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EXISTENCE THEOREM OF A WEAK SOLUTION FOR NAVIER-STOKES TYPE EQUATIONS ASSOCIATED WITH DE RHAM COMPLEX

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Abstract: Let $\{d_q, \Lambda^q\}$ be de Rham complex on a smooth compact closed manifold X over $\mathbb{R}^3$ with Laplacians Δ_q. We consider operator equations, associated with the parabolic differential operators $\partial_t + \Delta_2 + N^2$ on the second step of complex with nonlinear bi-differential operator of zero order N^2. Using by projection on the next step of complex we show that the equation has unique solution in special Bochner-Sobolev type functional spaces for some (small enough) time T^*.

Keywords: elliptic differential complexes, parabolic nonlinear equations, open mapping theorem.

1 Introduction

Consider the de Rham complex on a Riemannian n-dimensional smooth compact closed manifold X with vector bundles Λ^q of exterior forms of

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degree q over X,

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d_0} \Omega^1(X) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(X) \longrightarrow 0. \tag{1}$$

Here $\Omega_q(X)$ denotes the space of all differential forms of degree q with smooth coefficients on X. In this case the Laplacians $\Delta_q = d_q^* d_q + d_{q-1} d_{q-1}^*$, $q = 0, 1, \ldots, n$, of the complex are second order strongly elliptic differential operators on X, where operator d_q^* is a formal adjoint to d_q. As usual, for $q < 0$ and $q \geq n$ we assume that $d_q = 0$.

We want to study the non-linear problems, associated with the complex. With this purpose, we denote by $M_{i,j}$ two bilinear bi-differential operators of zero order (see [5] or [23]),

$$\begin{array}{l} M_{q,1}(\cdot,\cdot) : \big(\Omega^{q+1}(X),\ \Omega^q(X)\big) \to \Omega^q(X), \\ M_{q,2}(\cdot,\cdot) : \big(\Omega^q(X),\ \Omega^q(X)\big) \to \Omega^{q-1}(X). \end{array} \tag{2}$$

We set for a differential form u of the degree q

$$N^q(u) =: M_{q,1}(d_q u, u) + d_{q-1} M_{q,2}(u,u). \tag{3}$$

Note, that operator $N^q(u)$ is non-linear.

Let now time $T > 0$ is finite. Then for any fixed positive number μ the operators $\partial_t + \mu\Delta_q$ are parabolic on the cylinder $X \times (0,T)$ (see [7]). Consider the following initial problem: given sufficiently regular differential forms f of the induced bundle $\Lambda^q(t)$ (the variable t enters into this bundle as a parameter) and u_0 of the bundle Λ^q, find a differential forms u of the induced bundle $\Lambda^q(t)$ and p of the induced bundle $\Lambda^{q-1}(t)$ such that

$$\begin{cases} \partial_t u + \mu\Delta_q u + N^q(u) + d_{q-1} p = f & \text{in } X \times (0,T), \\ d_{q-1}^* u = 0 & \text{in } X \times [0,T], \\ d_{q-2}^* p = 0 & \text{in } X \times [0,T], \\ u(x,0) = u_0 & \text{in } X, \end{cases} \tag{4}$$

For general elliptic complexes this problem was considered in the works [21] and [27], where the open mapping theorems were proved in the special spaces of Hölder (see [21]) and Sobolev (see [27]) types. It means that the range of the non-linear operator $\mathcal{A}_q$, related to the problem, is open in the constructed spaces. However, obtaining an existence theorem for the solution (even the so-called weak one) and closedness of the range of the related non-linear operator in such spaces appears to be a more difficult task.

For example, if we take $q = 1$ and a suitable nonlinear term we may treat (4) as the initial problem for the well known Navier-Stokes equations for incompressible fluid over the manifold X (see, for instance, [16] or [28]). Note that the equation with respect to p is actually missing in this case, because $d_{-1}^* = 0$.

