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EXISTENCE THEOREM OF A WEAK SOLUTION FOR NAVIER-STOKES TYPE EQUATIONS ASSOCIATED WITH DE RHAM COMPLEX

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Abstract: Let $\{d_q, \Lambda^q\}$ be the de Rham complex on a smooth, compact, closed manifold X over $\mathbb{R}^3$ with Laplacians Δ_q. We consider operator equations associated with the parabolic differential operators $\partial_t + \Delta_2 + N^2$ on the second step of the complex with a nonlinear bi-differential operator of zero order N^2. Using projection on the next step of the complex, we show that the equation has a unique solution in special Bochner-Sobolev type functional spaces for some (sufficiently small) time T^*.

Keywords: elliptic differential complexes, parabolic nonlinear equations, open mapping theorem.

1 Introduction

The Navier-Stokes equations have remained one of the central challenges for both mathematicians and fluid dynamics specialists for many decades (see, for example, [13], [26]). These equations have also been studied within the framework of the theory of differential complexes, see works such as

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[6], [16], and many others. In [29], the problem was investigated for the de Rham complex in $\mathbb{R}^n$ within special Bochner-Sobolev spaces. In particular, at the first step of the complex, the problem coincides with the Navier-Stokes equations for incompressible fluids. For such problems, open mapping theorems were established; however, the question of the existence of solutions for arbitrary degrees of the complex remains unresolved. This paper focuses on studying the existence of solutions to the problem at the second step of the de Rham complex in $\mathbb{R}^3$.

Namely, consider the de Rham complex on a Riemannian n-dimensional smooth compact closed manifold X with vector bundles Λ^q of exterior forms of degree q over X,

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d_0} \Omega^1(X) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(X) \longrightarrow 0. \tag{1}$$

Here $\Omega_q(X)$ denotes the space of all differential forms of degree q with smooth coefficients on X. In this case the Laplacians $\Delta_q = d_q^* d_q + d_{q-1} d_{q-1}^*$, $q = 0, 1, \ldots, n$, of the complex are second-order strongly elliptic differential operators on X, where operator d_q^* is a formal adjoint to d_q. As usual, we assume that for $q < 0$ and $q \geq n$, d_q is equal to zero.

We want to study non-linear problems associated with the complex. To this end, we define two bilinear bi-differential operators of zero order $M_{i,j}$ (see [5] or [24]),

$$\begin{aligned} M_{q,1}(\cdot,\cdot) &: \big(\Omega^{q+1}(X),\ \Omega^q(X)\big) \to \Omega^q(X), \\ M_{q,2}(\cdot,\cdot) &: \big(\Omega^{q}(X),\ \Omega^q(X)\big) \to \Omega^{q-1}(X). \end{aligned} \tag{2}$$

We set for a differential form u of degree q

$$N^q(u) =: M_{q,1}(d_q u, u) + d_{q-1} M_{q,2}(u, u). \tag{3}$$

Note that the operator $N^q(u)$ is nonlinear.

Let time $T > 0$ be finite. Then for any fixed positive number μ the operators $\partial_t + \mu\Delta_q$ are parabolic on the cylinder $X \times (0, T)$ (see [7]). Consider the following initial problem: given sufficiently regular differential forms f of the induced bundle $\Lambda^q(t)$ (the variable t enters into this bundle as a parameter) and u_0 of the bundle Λ^q, find differential forms u of the induced bundle $\Lambda^q(t)$ and p of the induced bundle $\Lambda^{q-1}(t)$ such that

$$\begin{cases} \partial_t u + \mu\Delta_q u + N^q(u) + d_{q-1} p = f & \text{in } X \times (0, T), \\ d_{q-1}^* u = 0 & \text{in } X \times [0, T], \\ d_{q-2}^* p = 0 & \text{in } X \times [0, T], \\ u(x, 0) = u_0 & \text{in } X, \end{cases} \tag{4}$$

For general elliptic complexes, this problem was considered in the works [21], [22] and [28], where the open mapping theorems were proved in special spaces of Hölder (see [22], [21]) and Sobolev (see [28]) types. This means that the range of the non-linear operator $\mathcal{A}_q$, related to the problem, is open in these constructed spaces. However, obtaining an existence theorem for a solution (even a so-called weak one) and closedness of the range for the related non-linear operator in such spaces appears to be a more difficult task.

For example, if we take $q = 1$ and a suitable nonlinear term, we may treat (4) as the initial problem for the well-known Navier-Stokes equations for incompressible fluid over the manifold X (see, for instance, [16] or [29]). Note that the equation with respect to p is actually missing in this case, because $d^*_{-1} = 0$.

We consider problem (4) in the case $n = 3$, $q = 2$ and a special nonlinearity $M_{q,1}(d_q u, u) = (d_q u)u$. It is easy to see that in this case we can treat the de Rham differentials as $d_2 = \mathrm{div}$, $d_1 = \mathrm{rot}$, $d^*_2 = -\nabla$, $d^*_1 = \mathrm{rot}$ and then (4) transforms to

$$\begin{cases} \partial_t u + \mu\Delta_2 u + N^2(u) + \mathrm{rot}\, p = f & \text{in } X \times (0,T), \\ \mathrm{rot}\, u = 0 & \text{in } X \times [0,T], \\ \mathrm{div}\, p = 0 & \text{in } X \times [0,T], \\ u(x,0) = u_0 & \text{in } X, \end{cases} \tag{5}$$

where

$$N^2(u) = (\mathrm{div}\, u)u + \mathrm{rot}\,(M_{q,2}(u,u)), \tag{6}$$

and Laplacian

$$\Delta_2 u = d^*_2 d_2 + d_1 d^*_1 = -\nabla \mathrm{div}\, u + \mathrm{rot}\,\mathrm{rot}\, u = -\Delta u.$$

Here Δu is the standard Laplace operator applied component-wise to the differential form u in the space variable x.

Using projection to the next step of the complex (1), we prove an existence theorem for a weak (distributional) solution in the constructed Bochner-Sobolev type spaces for some (small enough) time T^*. Note that, considering general non-linear perturbations of linear parabolic equations, one has to impose essential restrictions on the non-linear term $N^2(u)$ in order to achieve the existence of weak solutions. For example, one such condition can be the positiveness of the nonlinear operator $N^2(u)$. However, we do not impose such strong conditions on the non-linear term, but still achieve the existence of weak solutions due to the special properties of the de Rham complex.

More specifically, Section 2 of this work is devoted to the construction of special Bochner-Sobolev spaces.The main theorem of this chapter is Theorem 2, which describes the well-posedness of the action of the main operators in the introduced spaces. The proof is based on Hölder's inequality and the Gagliardo-Nirenberg interpolation inequalities (see, for example, [29] for the de Rham complex or [28] for arbitrary elliptic complexes).

