T be a binary weakly o-minimal theory and $\mathcal{M},\mathcal{N}$ be models of T where $\mathcal{M} = \langle M,\Sigma\rangle$ and $\mathcal{N} = \langle N,\Sigma\rangle$ such that $\mathcal{N}$ is a 1-conservative extension of $\mathcal{M}$ and let $\bar c, d \in N\setminus M$. Suppose that $tp(\bar c/M)$ and $tp(d/M)$ are definable and $tp(d\cup\bar c/M)$ is non-definable. Then there exists an irrational type $r \in S_1(M)$ such that $tp(d\cup\bar c/M) \not\perp^w r$.*

Proof. Since the type $tp(d/M)$ is definable, it is quasirational. Let $A = M\cup\bar c$. Non-definability of $tp(d\cup\bar c/M)$ implies that $tp(d/M\cup\bar c)$ is also non-definable. By the Criterion of non-definability (Theorem 2), there exists A-definable formula $H(x,\bar{y},\bar{c})$ such that for any A-definable formula $\theta(\bar{y},\bar{c})$ the following holds:

$$C\big(H(x,\bar{y},\bar{c}),\theta(\bar{y},\bar{c}),tp(d/M\cup\bar{c})\big)\bigvee C\big(H(x,\bar{y},\bar{c}),\neg\theta(\bar{y},\bar{c}),tp(d/M\cup\bar{c})\big).$$

Assume that the condition of convergence holds for $\theta(\bar{y},\bar{c})$ (throughout the proof we write simply "condition of convergence" instead of "condition of two-side convergence"). We apply method of decomposition of sets into cells by D.Macpherson, D. Marker and Ch. Steinhorn [3]. We can partition $\theta(M,\bar{c})$ into a finite union of cells: $\theta(M,\bar{c})=\theta^1(M,\bar{c})\dot{\cup}\ldots\dot{\cup}\theta^t(M,\bar{c})$. Since $\theta(M,\bar{c})$ is A-definable, each cell is definable using only parameters from A. The condition of convergence holds for one of these cells, say for θ^s or its negetion. Assume that it holds for θ^s. For simplicity purposes, we can omit index s and work on a cell $\theta(M,\bar{c})$.

Denote $X:=\theta(M,\bar{c})\cap(M\cup\bar{c})^{l(\bar{y})}$ and let $X=\{\bar{b}_i|i\in I\}$. It follows from the proof [5] of the Criterion of non-definability (Theorem 2) that the formula H satisfies the following property: $(H(M,\bar{b},\bar{c})^+)^-=H(M,\bar{b},\bar{c})$. We introduce order into the set X in the following sense: for any $\bar{b}_1,\bar{b}_2\in X$ we denote $\bar{b}_1<_1\bar{b}_2$ if and only if $H(N,\bar{b}_1,\bar{c})^+\subset H(N,\bar{b}_2,\bar{c})^+$.

By the binarity of theory, every formula is equivalent to a boolean combination, and consequently, disjunctive normal form (DNF) of binary formulas. Without loss of generality, we can assume this DNF as disjoint disjunctions of conjunctions of binary formulas and their negations:

$H(x,\bar{y},\bar{c})\equiv H_1(x,\bar{y},\bar{c})\dot{\vee}\ldots\dot{\vee}H_i(x,\bar{y},\bar{c})\dot{\vee}\ldots\dot{\vee}H_n(x,\bar{y},\bar{c})$.

Lemma 1. *If two-side convergence holds on a formula $H(x,\bar{y},\bar{c})$ on a binary theory, then it holds on at least one of the formulas from its disjunctive normal form.*

The contrary of lemma 1 is impossible. Assume that the condition of convergence does not hold on any of the formulas from DNF. Consider realisation of given formula:

$H(M,\bar{b},\bar{c})\equiv H_1(M,\bar{b},\bar{c})\dot{\cup}\ldots\dot{\cup}H_i(M,\bar{b},\bar{c})\dot{\cup}\ldots\dot{\cup}H_n(M,\bar{b},\bar{c})$.

Since the condition of convergence does not hold on any of the formulas from given union, it does not hold on $H(M,\bar{b},\bar{c})$ which contradicts the clause of lemma.

We now return to the proof of the theorem. Assume that the condition of convergence holds on $H_i(x,\bar{y},\bar{c})$. Since $H_i(x,\bar{y},\bar{c})$ composes a cell, by [2], [3], it is monotonic in each coordinate. Assume that $H_i(x,\bar{y},\bar{c})$ monotone increases in coordinate $\bar{y}_i$. For simplicity we can omit the index of H_i, then take a deeper look into this formula:

$H(x,y_1,\ldots,y_n,c_1,\ldots,c_m)\equiv W_1(x,y_1)\wedge\ldots\wedge W_n(x,y_n)\wedge R_1(x,c_1)\wedge\ldots\wedge R_m(x,c_m)\wedge\bigwedge\limits_{i,j\leq n}L_{i,j}(y_i,y_j)\wedge\bigwedge\limits_{k\leq n}\bigwedge\limits_{l\leq m}G_{k,l}(y_k,c_l)$.