We consider problem (4) in the case $n = 3$, $q = 2$ and a special nonlinearity $M_{q,1}(d_q u, u) = (d_q u)u$. It easy to see that in this case we can treat the de

Rham differentials as $d_2 = \text{div}$, $d_1 = \text{rot}$, $d_2^* = -\nabla$, $d_1^* = \text{rot}$ and then (4) transforms to

$$\begin{cases} \partial_t u + \mu\Delta_2 u + N^2(u) + \text{rot}\, p = f & \text{in } X \times (0,T), \\ \text{rot}\, u = 0 & \text{in } X \times [0,T], \\ \text{div}\, p = 0 & \text{in } X \times [0,T], \\ u(x,0) = u_0 & \text{in } X, \end{cases} \tag{5}$$

where

$$N^2(u) = (\text{div}\, u)u + \text{rot}\,(M_{q,2}(u,u)), \tag{6}$$

and Laplacian

$$\Delta_2 u = d_2^* d_2 + d_1 d_1^* = -\nabla \text{div}\, u + \text{rot}\,\text{rot}\, u = -\Delta u.$$

Here Δu is a standard Laplace operator applied componentwise to the differential form u in the space variable x.

Using projection to the next step of complex (1), we prove an existence theorem of weak (distributional) solution in the constructed Bochner-Sobolev type spaces for some (small enough) time T^*. Note, that considering general non-linear perturbations of linear parbolic equations one have to impose essential restrictions on the non-linear term $N^2(u)$ in order to achieve existence of weak solutions. For example, one of such condition can be positiveness of non-linear operator $N^2(u)$. However, we do not impose such strong conditions on the non-linear term, but still have an existence of weak solutions due to special properties of the de Rham complex.

2 Functional spaces

Denote by $L^p_{\Lambda^q}$, $1 \le p \le \infty$, space of differential forms of the degree q with coefficients in the Lebesgue space $L^p(X)$. In a similar way we designate the spaces of forms on X whose components are of Sobolev class or have continuous partial derivatives. We denote it by $W^{s,p}_{\Lambda^q}$ and $C^s_{\Lambda^q}$ respectively with smoothness s. In particular case, for $p = 2$ we designate $H^s_{\Lambda^q} := W^{s,2}_{\Lambda^q}$.

For calculations, it is convenient to use the fractional powers of the Laplace operator. Namely, for differential form u of degree q we denote by

$$\nabla_q^m u := \begin{cases} \Delta_q^{m/2} u, & m \text{ is even}, \\ (d_q \oplus d_{q-1}^*)\Delta_q^{(m-1)/2} u, & m \text{ is odd}. \end{cases} \tag{7}$$

It easy to see that integration by parts yields

$$\sum_{|\alpha|=m} \|\partial^\alpha u\|^2_{L^2_{\Lambda^q}} = \|\nabla_q^m u\|^2_{L^2_{\Lambda^q}}.$$

Now, we want to recall the standard Hodge theorem for elliptic complexes. For this purpose denote by $\mathcal{H}^q$ the harmonic space of complex (1), i.e.

$$\mathcal{H}^q = \left\{ u \in C^\infty_{\Lambda^q} : d_q u = 0 \text{ and } d_{q-1}^* u = 0 \text{ in } X \right\}, \tag{8}$$

and by Π^i the orthogonal projection from $L^2_{\Lambda^q}$ onto $\mathcal{H}^q$.

Theorem 1. *Let $0 \le q \le n$, $s \in \mathbb{Z}_+$. Then operator*

$$\Delta_q : H^{s+2}_{\Lambda^q} \to H^s_{\Lambda^q} \tag{9}$$

is Fredholm:

(1) the kernel of operator (9) equals to the finite-dimensional space $\mathcal{H}^q$;

(2) given $v \in H^s_{\Lambda^q}$ there is a form $u \in H^{s+2}_{\Lambda^q}$ such that $\Delta_q u = v$ if and only if $(v,h)_{L^2_{\Lambda^q}} = 0$ for all $h \in \mathcal{H}^q$;

(3) there exists a pseudo-differential operator φ^i on X such that the operator

$$\varphi^q : H^s_{\Lambda^q} \to H^{s+2}_{\Lambda^q}, \tag{10}$$

induced by φ^q, is linear bounded and with the identity I we have

$$\varphi^q \Delta_q = I - \Pi^q \text{ on } H^{s+2}_{\Lambda^q}, \quad \Delta_q \varphi^q = I - \Pi^q \text{ on } H^s_{\Lambda^q} \tag{11}$$