The main results are presented in Section 3. Namely, by projecting problem (5) onto the next step of the complex, we obtain the problem (20), which is related to the original one. In Theorem 3, we prove that the projected problem admits a weak solution g, provided the right-hand sides are sufficiently regular and the time $t_0 \in (0,T]$ is sufficiently small. Then, in Theorem 4, we demonstrate that for any fixed weak solution of the projected problem, there exists a unique weak solution to the original problem (5). This result enables us to "reconstruct"the solution to the original problem from the solution to the projected problem. Next, in Theorem 5, we consider the projected problem in the Bochner-Sobolev spaces we have constructed and prove that, in these spaces, a sufficiently regular solution exists for sufficiently small time $T_k \in (0, T]$. Finally, in Theorem 6, we demonstrate that the original problem also has a sufficiently regular solution for bounded time, and, moreover, this solution is unique.

2 Functional spaces

Denote by $L^p_{\Lambda^q}$, $1 \le p \le \infty$, the space of differential forms of degree q with coefficients in the Lebesgue space $L^p(X)$. Similarly, we designate the spaces of forms on X whose components are of Sobolev class or have continuous partial derivatives. We denote these by $W^{s,p}_{\Lambda^q}$ and $C^s_{\Lambda^q}$ respectively with smoothness s. In the particular case of $p = 2$, we denote $H^s_{\Lambda^q} := W^{s,2}_{\Lambda^q}$.

For calculations, it is convenient to use the fractional powers of the Laplace operator. Namely, for a differential form u of degree q we denote by

$$\nabla^m_q u := \begin{cases} \Delta_q^{m/2} u, & m \text{ is even}, \\ (d_q \oplus d^*_{q-1}) \Delta_q^{(m-1)/2} u, & m \text{ is odd}. \end{cases} \tag{7}$$

It is easy to see that integration by parts yields

$$\sum_{|\alpha|=m} \|\partial^\alpha u\|^2_{L^2_{\Lambda^q}} = \|\nabla^m_q u\|^2_{L^2_{\Lambda^q}}.$$

Now, we want to recall the standard Hodge theorem for elliptic complexes. For this purpose, we denote by $\mathcal{H}^q$ the harmonic space of the complex (1), i.e.

$$\mathcal{H}^q = \left\{ u \in C^\infty_{\Lambda^q} : d_q u = 0 \text{ and } d^*_{q-1} u = 0 \text{ in } X \right\}, \tag{8}$$

and by Π^i the orthogonal projection from $L^2_{\Lambda^q}$ onto $\mathcal{H}^q$.

Theorem 1. *Let $0 \le q \le n$, $s \in \mathbb{Z}_+$. Then the operator*

$$\Delta_q : H^{s+2}_{\Lambda^q} \to H^s_{\Lambda^q} \tag{9}$$

is Fredholm:

(1) the kernel of the operator (9) equals the finite-dimensional space $\mathcal{H}^q$;

(2) given $v \in H^s_{\Lambda^q}$, there is a form $u \in H^{s+2}_{\Lambda^q}$ such that $\Delta_q u = v$ if and only if $(v, h)_{L^2_{\Lambda^q}} = 0$ for all $h \in \mathcal{H}^q$;

(3) there exists a pseudo-differential operator φ^i on X such that the operator

$$\varphi^q : H^s_{\Lambda^q} \to H^{s+2}_{\Lambda^q}, \tag{10}$$

induced by φ^q, is linear, bounded, and with the identity I we have

$$\varphi^q \Delta_q = I - \Pi^q \text{ on } H^{s+2}_{\Lambda^q}, \quad \Delta_q \varphi^q = I - \Pi^q \text{ on } H^s_{\Lambda^q} \tag{11}$$

Proof. See, for instance, [24, Theorem 2.2.2]. □

Denote by $V^s_{\Lambda^q} := H^s_{\Lambda^q} \cap S_{d^*_{q-1}}$ the space of all differential forms $u \in H^s_{\Lambda^q}$ satisfying $d^*_{q-1} u = 0$ in the sense of distributions in X. Let now $L^2(I, H^s_{\Lambda^q})$ be the Bochner space of L^2-mappings

$$u(t) : I \to H^s_{\Lambda^q},$$

where $I = [0, T]$, see, for instance, [14]. It is a Banach space with the norm

$$\|u\|^2_{L^2(I,H^s_{\Lambda^q})} = \int_0^T \|u\|^2_{H^s_{\Lambda^q}} dt.$$

We need to introduce suitable Bochner-Sobolev type spaces, see [29] for the de Rham complex and [28] for the general elliptic complexes. Namely, for $s \in \mathbb{Z}_+$ denote by $B^{k,2s,s}_{q,\mathrm{vel}}(X_T)$ the space of all differential forms of degree q over $X_T := X \times [0, T]$ with variable $t \in [0, T]$ as a parameter, such that

$$u \in C(I, V^{k+2s}_{\Lambda^q}) \cap L^2(I, V^{k+2s+1}_{\Lambda^q})$$

and

$$\nabla^m_q \partial^j_t u \in C(I, V^{k+2s-m-2j}_{\Lambda^q}) \cap L^2(I, V^{k+2s+1-m-2j}_{\Lambda^q})$$

for all $m + 2j \le 2s$. It is a Banach space with the norm

$$\|u\|^2_{B^{k,2s,s}_{q,\mathrm{vel}}} := \sum_{\substack{m+2j\le 2s\\ 0\le l\le k}} \|\nabla^l_q \nabla^m_q \partial^j_t u\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^{l+1}_q \nabla^m_q \partial^j_t u\|^2_{L^2(I,L^2_{\Lambda^q})}.$$

Similarly, for $s, k \in \mathbb{Z}_+$, we define the space $B^{k,2s,s}_{q,\mathrm{for}}(X_T)$ to consist of all differential forms

$$f \in C(I, H^{2s+k}_{\Lambda^q}) \cap L^2(I, H^{2s+k+1}_{\Lambda^q})$$

with the property that

$$\nabla^m_q \partial^j_t f \in C(I, H^{k+2s-m-2j}_{\Lambda^q}) \cap L^2(I, H^{k+2s-m-2j+1}_{\Lambda^q})$$

for all $m + 2j \le 2s$. We endow the space $B^{k,2s,s}_{q,\mathrm{for}}(X_T)$ with the natural norm

$$\|f\|^2_{B^{k,2s,s}_{q,\mathrm{for}}} := \sum_{\substack{m+2j\le 2s\\ 0\le l\le k}} \|\nabla^l_q \nabla^m_q \partial^j_t f\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^{l+1}_q \nabla^m_q \partial^j_t f\|^2_{L^2(I,L^2_{\Lambda^q})}.$$

Lastly, the space for the differential form p is denoted by $B^{k+1,2s,s}_{q-1,\mathrm{pre}}(X_T)$. This space consists of all forms p from the space $C(I, H^{2s+k+1}_{\Lambda^{q-1}}) \cap L^2(I, H^{2s+k+2}_{\Lambda^{q-1}})$ such that $d_{q-1} p \in B^{k,2s,s}_{q,\mathrm{for}}(X_T)$, $d^*_{q-2} p = 0$ and for all $h \in \mathcal{H}^{q-1}$

$$(p, h)_{L^2_{\Lambda^{q-1}}} = 0. \tag{12}$$

It is a Banach space with the norm

$$\|p\|_{B^{k+1,2s,s}_{q-1,\mathrm{pre}}} = \|d_{q-1} p\|_{B^{k,2s,s}_{q,\mathrm{for}}}.$$

Define now, for suitable forms v and w of degree q, a bi-differential operator

$$\mathbf{B}_q(w, v) = M_{q,1}(d_q w, v) + M_{q,1}(d_q v, w) + d_{q-1}\big(M_{q,2}(w, v) + M_{q,2}(v, w)\big), \tag{13}$$

with the operators $M_{q,1}$ and $M_{q,2}$ satisfying

$$|M_{q,1}(u,v)| \leq c_{q,1}|u|\,|v|, \quad |M_{q,2}(u,v)| \leq c_{q,2}|u|\,|v| \text{ on } X \tag{14}$$

with some positive constants $c_{i,j}$. The following theorem allows us to see the correctness of the operators in these spaces.