The definable sets corresponding to the formulas $\bigwedge\limits_{i,j\leq n}L_{i,j}(y_i,y_j)$, $\bigwedge\limits_{k\leq n}\bigwedge\limits_{l\leq m}G_{k,l}(y_k,c_l)$ and $R_1(x,c_1),\ldots,R_m(x,c_m)$ are fixed, i.e. boundary of the formula H does not depend on the behavior of these formulas. Therefore, $H(M, \bar{b}, \bar{c})^+ \equiv W_i(M, b_i)^+$ for some $i \in \{1, \ldots, n\}$. So we look into right boundaries of the formulas $W_1(x, y_1), \ldots, W_n(x, y_n)$. The boundary for each W_t can be expressed by the formula:

$$B_t(x, y_t, \bar{c}) := W_t(x, y_t) \wedge R_1(x, c_1) \wedge \ldots \wedge R_m(x, c_m) \wedge \bigwedge_{k \le n} \bigwedge_{l \le m} G_{k,l}(y_k, c_l)$$

for $t \in \{1, \ldots, n\}$.

Again, the condition of convergence holds on at least one of these formulas. Denote it by B_j. Then $C(B_j, X_j, tp(d/M \cup \bar{c}))$. Then we have

$X_j := \exists y_1 \ldots \exists y_{j-1} \exists y_{j+1} \ldots \exists y_n \theta(y_1, \ldots y_{j-1}, N, y_{j+1}, \ldots, y_n, \bar{c}) \cap (M \cup \bar{c})$. The set X_j can be partitioned into finite union of cells: $X_j = X_j^1 \dot{\cup} \ldots \dot{\cup} X_j^s$. The condition of convergence holds at least on one of them, say X_j^l: $C(B_j, X_j^l, tp(d/M \cup \bar{c}))$. Since a cell of dimension 1 defines a convex set in a weakly o-minimal theory, X_j^l is a convex set where W_j monotone increases. Then X_j^l contains an irrational type $r \in S_1(M)$. Since the type of $\bar{c}$ is definable, $tp(\bar{c}/M) \perp^w r$.

Let $r' = tp(\delta/M \cup \bar{c})$ where $\delta \in r(N)$. Notice that $r'(N) = r(N)$. Then $\forall b \in X_j^l$, $\models W_j(d, b, \bar{c}) \Rightarrow b > r'(N)$ and $\models \neg W_j(d, b, \bar{c}) \Rightarrow b < r'(N)$. Thus, $tp(d/M \cup \bar{c}) \not\perp^w r'$ which implies that $tp(d \cup \bar{c}/M) \not\perp^w r$.

□

Let $\mathcal{N}$ be a sufficiently saturated model of the theory T, $A \subset N$, $p, q \in S_1(A)$. We say [6] that the type p is *not almost orthogonal* to q ($p \not\perp^a q$) if there exists A-definable formula $\phi(x, y)$ such that $\forall \alpha \in p(N)$, $\exists \beta_1, \beta_2 \in q(N)$ so that $\beta_1 < \phi(N, \alpha) < \beta_2$, $\emptyset \neq \phi(N, \alpha) \subset q(N)$ holds. The set $A \subset N \models T$ of weakly o-minimal theory T is said to be *almost o-minimal* if for every $p, q \in S_1(A)$ we have $q \not\perp^w p \iff q \not\perp^a p$. We say that weakly o-minimal theory is *almost o-minimal* if any of its sets is almost o-minimal.

We say [9] that the type p is *not quite orthogonal* to the type q ($p \not\perp^q q$) if there exists A-definable bijection $f : p(M) \to q(M)$. We say that weakly o-minimal theory is *quite o-minimal* if the notions of weakly and quite orthogonality of 1-types coincide.

Theorem 3. *Let T be a binary weakly ordered minimal theory and $\mathcal{M} \prec_{1,c} \mathcal{N}$. Suppose $\mathcal{M} \not\prec_c \mathcal{N}$ and, consequently, there exists $\bar{\alpha} \in N \setminus M$ such that $tp(\bar{\alpha}|M)$ is non-definable, length of tuple $\bar{\alpha}$ is minimal. Then there exists irrational 1-type over M, $r \in S_1(M)$ such that $tp(\bar{\alpha}|M) \not\perp^w r$, $tp(\bar{\alpha}|M) \perp^a r$.*

Corollary 1 (Analogy of Marker-Steinhorn Theorem)**.** *Let T be a binary weakly ordered minimal theory. If T is quite o-minimal or almost o-minimal, then for any elementary pair of models $\mathcal{M} \prec \mathcal{N}$ the following holds: $\mathcal{M} \prec_{1,c} \mathcal{N}$ implies $\mathcal{M} \prec_c \mathcal{N}$.*

Proof. Consider the irrational type $r \in S_1(M)$ and $\delta \in r(N)$ from Theorem 2. Again let $r' = tp(\delta/M \cup \bar{c})$. Then $tp(d/M \cup \bar{c}) \not\perp^w r'$. For quite and almost o-minimal theories the notions of not quite and not almost orthogonality coincide with not weakly orthogonality. So there exists A-definable formula $\varphi(x, d, \bar{c})$ such that $\varphi(N, d, \bar{c}) \subset r(N)$, that is $\mathcal{N} \models \exists x \varphi(x, d, \bar{c})$ which implies that $\delta \in N$. Since the type $tp(\delta/M)$ is irrational, it is non-definable. It contradicts the assumption that $\mathcal{M} \prec_{1,c} \mathcal{N}$, so there does not exist a non-definable type over M realised in N. □

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