Proof. See, for instance, [23, Theorem 2.2.2]. □

Denote by $V^s_{\Lambda^q} := H^s_{\Lambda^q} \cap S_{d^*_{q-1}}$ the space of all differential forms $u \in H^s_{\Lambda^q}$ satisfying $d^*_{q-1} u = 0$ in the sense of distributions in X. Let now $L^2(I, H^s_{\Lambda^q})$ be the Bochner space of L^2-mappings

$$u(t) : I \to H^s_{\Lambda^q},$$

where $I = [0,T]$, see, for instance, [14]. It is a Banach space with the norm

$$\|u\|^2_{L^2(I,H^s_{\Lambda^q})} = \int_0^T \|u\|^2_{H^s_{\Lambda^q}} dt.$$

We need to introduce suitable Bocner-Sobolev type spaces, see [**?**] or [28] for the de Rham complex and [27] for the general elliptic complexes. Namely, for $s \in \mathbb{Z}_+$ denote by $B^{k,2s,s}_{q,\mathrm{vel}}(X_T)$ the space of all differential forms of degree q over $X_T := X \times [0,T]$ with variable $t \in [0,T]$ as a parameter, such that

$$u \in C(I, V^{k+2s}_{\Lambda^q}) \cap L^2(I, V^{k+2s+1}_{\Lambda^q})$$

and

$$\nabla^m_q \partial^j_t u \in C(I, V^{k+2s-m-2j}_{\Lambda^q}) \cap L^2(I, V^{k+2s+1-m-2j}_{\Lambda^q})$$

for all $m + 2j \le 2s$. It is a Banach space with the norm

$$\|u\|^2_{B^{k,2s,s}_{q,\mathrm{vel}}} := \sum_{\substack{m+2j\le 2s \\ 0\le l\le k}} \|\nabla^l_q \nabla^m_q \partial^j_t u\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^{l+1}_q \nabla^m_q \partial^j_t u\|^2_{L^2(I,L^2_{\Lambda^q})}.$$

Similarly, for $s,k \in \mathbb{Z}_+$, we define the space $B^{k,2s,s}_{q,\mathrm{for}}(X_T)$ to consist of all differential forms

$$f \in C(I, H^{2s+k}_{\Lambda^q}) \cap L^2(I, H^{2s+k+1}_{\Lambda^q})$$

with the property that

$$\nabla^m_q \partial^j_t f \in C(I, H^{k+2s-m-2j}_{\Lambda^q}) \cap L^2(I, H^{k+2s-m-2j+1}_{\Lambda^q})$$

for all $m+2j \le 2s$. We endow the space $B^{k,2s,s}_{q,\mathrm{for}}(X_T)$ with the natural norm

$$\|f\|^2_{B^{k,2s,s}_{q,\mathrm{for}}} := \sum_{\substack{m+2j\le 2s\\ 0\le l\le k}} \|\nabla^l_q \nabla^m_q \partial^j_t f\|^2_{C(I,L^2_{\Lambda^q})} + \|\nabla^{l+1}_q \nabla^m_q \partial^j_t f\|^2_{L^2(I,L^2_{\Lambda^q})}.$$

Lastly, the space for the differential form p we denote by $B^{k+1,2s,s}_{q-1,\mathrm{pre}}(X_T)$. This space consists of all forms p from the space $C(I,H^{2s+k+1}_{\Lambda^{q-1}})\cap L^2(I,H^{2s+k+2}_{\Lambda^{q-1}})$ such that $d_{q-1}p \in B^{k,2s,s}_{q,\mathrm{for}}(X_T)$, $d^*_{q-2}p = 0$ and for all $h \in \mathcal{H}^{q-1}$

$$(p,h)_{L^2_{\Lambda^{q-1}}} = 0. \tag{12}$$

It is a Banach space with the norm

$$\|p\|_{B^{k+1,2s,s}_{q-1,\mathrm{pre}}} = \|d_{q-1}p\|_{B^{k,2s,s}_{q,\mathrm{for}}}.$$

Define now for suitable forms v and w of degree q a bi-differential operator

$$\mathbf{B}_q(w,v) = M_{q,1}(d_q w, v) + M_{q,1}(d_q v, w) + d_{q-1}\big(M_{q,2}(w,v) + M_{q,2}(v,w)\big), \tag{13}$$

with the operators $M_{q,1}$ and $M_{q,2}$ satisfying

$$|M_{q,1}(u,v)| \le c_{q,1}|u|\,|v|, \quad |M_{q,2}(u,v)| \le c_{q,2}|u|\,|v| \text{ on } X \tag{14}$$

with some positive constants $c_{i,j}$. Following theorem allows us to see the correctness of operators in this spaces.