Theorem 2. *Suppose that $s \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and $2s+k > \frac{n}{2}-1$. Then the mappings*

$$\begin{array}{lrcl} \nabla_q^m : & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T) & \to & B^{k-m,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \ m \leq k \\ \Delta_q : & B^{k,2s,s}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \\ \partial_t : & B^{k,2s,s}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \end{array}$$

are continuous. Besides, if $w, v \in B^{k+2,2(s-1),s-1}_{i,\mathrm{vel}}(X_T)$ then the mappings

$$\begin{array}{lrcl} \mathbf{B}_q(w,\cdot) : & B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \\ \mathbf{B}_q(w,\cdot) : & B^{k,2s,s}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \end{array} \tag{15}$$

are continuous, too. In particular, for all $w, v \in B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}(X_T)$ there is a positive constant $c_{s,k}$, independent of v and w, such that

$$\|\mathbf{B}_q(w,v)\|_{B^{k,2(s-1),s-1}_{q,\mathrm{for}}} \leq c_{s,k}\|w\|_{B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}}\|v\|_{B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}}. \tag{16}$$

Proof. See, for instance, [29] or [28]. □

Let us introduce now the Helmholtz type projection P^q from $B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T)$ to the kernel of the operator d_q^*.

Lemma 1. *If $s, k \in \mathbb{Z}_+$, then for each q, the pseudo-differential operator $\mathrm{P}^q = d_q^* d_q \varphi^q + \Pi^q$ on X induces a continuous map*

$$\mathrm{P}^q : B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T) \to B^{k,2(s-1),s-1}_{q,\mathrm{vel}}(X_T), \tag{17}$$

such that

$$\mathrm{P}^q \circ \mathrm{P}^q u = \mathrm{P}^q u, \quad (\mathrm{P}^q u, v)_{L^2_{\Lambda^q}(X)} = (u, \mathrm{P}^q v)_{L^2_{\Lambda^q}(X)}, \quad (\mathrm{P}^q u, (I-\mathrm{P}^q)u)_{L^2_{\Lambda^q}(X)} = 0$$

for all $u, v \in B^{k,2(s-1),s-1}_{q,\mathrm{for}}$.

Proof. See, for instance, [28]. □

The following lemma is just a consequence of Hodge Theorem 1.

Lemma 2. *Let $F \in B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T)$ satisfy $\mathrm{P}^q F = 0$ in X_T. Then there is a unique section $p \in B^{k+1,2(s-1),s-1}_{q-1,\mathrm{pre}}(X_T)$ such that (12) holds and*

$$d_{q-1} p = F \text{ in } X \times [0,T]. \tag{18}$$

Now we are ready to move on to the main part of this paper.

3 Existence theorem

In order to get an existence theorem for Problem (5) we use a projection to the next step of the complex (1). Namely, applying the operator $d_2 = \mathrm{div}$ to equation (5) we have

$$\begin{cases} \partial_t \mathrm{div} u - \mu\, \mathrm{div}(\nabla \mathrm{div}\, u) + \mathrm{div}((\mathrm{div}\, u)u) = \mathrm{div} f & \text{in } X \times (0,T), \\ \mathrm{div}\, u(x,0) = \mathrm{div}\, u_0 & \text{in } X, \end{cases} \tag{19}$$

because $\mathrm{rot}\, u = 0$ and $\mathrm{div} \circ \mathrm{rot} \equiv 0$. Now,

$$\mathrm{div}((\mathrm{div}\, u)u) = (\mathrm{div}\, u)^2 + \Delta u \cdot u = (\mathrm{div}\, u)^2 + \nabla \mathrm{div}\, u \cdot u.$$

By Theorem 1

$$u = \varphi^2 \Delta_2 u + \Pi^2 u = \varphi^2 \nabla \mathrm{div}\, u + \Pi^2 u.$$

Denote

$$g = \mathrm{div}\, u,$$

then we can rewrite (19) in the following way

$$\begin{cases} \partial_t g - \mu\, \mathrm{div}(\nabla g) + g^2 + \nabla g \cdot (\varphi^2 \nabla g + \Pi^2 u) = \mathrm{div} f & \text{in } X \times (0,T), \\ g(x,0) = \mathrm{div}\, u_0 & \text{in } X. \end{cases} \tag{20}$$

Theorem 3. *Given any pair $(f, u_0) \in L^2(I, (V^0_{\Lambda^2})') \times V^1_{\Lambda^2}$, there exists a time $t_0 \in (0,T]$ such that for all $t \in [0,t_0]$ there exists a differential form $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ with $\partial_t g \in L^2(I, (H^1_{\Lambda^3})')$, satisfying*

$$\left\{ \begin{array}{rcl} \frac{d}{dt}(g,v)_{L^2_{\Lambda^3}} + \mu(\nabla g, \nabla v)_{L^2_{\Lambda^2}} & = & \langle \mathrm{div}\, f - g^2 - \nabla g \cdot (\varphi^2\, \nabla g + \Pi^2 u), v \rangle, \\ g(\cdot, 0) & = & \mathrm{div}\, u_0 \end{array} \right. \tag{21}$$

for all $v \in H^k_{\Lambda^3}$ with $k \geq 2$.

Proof. Let $\{u_m\}$ be the sequence of Faedo-Galerkin approximations, namely,

$$u_m = \sum_{j=1}^{M} c_j^{(m)}(t) b_j(x), \tag{22}$$

then

$$g_m = \mathrm{div}\, u_m = \sum_{j=1}^{M} c_j^{(m)}(t)\, \mathrm{div}\, b_j(x), \tag{23}$$

where the system $\{b_j\}_{j \in \mathbb{N}}$ is a $L^2_{\Lambda^2}(X)$-orthogonal basis in $V^1_{\Lambda^2}$ and the functions u_m satisfy the following relations

$$\frac{d}{dt}(g_m, \mathrm{div}\, b_j)_{L^2_{\Lambda^3}} + \mu(\nabla g_m, \nabla \mathrm{div}\, b_j)_{L^2_{\Lambda^2}} = \tag{24}$$

$$\langle \mathrm{div}\, f - g_m^2 - \nabla g_m \cdot \varphi^2\, \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, \mathrm{div}\, b_j \rangle,$$

$$g_m(x,0) = \mathrm{div}\, u_{0,m}(x),$$

for all $0 \le j \le m$ with the initial data $u_{0,m}$ from the linear span $\mathcal{L}(\{b_j\}_{j=1}^m)$, such that the sequence $\{u_{0,m}\}$ converges to u_0 in $V^1_{\Lambda^2}$. For instance, as $\{u_{0,m}\}$ we may take the orthogonal projection onto the linear span $\mathcal{L}(\{b_j\}_{j=1}^m)$.