Theorem 2. *Suppose that $s\in\mathbb{N}$, $k\in\mathbb{Z}_+$ and $2s+k > \frac{n}{2}-1$. Then the mappings*

$$\begin{array}{lrcl} \nabla^m_q: & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T) & \to & B^{k-m,2(s-1),s-1}_{q,\mathrm{for}}(X_T),\ m\le k \\ \Delta_q: & B^{k,2s,s}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \\ \partial_t: & B^{k,2s,s}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \end{array}$$

are continuous. Besides, if $w,v\in B^{k+2,2(s-1),s-1}_{i,\mathrm{vel}}(X_T)$ then the mappings

$$\begin{array}{lrcl} \mathbf{B}_q(w,\cdot): & B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \\ \mathbf{B}_q(w,\cdot): & B^{k,2s,s}_{q,\mathrm{vel}}(X_T) & \to & B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T), \end{array} \tag{15}$$

are continuous, too. In particular, for all $w,v\in B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}(X_T)$ there is positive constant $c_{s,k}$, independent on v and w, such that

$$\|\mathbf{B}_q(w,v)\|_{B^{k,2(s-1),s-1}_{q,\mathrm{for}}} \le c_{s,k}\|w\|_{B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}}\|v\|_{B^{k+2,2(s-1),s-1}_{q,\mathrm{vel}}}. \tag{16}$$

Proof. See, for instance, [28] or [27]. □

Let us introduce now the Helmholtz type projection P^q from $B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T)$ to the kernel of operator d^*_q.

Lemma 1. *If $s,\ k \in \mathbb{Z}_+$, then for each q the pseudo-differential operator $\mathrm{P}^q = d_q^* d_q \varphi^q + \Pi^q$ on X induce continuous map*

$$\mathrm{P}^q : B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T) \to B^{k,2(s-1),s-1}_{q,\mathrm{vel}}(X_T), \tag{17}$$

such that

$$\mathrm{P}^q \circ \mathrm{P}^q u = \mathrm{P}^q u, \quad (\mathrm{P}^q u, v)_{L^2_{\Lambda^q}(X)} = (u, \mathrm{P}^q v)_{L^2_{\Lambda^q}(X)}, \quad (\mathrm{P}^q u, (I - \mathrm{P}^q)u)_{L^2_{\Lambda^q}(X)} = 0$$

for all $u, v \in B^{k,2(s-1),s-1}_{q,\mathrm{for}}$.

Proof. See, for instance, [27]. □

The following Lemma is just a consequence of Hodge Theorem 1.

Lemma 2. *Let $F \in B^{k,2(s-1),s-1}_{q,\mathrm{for}}(X_T)$ satisfy $\mathrm{P}^q F = 0$ in X_T. Then there is a unique section $p \in B^{k+1,2(s-1),s-1}_{q-1,\mathrm{pre}}(X_T)$ such that (12) holds and*

$$d_{q-1} p = F \text{ in } X \times [0, T]. \tag{18}$$

Now we are ready to go to the main section of this paper.

3 Existence theorem

In order to get existence theorem to the Problem (5) we use a projection to the next step of complex (1). Namely, applying an operator $d_2 = \mathrm{div}$ to the equation (5) we have

$$\begin{cases} \partial_t \mathrm{div} u - \mu\, \mathrm{div}(\nabla \mathrm{div}\, u) + \mathrm{div}((\mathrm{div}\, u) u) = \mathrm{div} f & \text{in } X \times (0, T), \\ \mathrm{div}\, u(x, 0) = \mathrm{div}\, u_0 & \text{in } X, \end{cases} \tag{19}$$

because of $\mathrm{rot}\, u = 0$ and $\mathrm{div} \circ \mathrm{rot} \equiv 0$. Now,

$$\mathrm{div}((\mathrm{div}\, u)u) = (\mathrm{div}\, u)^2 + \Delta u \cdot u = (\mathrm{div}\, u)^2 + \nabla \mathrm{div}\, u \cdot u.$$

By Theorem 1

$$u = \varphi^2 \Delta_2 u + \Pi^2 u = \varphi^2 \nabla \mathrm{div}\, u + \Pi^2 u.$$

Denote

$$g = \mathrm{div}\, u,$$

then we can rewrite (19) by the next way

$$\begin{cases} \partial_t g - \mu\, \mathrm{div}(\nabla g) + g^2 + \nabla g \cdot (\varphi^2 \nabla g + \Pi^2 u) = \mathrm{div} f & \text{in } X \times (0, T), \\ g(x, 0) = \mathrm{div}\, u_0 & \text{in } X. \end{cases} \tag{20}$$