Multiplying (24) by $c_j^{(m)}(t)$ and summing over j we have

$$(\partial_t g_m, g_m)_{L^2_{\Lambda^3}} + \mu(\nabla g_m, \nabla g_m)_{L^2_{\Lambda^2}} = \langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2\, \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, g_m\rangle. \tag{25}$$

It follows from the Lemma by J.-L. Lions (see, for instance, [26, Ch. III, § 1, Lemma 1.2]) that

$$\frac{d}{dt}\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} = 2\,\langle \partial_t g_m, g_m\rangle.$$

Then, integrating over $t \in [0,T]$, we see that

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} + 2\mu \int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}}\, dt = \tag{26}$$

$$\|g_m(\cdot,0)\|^2_{L^2_{\Lambda^3}} + 2\int_0^t \langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2\, \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, g_m\rangle dt.$$

Since $f \in L^2(I, L^2_{\Lambda^2})$, then $\operatorname{div} f \in L^2(I, (V^1_{\Lambda^3})')$ and

$$2\left|\int_0^t \langle \operatorname{div} f, g_m\rangle dt\right| \le 2\int_0^t \|\operatorname{div} f\|_{(V^1_{\Lambda^3})'} \|g_m\|_{V^1_{\Lambda^3}}\, dt \le \tag{27}$$

$$\frac{4}{\mu}\int_0^t \|\operatorname{div} f\|^2_{(V^1_{\Lambda^3})'} dt + \frac{\mu}{4}\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt + \frac{\mu}{4}\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt.$$

On the other hand

$$2\left|\int_0^t \langle g_m^2, g_m\rangle dt\right| \le 2\int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt. \tag{28}$$

Note that in our case $\nabla_3 = -\nabla$ with $n = 3$. Then, from the Gagliardo-Nirenberg inequality (see [20] or [4, Theorem 3.70]) we have

$$2\int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt \le \tag{29}$$

$$c\int_0^t \left[\left(\|\nabla g_m\|_{L^2_{\Lambda^2}} + \|g_m\|_{L^2_{\Lambda^3}}\right)^{\frac{1}{2}} \|g_m\|^{\frac{1}{2}}_{L^2_{\Lambda^3}} + \|g_m\|_{L^2_{\Lambda^3}}\right]^3 dt \le$$

$$c_1\int_0^t \left[\|\nabla g_m\|^{\frac{1}{2}}_{L^2_{\Lambda^2}} \|g_m\|^{\frac{1}{2}}_{L^2_{\Lambda^3}} + \|g_m\|_{L^2_{\Lambda^3}}\right]^3 dt \le$$

$$c_2\int_0^t \left(\|\nabla g_m\|^{\frac{3}{2}}_{L^2_{\Lambda^2}} \|g_m\|^{\frac{3}{2}}_{L^2_{\Lambda^3}} + \|g_m\|^3_{L^2_{\Lambda^3}}\right) dt \le$$

$$\frac{\mu}{2}\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt + c_3\int_0^t \left(\|g_m\|^3_{L^2_{\Lambda^3}} + \|g_m\|^6_{L^2_{\Lambda^3}}\right) dt$$

with positive constants c, c_1, and c_2. The last expression is a consequence of the standard Young's inequality. Moreover, there are positive constants c and c_1 such that

$$\int_0^t \left(\|g_m\|^3_{L^2_{\Lambda^3}} + \|g_m\|^6_{L^2_{\Lambda^3}}\right) dt \leq c \int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} \left(1 + \|g_m\|_{L^2_{\Lambda^3}}\right)^4 dt \leq$$

$$c_1 \left(\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt + \int_0^t \|g_m\|^6_{L^2_{\Lambda^3}} dt\right).$$

Then we conclude that

$$2\int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt \leq \frac{\mu}{2}\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt + c\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt + c\int_0^t \|g_m\|^6_{L^2_{\Lambda^3}} dt \tag{30}$$

with some constant $c > 0$. Next,

$$\int_0^t \langle \nabla g_m \cdot \varphi^2\, \nabla g_m, g_m\rangle dt = \sum_{j=1}^3 \int_0^t \int\limits_X \partial_j g_m (\varphi^2 \partial_j g_m) g_m \, dx\, dt =$$

$$-\sum_{j=1}^3 \int_0^t \int\limits_X g_m (\varphi^2 \partial_j g_m) \partial_j g_m \, dx\, dt - \int_0^t \int\limits_X g_m^3 \, dx\, dt,$$

because $\varphi^2 \Delta g_m = g_m$. It means that

$$\int_0^t \langle \nabla g_m \cdot \varphi^2\, \nabla g_m, g_m\rangle dt = -\frac{1}{2}\int_0^t \int\limits_X g_m^3\, dx\, dt,$$

and hence

$$2\left|\int_0^t \langle \nabla g_m \cdot \varphi^2\, \nabla g_m, g_m\rangle dt\right| \leq \int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt. \tag{31}$$

Finally,

$$\int_0^t \langle \nabla g_m \cdot \Pi^2 u_m, g_m\rangle dt = \sum_{j=1}^3 \int_0^t \int\limits_X \partial_j g_m (\Pi^2 u_m^j) g_m \, dx\, dt =$$

$$-\sum_{j=1}^3 \int_0^t \int\limits_X g_m (\Pi^2 u_m^j) \partial_j g_m \, dx\, dt - \sum_{j=1}^3 \int_0^t \int\limits_X g_m^2 \partial_j (\Pi^2 u_m^j)\, dx\, dt,$$

and then

$$\int_0^t \langle \nabla g_m \cdot \Pi^2 u_m, g_m\rangle dt = 0, \tag{32}$$

because $\operatorname{div} \Pi^2 u_m = 0$.

Now, inequalities (26) - (32) give

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} + 2\mu\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^3}} dt \leq \|g_m(\cdot,0)\|^2_{L^2_{\Lambda^3}} + \tag{33}$$

$$\frac{4}{\mu}\int_0^t \|\operatorname{div} f\|^2_{(V^1_{\Lambda^3})'} dt + \mu\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} + \frac{\mu}{4}\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt +$$

$$2c\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}}dt + 2c\int_0^t \|g_m\|^6_{L^2_{\Lambda^3}}dt,$$

and then

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} + \mu\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^3}}dt \le \|g_m(\cdot,0)\|^2_{L^2_{\Lambda^3}} + \tag{34}$$

$$\frac{4}{\mu}\|\operatorname{div} f\|^2_{L^2\left(I,(V^1_{\Lambda^3})'\right)} + \left(\frac{\mu}{4}+2c\right)\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}}dt + 2c\int_0^t \|g_m\|^6_{L^2_{\Lambda^3}}dt.$$

It follows from the Gronwall-Perov's Lemma (see, for instance [18, p. 360]) that there exists a time $t_0 \in (0,T]$ and a positive constant C_{t_0} such that

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} \le C_{t_0} \tag{35}$$

for all $t \in [0,t_0]$. Then the sequence g_m is bounded in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$, where $I_{t_0} = [0,t_0]$. Moreover, it follows from (34) and (35) that $\|\nabla g_m(\cdot,t)\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})}$ is also bounded. This means that there exists a subsequence that converges weakly-$*$ in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$ and weakly in $L^2(I_{t_0}, H^1_{\Lambda^3})$ to some $g \in L^\infty(I_{t_0}, L^2_{\Lambda^3}) \cap L^2(I_{t_0}, H^1_{\Lambda^3})$. We use the same designation g_m for such a subsequence. Then the standard arguments show (see, for instance, [15], [26] or [13]) that we can pass to the limit in (24) with respect to $m \to \infty$ and conclude that the element g satisfies equation (21).

□

Let us now return to the Problem (5). Denoting again $g = \operatorname{div} u$ and multiplying (5) scalar in $L^2_{\Lambda^2}$ by a differential form $v \in V^k_{\Lambda^2}$ we get

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}(u,v)_{L^2_{\Lambda^2}} + \mu(g,\operatorname{div} v)_{L^2_{\Lambda^3}} & = & \langle f - gu, v\rangle, \\ u(x,0) & = & u_0. \end{array}\right. \tag{36}$$

Theorem 4. *Let $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ be the solution to (21) for any given pair $(f,u_0) \in L^2(I, V^0_{\Lambda^2}) \times V^1_{\Lambda^2}$. Then there exists a unique differential form $u \in C(I, V^1_{\Lambda^2}) \cap L^2(I, V^2_{\Lambda^2})$ satisfying (36) for all $v \in V^k_{\Lambda^3}$ with $k \ge 2$.*

Proof. Indeed, let $\{u_m\}$ be the sequence of Faedo-Galerkin approximations (see (22)) such that the sequence $\{g_m\} = \{\operatorname{div} u_m\}$ converges to $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$. Substituting u_m into (36) instead of u, v and integrating by $t \in [0,t_0]$ we have

$$\|u_m(\cdot,t)\|^2_{L^2_{\Lambda^2}} + 2\mu\|g_m\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})} = \|u_m(x,0)\|^2_{L^2_{\Lambda^2}} + 2\int_0^{t_0}\langle f - g_m u_m, u_m\rangle dt. \tag{37}$$

As usual, we evaluate using the Hölder inequality

$$2\left|\int_0^{t_0}\langle f, u_m\rangle dt\right| \le \int_0^{t_0}\|u_m\|^2_{L^2_{\Lambda^2}}dt + \int_0^{t_0}\|f\|^2_{L^2_{\Lambda^2}}dt$$

and by the Gagliardo-Nirenberg inequality

$$2\left|\int_0^{t_0}\langle g_m u_m, u_m\rangle dt\right| \le \int_0^{t_0}\|u_m\|^2_{L^4_{\Lambda^2}}\|g_m\|_{L^2_{\Lambda^3}}dt \le$$

$$c\int_0^{t_0}\|g_m\|_{L^2_{\Lambda^3}}\left(\|g_m\|^{3/2}_{L^2_{\Lambda^3}}\|u_m\|^{1/2}_{L^2_{\Lambda^2}} + \|u_m\|^2_{L^2_{\Lambda^2}}\right)dt \le$$

$$2\mu\int_0^{t_0}\|g_m\|^2_{L^2_{\Lambda^3}}dt + c_1\int_0^{t_0}\|g_m\|^4_{L^2_{\Lambda^3}}\|u_m\|^2_{L^2_{\Lambda^2}}dt + c_2\int_0^{t_0}\|g_m\|_{L^2_{\Lambda^3}}\|u_m\|^2_{L^2_{\Lambda^2}}dt$$

with some positive constants c, c_1 and c_2. Then

$$\|u_m(\cdot,t)\|^2_{L^2_{\Lambda^2}} \le \|u_0\|^2_{L^2_{\Lambda^2}} + \int_0^{t_0}\|f\|^2_{L^2_{\Lambda^2}}dt + c\int_0^{t_0}\|u_m\|^2_{L^2_{\Lambda^2}}dt \tag{38}$$

with a constant $c > 0$ independent of m. It follows from Gronwall's Lemma that

$$\|u_m(\cdot,t)\|^2_{L^2_{\Lambda^2}} \le C, \tag{39}$$

where constant C depends on the norms $\|f\|^2_{L^2(I_{t_0},L^2_{\Lambda^2})}$, $\|u_0\|^2_{L^2_{\Lambda^2}}$ and $\|g\|_{C(I_{t_0},L^2_{\Lambda^2})}$, but is independent of m.

It follows that the sequence u_m is bounded in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$ and there is a subsequence that converges weakly-$*$ in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$ to some $u \in L^\infty(I_{t_0}, L^2_{\Lambda^3})$. We again use the same designation u_m for such a subsequence. Under the hypothesis of this Theorem, the sequence $g_m = \operatorname{div} u_m$ converges to $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$, then actually $u \in C(I, V^1_{\Lambda^3}) \cap L^2(I, V^2_{\Lambda^3})$. Passing to the limit in (37) with respect to $m \to \infty$ we conclude that the element u satisfies (36).

Let now u' and u'' be two solutions of (36) such that $\operatorname{div} u' = \operatorname{div} u'' = g$. Hence, the differential form $u = u' - u''$ satisfies (36) with zero data $(f, u_0) = (0,0)$. It follows from (38) and Gronwall-Perov's Lemma that $\|g(\cdot,t)\|_{L^2_{\Lambda^3}} = 0$. Therefore, the Problem (36) has a unique solution.

Moreover, if u_1, u_2 are two solutions to (36), corresponding to the solutions $g_1 = \operatorname{div} u_1$ and $g_2 = \operatorname{div} u_2$ of (21), then the differential form $u = u_1 - u_2$ satisfies

$$\begin{cases} \dfrac{d}{dt}(u,v)_{L^2_{\Lambda^2}} + \mu(g, \operatorname{div} v)_{L^2_{\Lambda^3}} &= \langle -gu, v\rangle, \\ u(x,0) &= 0, \end{cases} \tag{40}$$

where $g = g_1 - g_2$.

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^2}} + 2\mu\|g\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})} = -2\int_0^{t_0}\langle gu, u\rangle dt. \tag{41}$$

Applying the Gagliardo-Nirenberg inequality, we have

$$2\left|\int_0^{t_0}\langle gu, u\rangle dt\right| \le c_1\int_0^{t_0}\|g\|_{L^2_{\Lambda^3}}\left(\|\nabla u\|^{3/4}_{L^2_{\Lambda^2}}\|u\|^{1/4}_{L^2_{\Lambda^2}} + \|u\|_{L^2_{\Lambda^2}}\right)^2 dt \le$$

$$c_2\int_0^{t_0}\left(\|g\|^{5/2}_{L^2_{\Lambda^2}}\|u\|^{1/2}_{L^2_{\Lambda^2}} + \|u\|^2_{L^2_{\Lambda^2}}\|g\|_{L^2_{\Lambda^3}}\right)dt \le$$

$$2\mu\|g\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})} + c_3\left(\|g\|^4_{C(I_{t_0},L^2_{\Lambda^3})} + \|g\|_{C(I_{t_0},L^2_{\Lambda^3})}\right)\int_0^{t_0}\|u\|_{L^2_{\Lambda^2}}dt$$

with positive constants c_1, c_2, and c_3. The last inequality, (41), and Gronwall-Perov's Lemma yield

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^2}} \le 0,$$

then $u_1 = u_2$ and the Problem (36) has a unique solution.