Theorem 3. *Given any pair $(f, u_0) \in L^2(I, (V^0_{\Lambda^2})') \times V^1_{\Lambda^2}$. There is time $t_0 \in (0, T]$ such that for all $t \in [0, t_0]$ there exist a differential form $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ with $\partial_t g \in L^2(I, (H^1_{\Lambda^3})')$, satisfying*

$$\begin{cases} \dfrac{d}{dt}(g, v)_{L^2_{\Lambda^3}} + \mu (\nabla g, \nabla v)_{L^2_{\Lambda^2}} = \langle \mathrm{div}\, f - g^2 - \nabla g \cdot (\varphi^2\, \nabla g + \Pi^2 u), v \rangle, \\ g(\cdot, 0) = \mathrm{div}\, u_0 \end{cases} \tag{21}$$

for all $v \in H^k_{\Lambda^3}$ with $k \geq 2$.

Proof. Let $\{u_m\}$ be the sequence of Faedo-Galerkin approximations, namely,

$$u_m = \sum_{j=1}^{M} c_j^{(m)}(t) b_j(x), \tag{22}$$

then

$$g_m = \operatorname{div} u_m = \sum_{j=1}^{M} c_j^{(m)}(t) \operatorname{div} b_j(x), \tag{23}$$

where the system $\{b_j\}_{j\in\mathbb{N}}$ is a $L^2_{\Lambda^2}(X)$-orthogonal basis in $V^1_{\Lambda^2}$ and the functions u_m satisfy the following relations

$$\frac{d}{dt}(g_m, \operatorname{div} b_j)_{L^2_{\Lambda^3}} + \mu(\nabla g_m, \nabla \operatorname{div} b_j)_{L^2_{\Lambda^2}} = \tag{24}$$
$$\langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2\, \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, \operatorname{div} b_j \rangle,$$
$$g_m(x, 0) = \operatorname{div} u_{0,m}(x),$$

for all $0 \leq j \leq m$ with the initial date $u_{0,m}$ from the linear span $\mathcal{L}(\{b_j\}_{j=1}^m)$ such that the sequence $\{u_{0,m}\}$ converges to u_0 in $V^1_{\Lambda^2}$. For instance, as $\{u_{0,m}\}$ we may take the orthogonal projection onto the linear span $\mathcal{L}(\{b_j\}_{j=1}^m)$.

Multiplying (24) by $c_j^{(m)}(t)$ and summing by j we have

$$(\partial_t g_m, g_m)_{L^2_{\Lambda^3}} + \mu(\nabla g_m, \nabla g_m)_{L^2_{\Lambda^2}} = \langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2\, \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, g_m \rangle. \tag{25}$$

It follows from Lemma by J.-L. Lions (see, for instance, [25, Ch. III, § 1, Lemma 1.2]) that

$$\frac{d}{dt}\|g_m(\cdot, t)\|^2_{L^2_{\Lambda^3}} = 2\,\langle \partial_t g_m, g_m \rangle.$$

Then integrating by $t \in [0, T]$ we see that

$$\|g_m(\cdot, t)\|^2_{L^2_{\Lambda^3}} + 2\mu \int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt = \tag{26}$$
$$\|g_m(\cdot, 0)\|^2_{L^2_{\Lambda^3}} + 2\int_0^t \langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2\, \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, g_m \rangle dt.$$

Since $f \in L^2(I, L^2_{\Lambda^2})$ then $\operatorname{div} f \in L^2(I, (V^1_{\Lambda^3})')$ and

$$2\left|\int_0^t \langle \operatorname{div} f, g_m \rangle dt\right| \leq 2\int_0^t \|\operatorname{div} f\|_{(V^1_{\Lambda^3})'} \|g_m\|_{V^1_{\Lambda^3}} dt \leq \tag{27}$$
$$\frac{4}{\mu}\int_0^t \|\operatorname{div} f\|^2_{(V^1_{\Lambda^3})'} dt + \frac{\mu}{4}\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt + \frac{\mu}{4}\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt.$$

On the other hand

$$2\left|\int_0^t \langle g_m^2, g_m \rangle dt\right| \leq 2\int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt. \tag{28}$$