□

Corollary 1. *Under the hypothesis of Theorem 3, let $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ be a solution of (21) and let $u \in C(I, V^1_{\Lambda^2}) \cap L^2(I, V^2_{\Lambda^2})$ be the solution of (36), corresponding to g. If moreover $g \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$, then the solution g is unique.*

Proof. Indeed, let $g_1, g_2 \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ be two solutions of (21) with corresponding forms $u_1, u_2 \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ satisfying (36). Hence, the differential form $g = g_1 - g_2$ satisfies

$$\left\{\begin{array}{rcl} \frac{d}{dt}\|g\|^2_{L^2_{\Lambda^3}} + \mu\|\nabla g\|^2_{L^2_{\Lambda^2}} &=& \left\langle -(g_1^2 - g_2^2) - \left(\nabla g_1\cdot(\varphi^2\,\nabla g_1 + \Pi^2 u_1) - \right.\right. \\ && \left.\left.\nabla g_2\cdot(\varphi^2\,\nabla g_2 + \Pi^2 u_2)\right), v\right\rangle, \\ g(\cdot,0) &=& 0. \end{array}\right. \tag{42}$$

Integrating by $t \in I_{t_0}$ we get

$$\|g(\cdot,t\|^2_{L^2_{\Lambda^3}} + 2\mu\int_0^{t_0}\|\nabla g\|^2_{L^2_{\Lambda^2}}dt \le 2\int_0^{t_0}|\langle g(g_1+g_2) + \tag{43}$$
$$\left(\nabla g_1\cdot(\varphi^2\,\nabla g_1 + \Pi^2 u_1) - \nabla g_2\cdot(\varphi^2\,\nabla g_2 + \Pi^2 u_2)\right), v\rangle|\,dt,$$

We have to estimate the right side of (43). First, it follows from the Gagliardo-Nirenberg interpolation inequality that

$$2\int_0^{t_0}|\langle g(g_1+g_2), g\rangle|\,dt \le \tag{44}$$
$$c\|(g_1+g_2)\|_{C(I_{t_0},L^2_{\Lambda^3})}\int_0^{t_0}\left(\|\nabla g\|^{3/2}_{L^2_{\Lambda^3}}\|g\|^{1/2}_{L^2_{\Lambda^3}} + \|g\|^2_{L^2_{\Lambda^3}}\right)dt \le$$
$$\mu\int_0^{t_0}\|\nabla g\|^2_{L^2_{\Lambda^3}}dt + c_1\int_0^{t_0}\|g\|^2_{L^2_{\Lambda^3}}dt$$

with positive constants c and c_1. Next,

$$2\int_0^{t_0}\left|\left\langle\nabla g_1\cdot\varphi^2\,\nabla g_1 - \nabla g_2\cdot\varphi^2\,\nabla g_2 + \nabla g_1\cdot\varphi^2\,\nabla g_2 - \nabla g_1\cdot\varphi^2\,\nabla g_2, g\right\rangle\right|dt \le \tag{45}$$
$$2\int_0^{t_0}\left|\left\langle\nabla g_1\cdot\varphi^2\,\nabla g, g\right\rangle\right|dt + 2\int_0^{t_0}\left|\left\langle\nabla g\cdot\varphi^2\,\nabla g_2, g\right\rangle\right|dt \le$$
$$c_1\|g_1\|_{C(I_{t_0},H^1_{\Lambda^3})}\int_0^{t_0}\left(\|\nabla g\|^{3/4}_{L^2_{\Lambda^3}}\|g\|^{5/4}_{L^2_{\Lambda^3}} + \|g\|^2_{L^2_{\Lambda^3}}\right)dt +$$

$$c_2\|g_1\|_{C(I_{t_0},H^1_{\Lambda^3})}\int_0^{t_0}\left(\|\nabla g\|^{7/4}_{L^2_{\Lambda^3}}\|g\|^{1/4}_{L^2_{\Lambda^3}}+\|g\|^2_{L^2_{\Lambda^3}}\right)dt\le$$
$$\mu\int_0^{t_0}\|\nabla g\|^2_{L^2_{\Lambda^3}}dt+c\int_0^{t_0}\|g\|^2_{L^2_{\Lambda^3}}dt$$

with some positive constants c, c_1, and c_2. Finally,

$$2\int_0^{t_0}\left|\left\langle\nabla g_1\cdot\Pi^2 u_1-\nabla g_2\cdot\Pi^2 u_2+\nabla g_2\cdot\Pi^2 u_1-\nabla g_2\cdot\Pi^2 u_1,g\right\rangle\right|dt\le \quad (46)$$
$$2\int_0^{t_0}\left|\left\langle\nabla g\cdot\Pi^2 u_1,g\right\rangle\right|dt+2\int_0^{t_0}\left|\left\langle\nabla g_2\cdot\Pi^2(u_1-u_2),g\right\rangle\right|dt$$

The Theorem 4 implies that $u_1=u_2$. On the other hand, integrating by parts, we easily see that

$$2\int_0^{t_0}\left|\left\langle\nabla g\cdot\Pi^2 u_1,g\right\rangle\right|dt=0,$$

and then (46) equals to zero.

Finally, using (43) - (46), we get

$$\|g(\cdot,t\|^2_{L^2_{\Lambda^3}}\le c\int_0^{t_0}\|g\|^2_{L^2_{\Lambda^3}}dt$$

with some constant $c>0$. Then, it follows from the Gronwall-Perov's Lemma that $\|g(\cdot,t)\|_{L^2_{\Lambda^3}}=0$, and thus the Problem (21) has a unique solution. □

Theorem 5. *Let $s\in\mathbb{N}$ and $k\in\mathbb{Z}_+$ with $k>3/2$. Then for all*

$$(f,u_0)\in B^{k+1,2(s-1),s-1}_{\Lambda^2,\mathrm{for}}(X_T)\times V^{2s+k+1}_{\Lambda^2}$$

there exists a time $T_k\in(0,T]$ such that the Problem (20) has a solution

$$g\in B^{k,2s,s}_{\Lambda^3,\mathrm{for}}(X_{T_k}).$$

Moreover, the solution g is unique, if the form u in (19) satisfies (36).

Proof. First of all, denote by

$$\Lambda_r=\begin{cases}\Lambda^3, & r\text{ is even},\\ \Lambda^2, & r\text{ is odd}.\end{cases}$$

As before, let g_m be the Faedo-Galerkin approximations (see (23)). We start with the following apriori estimates.

Lemma 3. *Under the hypothesis of Theorem 5, if $(f,u_0)\in B^{k+1,0,0}_{2,\mathrm{for}}(X_T)\times V^{k+3}_{\Lambda^2}$ with some $k\in\mathbb{Z}_+$, then there exists a time $T_k\in(0,T]$ such that*

$$\|\nabla_2^{k'}g_m\|^2_{C(I_{T_k},L^2_{\Lambda_{k'}})}+\mu\,\|\nabla_2^{k'+1}g_m\|^2_{L^2(I_{T_k},L^2_{\Lambda_{k'+1}})}\le C_{k'} \quad (47)$$

for any $0\le k'\le k+2$, where $I_{T_k}=[0,T_k]$ and the constants $C_{k'}=C_{k'}(\mu,f,u_0)>0$ depend on k', μ, and the norms $\|f\|_{B^{k+1,0,0}_{2,\mathrm{for}}(X_{T_k})}$, $\|u_0\|_{V^{k+3}_{\Lambda^2}}$ but not on m.

Proof. Indeed, if $k' = 0$ then (47) follows immediately from (34) and Gronwall-Perov's Lemma. Now, substituting g_m and $\nabla_3^{2r} g_m$ in (20) instead of g and v respectively, with some $r \in \mathbb{N}$, and integrating by $t \in [0,T]$ we get

$$\|\nabla_3^r g_m(\cdot,t)\|^2_{L^2_{\Lambda_r}} + 2\mu \int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda^3}} dt = \tag{48}$$

$$\|\nabla_3^r g_m(\cdot,0)\|^2_{L^2_{\Lambda_r}} + 2\int_0^t \langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2 \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, \nabla_3^{2r} g_m\rangle dt.$$

We have to estimate the right side of (48). First,

$$2\left|\int_0^t \langle \operatorname{div} f, \nabla_3^{2r} g_m\rangle dt\right| \le 2\int_0^t \|\nabla_3^{r-1} \operatorname{div} f\|_{L^2_{\Lambda_{r-1}}} \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}} dt \le \tag{49}$$

$$\frac{4}{\mu}\int_0^t \|\nabla_3^{r-1} \operatorname{div} f\|^2_{L^2_{\Lambda_{r-1}}} dt + \frac{\mu}{4}\int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda_{r+1}}} dt.$$

Further,

$$2\left|\int_0^t \langle g_m^2, \nabla_3^{2r} g_m\rangle dt\right| \le 2\int_0^t \|\nabla_3^{r-1}(g_m^2)\|_{L^2_{\Lambda_{r-1}}} \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}} dt. \tag{50}$$

Let $r \ge 2$, using Hölder and Gagliardo-Nirenberg inequalities, we get

$$\|\nabla_3^{r-1}(g_m^2)\|_{L^2_{\Lambda_{r-1}}} \le \sum_{|\alpha|+|\beta|=r-1} c_{\alpha\beta} \|\partial^\alpha g_m\|_{L^4_{\Lambda^3}} \|\partial^\beta g_m\|_{L^4_{\Lambda^3}} \le \tag{51}$$

$$\sum_{|\alpha|+|\beta|=r-1} c_{\alpha\beta}\left(\left(\|\nabla_3^{|\alpha|+1} g_m\|_{L^2_{\Lambda_{|\alpha|+1}}} + \|\nabla_3^{|\alpha|} g_m\|_{L^2_{\Lambda_{|\alpha|}}}\right)^{3/4} \|\nabla_3^{|\alpha|} g_m\|^{1/4}_{L^2_{\Lambda_{|\alpha|}}} + \right.$$

$$\left. + \|\nabla_3^{|\alpha|} g_m\|_{L^2_{\Lambda_{|\alpha|}}}\right)\left(\left(\|\nabla_3^{|\beta|+1} g_m\|_{L^2_{\Lambda_{|\beta|+1}}} + \|\nabla_3^{|\beta|} g_m\|_{L^2_{\Lambda_{|\beta|}}}\right)^{3/4} \|\nabla_3^{|\beta|} g_m\|^{1/4}_{L^2_{\Lambda_{|\beta|}}} + \right.$$

$$\left. + \|\nabla_3^{|\beta|} g_m\|_{L^2_{\Lambda_{|\beta|}}}\right) \le c\left(\|g_m\|^2_{H^{r-1}_{\Lambda^3}} + \|g_m\|^{5/4}_{H^{r-1}_{\Lambda^3}} \|\nabla_3^r g_m\|^{3/4}_{L^2_{\Lambda_r}}\right)$$

with some positive constants c and $c_{\alpha\beta}$. For the exception case $r = 1$, the last inequality takes the form

$$\|\nabla_3(g_m^2)\|_{L^2_{\Lambda_{r-1}}} \le c\left(\|g_m\|^2_{L^2_{\Lambda^3}} + \|g_m\|^{1/2}_{L^2_{\Lambda^3}} \|\nabla_3 g_m\|^{3/2}_{L^2_{\Lambda^2}}\right), \tag{52}$$

because in (51) there arises a case when $|\alpha| = |\beta| = r - 1 = 0$. It follows from (50), (51) and Young's inequality that

$$2\left|\int_0^t \langle g_m^2, \nabla_3^{2r} g_m\rangle dt\right| \le \frac{\mu}{4}\int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda_{r+1}}} dt + \tag{53}$$

$$c\|g_m\|^4_{C(I,H^{r-1}_{\Lambda^3})} + c\|g_m\|^{5/2}_{C(I,H^{r-1}_{\Lambda^3})} \int_0^t \|\nabla_3^r g_m\|^{3/2}_{L^2_{\Lambda_r}} dt$$

for $r \ge 2$ and

$$2\left|\int_0^t \langle g_m^2, \nabla_3^2 g_m\rangle dt\right| \le \frac{\mu}{4}\int_0^t \|\nabla_3^2 g_m\|^2_{L^2_{\Lambda^3}} dt + \tag{54}$$

$$c\|g_m\|^4_{C(I,L^2_{\Lambda^2})} + c\|g_m\|_{C(I,L^2_{\Lambda^2})}\int_0^t \|\nabla_3 g_m\|^3_{L^2_{\Lambda^2}}dt$$

for $r = 1$, with some constant $c > 0$.

Next,

$$2\left|\int_0^t \langle \nabla g_m \cdot \varphi^2\, \nabla g_m, \nabla_3^{2r} g_m\rangle dt\right| \le \tag{55}$$

$$2\int_0^t \|\nabla_3^{r-1}\left(\nabla g_m \cdot \varphi^2\, \nabla g_m\right)\|_{L^2_{\Lambda_{r-1}}} \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}}dt.$$