Note that in our case $\nabla_3 = -\nabla$ with $n = 3$. Then from the Gagliardo-Nirenberg inequality (see [20] or [4, Theorem 3.70]) we have

$$2\int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt \leq \tag{29}$$

$$c\int_0^t \left[\left(\|\nabla g_m\|_{L^2_{\Lambda^2}} + \|g_m\|_{L^2_{\Lambda^3}}\right)^{\frac{1}{2}} \|g_m\|^{\frac{1}{2}}_{L^2_{\Lambda^3}} + \|g_m\|_{L^2_{\Lambda^3}}\right]^3 dt \leq$$

$$c_1\int_0^t \left[\|\nabla g_m\|^{\frac{1}{2}}_{L^2_{\Lambda^2}} \|g_m\|^{\frac{1}{2}}_{L^2_{\Lambda^3}} + \|g_m\|_{L^2_{\Lambda^3}}\right]^3 dt \leq$$

$$c_2\int_0^t \left(\|\nabla g_m\|^{\frac{3}{2}}_{L^2_{\Lambda^2}} \|g_m\|^{\frac{3}{2}}_{L^2_{\Lambda^3}} + \|g_m\|^3_{L^2_{\Lambda^3}}\right) dt \leq$$

$$\frac{\mu}{2}\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt + c_3\int_0^t \left(\|g_m\|^3_{L^2_{\Lambda^3}} + \|g_m\|^6_{L^2_{\Lambda^3}}\right) dt$$

with positive constants c, c_1 and c_2. The last expression is consequence of standard Young's inequality. Moreover, there are positive constants c and c_1 such that

$$\int_0^t \left(\|g_m\|^3_{L^2_{\Lambda^3}} + \|g_m\|^6_{L^2_{\Lambda^3}}\right) dt \leq c\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} \left(1 + \|g_m\|_{L^2_{\Lambda^3}}\right)^4 dt \leq$$

$$c_1\left(\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt + \int_0^t \|g_m\|^6_{L^2_{\Lambda^3}} dt\right).$$

Then we conclude that

$$2\int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt \leq \frac{\mu}{2}\int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} dt + c\int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt + c\int_0^t \|g_m\|^6_{L^2_{\Lambda^3}} dt \tag{30}$$

with some constant $c > 0$. Next,

$$\int_0^t \langle \nabla g_m \cdot \varphi^2 \nabla g_m, g_m\rangle dt = \sum_{j=1}^3 \int_0^t \int_X \partial_j g_m(\varphi^2 \partial_j g_m) g_m \, dx\, dt =$$

$$-\sum_{j=1}^3 \int_0^t \int_X g_m(\varphi^2 \partial_j g_m)\partial_j g_m \, dx\, dt - \int_0^t \int_X g_m^3 \, dx\, dt,$$

because $\varphi^2 \Delta g_m = g_m$. It means that

$$\int_0^t \langle \nabla g_m \cdot \varphi^2 \nabla g_m, g_m\rangle dt = -\frac{1}{2}\int_0^t \int_X g_m^3 \, dx\, dt,$$

and hence

$$2\left|\int_0^t \langle \nabla g_m \cdot \varphi^2 \nabla g_m, g_m\rangle dt\right| \leq \int_0^t \|g_m\|^3_{L^3_{\Lambda^3}} dt. \tag{31}$$

Finally,

$$\int_0^t \langle \nabla g_m \cdot \Pi^2 u_m, g_m \rangle dt = \sum_{j=1}^{3} \int_0^t \int\limits_X \partial_j g_m (\Pi^2 u_m^j) g_m \, dx \, dt =$$

$$-\sum_{j=1}^{3} \int_0^t \int\limits_X g_m (\Pi^2 u_m^j) \partial_j g_m \, dx \, dt - \sum_{j=1}^{3} \int_0^t \int\limits_X g_m^2 \partial_j (\Pi^2 u_m^j) \, dx \, dt,$$

and then

$$\int_0^t \langle \nabla g_m \cdot \Pi^2 u_m, g_m \rangle dt = 0, \tag{32}$$

because $\operatorname{div} \Pi^2 u_m = 0$.