Analogous to (51), we have

$$\|\nabla_3^{r-1}\left(\nabla g_m \cdot \varphi^2\, \nabla g_m\right)\|_{L^2_{\Lambda_{r-1}}} \le \tag{56}$$

$$c\Bigg(\|g_m\|_{H^{r-1}_{\Lambda^3}}\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}} + \|g_m\|^{1/4}_{H^{r-1}_{\Lambda^3}}\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}}\|\nabla_3^r g_m\|^{3/4}_{L^2_{\Lambda_r}} +$$

$$\|\nabla_3^r g_m\|^{1/4}_{L^2_{\Lambda_r}}\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}}\|\nabla_3^{r+1} g_m\|^{3/4}_{L^2_{\Lambda_r}}\Bigg)$$

with $r \in \mathbb{N}$ and some constant $c > 0$. Theorem 1 implies that $\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}} \le c\|g_m\|_{H^{r-1}_{\Lambda^3}}$ with some positive constant c, then

$$2\left|\int_0^t \langle \nabla g_m \cdot \varphi^2\, \nabla g_m, \nabla_3^{2r} g_m\rangle dt\right| \le \frac{\mu}{4}\int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda_{r+1}}}dt + \tag{57}$$

$$c\|g_m\|^4_{C(I,H^{r-1}_{\Lambda^3})} + c\|g_m\|^{5/2}_{C(I,H^{r-1}_{\Lambda^3})}\int_0^t \|\nabla_3^r g_m\|^{3/2}_{L^2_{\Lambda_r}}dt +$$

$$c\|g_m\|^{10}_{C(I,H^{r-1}_{\Lambda^3})}\int_0^t \|\nabla_3^r g_m\|^2_{L^2_{\Lambda_r}}dt$$

with $c > 0$.

Finally,

$$2\left|\int_0^t \langle \nabla g_m \Pi^2 u_m, \nabla_3^{2r} g_m\rangle dt\right| \le \tag{58}$$

$$2\int_0^t \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}}\|\nabla_3^{r-1}\left(\nabla g_m \cdot \Pi^2 u_m\right)\|_{L^2_{\Lambda_{r-1}}}dt,$$

and we have again

$$\|\nabla_3^{r-1}\left(\nabla g_m \cdot \Pi^2 u_m\right)\|_{L^2_{\Lambda_{r-1}}} \le \tag{59}$$

$$c\Bigg(\|g_m\|_{H^{r-1}_{\Lambda^3}}\|\Pi^2 u_m\|_{H^r_{\Lambda^2}} + \|g_m\|^{1/4}_{H^{r-1}_{\Lambda^3}}\|\Pi^2 u_m\|_{H^r_{\Lambda^2}}\|\nabla_3^r g_m\|^{3/4}_{L^2_{\Lambda_r}} +$$

$$\|\nabla_3^r g_m\|^{1/4}_{L^2_{\Lambda_r}}\|\Pi^2 u_m\|_{H^r_{\Lambda^3}}\|\nabla_3^{r+1} g_m\|^{3/4}_{L^2_{\Lambda_r}}\Bigg)$$

with positive constant c. Operator Π^2 is bounded in $L^2_{\Lambda^2}$ by the Hodge Theorem 1. On the other hand, Theorem 4 yields that the sequence $\{u_m\}$ is bounded in $L^2_{\Lambda^2}$ (see (39)), then $\|\Pi^2 u_m\|_{H^r_{\Lambda^2}} \le c\|g_m\|_{H^{r-1}_{\Lambda^3}}$ and we obtain

$$2\left|\int_0^t \langle \nabla g_m \Pi^2 u_m, \nabla_3^{2r} g_m\rangle dt\right| \leq \frac{\mu}{4}\int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda_{r+1}}} dt + \tag{60}$$
$$c\|g_m\|^4_{C(I,H^{r-1}_{\Lambda^3})} + c\|g_m\|^{5/2}_{C(I,H^{r-1}_{\Lambda^3})}\int_0^t \|\nabla_3^r g_m\|^{3/2}_{L^2_{\Lambda_r}} dt +$$
$$c\|g_m\|^{10}_{C(I,H^{r-1}_{\Lambda^3})}\int_0^t \|\nabla_3^r g_m\|^2_{L^2_{\Lambda_r}} dt$$

with $c > 0$.

It follows from (48) - (60) and Gronwall-Perov's Lemma that if $(f, u_0) \in B^{k+1,0,0}_{2,\text{for}}(X_T) \times V^{k+3}_{\Lambda^2}$ and the norm $\|g_m\|_{C(I,H^{r-1}_{\Lambda^3})}$ is bounded for some $r \in \mathbb{N}$, $r \leq k+2$, then there exists a time $t_r \in (0, t_0]$ and a positive constant C_r, which depends on the norms $\|f\|_{B^{r+1,0,0}_{2,\text{for}}(X_{T_k})}$ and $\|u_0\|_{V^{r+3}_{\Lambda^2}}$, such that

$$\|\nabla_3^r g_m(\cdot,t)\|^2_{L^2_{\Lambda_r}} + \mu\int_0^{t_r} \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda^3}} dt \leq C_r(\mu, f, u_0). \tag{61}$$

Using (61) consistently for $r = 1, \dots, k+2$ we get a family of times t_r. Denote $T_k = \min\limits_{r \leq k+2} t_r$, then, (61) yields that for any $k \in \mathbb{Z}_+$ there exists a time T_k such that (3) is fulfilled. □

Theorem 3 implies that there exists a solution $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ of (21). On the other hand, it follows from Lemma 3 that for each $(f, u_0) \in B^{k+1,2(s-1),s-1}_{\Lambda^2,\text{for}}(X_T) \times V^{2s+k+1}_{\Lambda^2}$ there exists a time $T_k \in (0, T]$ and a subsequence $\{g_{m'} = \operatorname{div} u_{m'}\}$ such that $\{g_{m'}\}$ converges weakly in $L^2(I_{T_k}, L^2_{\Lambda^3})$ and *-weakly in $L^\infty(I_{T_k}, H^{k+2}_{\Lambda^3}) \cap L^2(I, H^{k+3}_{\Lambda^3})$ to an element g, then $g \in B^{k,2s,s}_{\Lambda^3,\text{for}}(X_{T_k})$. Moreover, the uniqueness of g immediately follows from Corollary 1.

□

Theorem 6. *Let $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ with $k \geq 2$. Then for all*

$$(f, u_0) \in B^{k+1,2(s-1),s-1}_{\Lambda^2,\text{for}}(X_T) \times V^{2s+k+1}_{\Lambda^2}$$

there exists a time $T^* \in (0, T]$ such that the Problem (5) has a unique solution

$$(u, p) \in B^{k+1,2s,s}_{\Lambda^2,\text{vel}}(X_{T_k}) \times B^{k+2,2(s-1),s-1}_{\Lambda^2,\text{pre}}(X_{T_k}).$$

Proof. Indeed, applying the projection P^2 (see Lemma 1 above) to the equation (5) we have

$$\begin{cases} \partial_t u + \mu\Delta_2 u + P^2 N^2(u) = P^2 f & \text{in } X \times (0,T), \\ u(x,0) = u_0 & \text{in } X, \end{cases} \tag{62}$$

then the form p actually has to satisfy the equation

$$\operatorname{rot} p = (I - P^2)(f - N^2(u)) \quad \text{in } X \times (0,T). \tag{63}$$

Multiplying (62) by $v \in V^k_{\Lambda^3}$ we get the Problem (36). Then, the existence and regularity of the solution u follows immediately from Theorems 4 and

5. On the other hand, it follows from Lemma 2 that there exists a unique differential form $p \in B^{k+2,2(s-1),s-1}_{\Lambda^2,\mathrm{pre}}(X_{T_k})$, satisfying (63).

□

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