Now, inequalities (26) - (32) give

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} + 2\mu \int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^3}} dt \leq \|g_m(\cdot,0)\|^2_{L^2_{\Lambda^3}} + \tag{33}$$

$$\frac{4}{\mu} \int_0^t \|\operatorname{div} f\|^2_{(V^1_{\Lambda^3})'} dt + \mu \int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^2}} + \frac{\mu}{4} \int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt +$$

$$2c \int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt + 2c \int_0^t \|g_m\|^6_{L^2_{\Lambda^3}} dt,$$

and then

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} + \mu \int_0^t \|\nabla g_m\|^2_{L^2_{\Lambda^3}} dt \leq \|g_m(\cdot,0)\|^2_{L^2_{\Lambda^3}} + \tag{34}$$

$$\frac{4}{\mu} \|\operatorname{div} f\|^2_{L^2\left(I,(V^1_{\Lambda^3})'\right)} + \left(\frac{\mu}{4} + 2c\right) \int_0^t \|g_m\|^2_{L^2_{\Lambda^3}} dt + 2c \int_0^t \|g_m\|^6_{L^2_{\Lambda^3}} dt.$$

It follows from the Gronwall-Perov's Lemma (see, for instance [18, p. 360]) that there exist a time $t_0 \in (0,T]$ and a positive constant C_{t_0} such that

$$\|g_m(\cdot,t)\|^2_{L^2_{\Lambda^3}} \leq C_{t_0} \tag{35}$$

for all $t \in [0,t_0]$. Then the sequence g_m is bounded in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$, where $I_{t_0} = [0,t_0]$. Moreover it follows from (34) and (35) that $\|\nabla g_m(\cdot,t)\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})}$ is bounded too. It means that there is a subsequence that converges weakly-$*$ in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$ and weakly in $L^2(I_{t_0}, H^1_{\Lambda^3})$ to some $g \in L^\infty(I_{t_0}, L^2_{\Lambda^3}) \cap L^2(I_{t_0}, H^1_{\Lambda^3})$. We use the same designation g_m for such a subsequence. Then the standard argument show (see, for instance, [15], [25] or [13]) that we can to pass to the limit in (24) with respect to $m \to \infty$ and to conclude that the element g satisfies (21).

$\square$

Let us now return to the Problem (5). Denoting again $g = \operatorname{div} u$ and multiplying (5) scalar in $L^2_{\Lambda^2}$ by differential form $v \in V^k_{\Lambda^2}$ we get

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}(u,v)_{L^2_{\Lambda^2}} + \mu(g, \operatorname{div} v)_{L^2_{\Lambda^3}} & = & \langle f - gu, v\rangle, \\ u(x,0) & = & u_0. \end{array} \right. \tag{36}$$

Theorem 4. *Let $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ be the solution of (21) for given any pair $(f, u_0) \in L^2(I, V^0_{\Lambda^2}) \times V^1_{\Lambda^2}$. Then there exist a unique differential form $u \in C(I, V^1_{\Lambda^2}) \cap L^2(I, V^2_{\Lambda^2})$ satisfying (36) for all $v \in V^k_{\Lambda^3}$ with $k \geq 2$.*

Proof. Indeed, let $\{u_m\}$ be the sequence of Faedo-Galerkin approximations (see (22)) such that the sequence $\{g_m\} = \{\operatorname{div} u_m\}$ converges to $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$. Substituting u_m to (36) instead of u, v and integrating by $t \in [0, t_0]$ we have

$$\|u_m(\cdot,t)\|^2_{L^2_{\Lambda^2}} + 2\mu\|g_m\|^2_{L^2(I_{t_0}, L^2_{\Lambda^3})} = \|u_m(x,0)\|^2_{L^2_{\Lambda^2}} + 2\int_0^{t_0} \langle f - g_m u_m, u_m\rangle dt. \tag{37}$$

As usual, we evaluate by Hölder inequality

$$2\left|\int_0^{t_0} \langle f, u_m\rangle dt\right| \leq \int_0^{t_0} \|u_m\|^2_{L^2_{\Lambda^2}} dt + \int_0^{t_0} \|f\|^2_{L^2_{\Lambda^2}} dt$$

and by Gagliardo-Nirenberg inequality

$$2\left|\int_0^{t_0} \langle g_m u_m, u_m\rangle dt\right| \leq \int_0^{t_0} \|u_m\|^2_{L^4_{\Lambda^2}} \|g_m\|_{L^2_{\Lambda^3}} dt \leq$$

$$c\int_0^{t_0} \|g_m\|_{L^2_{\Lambda^3}} \left(\|g_m\|^{3/2}_{L^2_{\Lambda^3}} \|u_m\|^{1/2}_{L^2_{\Lambda^2}} + \|u_m\|^2_{L^2_{\Lambda^2}} \right) dt \leq$$

$$2\mu\int_0^{t_0} \|g_m\|^2_{L^2_{\Lambda^3}} dt + c_1\int_0^{t_0} \|g_m\|^4_{L^2_{\Lambda^3}} \|u_m\|^2_{L^2_{\Lambda^2}} dt + c_2\int_0^{t_0} \|g_m\|_{L^2_{\Lambda^3}} \|u_m\|^2_{L^2_{\Lambda^2}} dt$$

with some positive constants c, c_1 and c_2. Then

$$\|u_m(\cdot,t)\|^2_{L^2_{\Lambda^2}} \leq \|u_0\|^2_{L^2_{\Lambda^2}} + \int_0^{t_0} \|f\|^2_{L^2_{\Lambda^2}} dt + c\int_0^{t_0} \|u_m\|^2_{L^2_{\Lambda^2}} dt \tag{38}$$

with constant $c > 0$, independent on m. It follows from Gronwall's Lemma that

$$\|u_m(\cdot,t)\|^2_{L^2_{\Lambda^2}} \leq C, \tag{39}$$

where constant C depends on norms $\|f\|^2_{L^2(I_{t_0}, L^2_{\Lambda^2})}$, $\|u_0\|^2_{L^2_{\Lambda^2}}$ and $\|g\|_{C(I_{t_0}, L^2_{\Lambda^2})}$, but not on m.

It follows that the sequence u_m is bounded in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$ and there is a subsequence that converges weakly-$*$ in $L^\infty(I_{t_0}, L^2_{\Lambda^3})$ to some $u \in L^\infty(I_{t_0}, L^2_{\Lambda^3})$. We again use the same designation u_m for such a subsequence. Under hypothesis of this Theorem the sequence $g_m = \operatorname{div} u_m$ converges to $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$, then actually $u \in C(I, V^1_{\Lambda^3}) \cap L^2(I, V^2_{\Lambda^3})$. Passing to the limit in (37) with respect to $m \to \infty$ we conclude that the element u satisfies (36).

Let now u' and u'' are two solutions of (36) such that $\operatorname{div} u' = \operatorname{div} u'' = g$. Hence differential form $u = u' - u''$ satisfies (36) with zero date $(f, u_0) = (0,0)$. It follows from (38) and Gronwall-Perov's Lemma that $\|g(\cdot,t)\|_{L^2_{\Lambda^3}} = 0$, then the Problem (36) has unique solution.

Moreover, if u_1, u_2 are two solutions of (36), corresponding to the solutions $g_1 = \operatorname{div} u_1$ and $g_2 = \operatorname{div} u_2$ of (21), then differential form $u = u_1 - u_2$ satisfies

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}(u,v)_{L^2_{\Lambda^2}} + \mu(g, \operatorname{div} v)_{L^2_{\Lambda^3}} & = & \langle -gu, v\rangle, \\ u(x,0) & = & 0, \end{array} \right. \tag{40}$$

where $g = g_1 - g_2$.

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^2}} + 2\mu\|g\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})} = -2\int_0^{t_0} \langle gu, u\rangle dt. \tag{41}$$

Applying the Gagliardo-Nirenberg inequality we have

$$2\left|\int_0^{t_0} \langle gu, u\rangle dt\right| \le c_1 \int_0^{t_0} \|g\|_{L^2_{\Lambda^3}} \left(\|\nabla u\|^{3/4}_{L^2_{\Lambda^2}} \|u\|^{1/4}_{L^2_{\Lambda^2}} + \|u\|_{L^2_{\Lambda^2}}\right)^2 dt \le$$

$$c_2 \int_0^{t_0} \left(\|g\|^{5/2}_{L^2_{\Lambda^2}} \|u\|^{1/2}_{L^2_{\Lambda^2}} + \|u\|^2_{L^2_{\Lambda^2}} \|g\|_{L^2_{\Lambda^3}}\right) dt \le$$

$$2\mu\|g\|^2_{L^2(I_{t_0},L^2_{\Lambda^3})} + c_3 \left(\|g\|^4_{C(I_{t_0},L^2_{\Lambda^3})} + \|g\|_{C(I_{t_0},L^2_{\Lambda^3})}\right) \int_0^{t_0} \|u\|_{L^2_{\Lambda^2}} dt$$

with positive constants c_1, c_2 and c_3. The last inequality, (41) and Gronwall-Perov's Lemma yields

$$\|u(\cdot,t)\|^2_{L^2_{\Lambda^2}} \le 0,$$

then $u_1 = u_2$ and the Problem (36) has unique solution.

□

Corollary 1. *Under hypothesis of the Theorem 3, let* $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ *is a solution of* (21) *and* $u \in C(I, V^1_{\Lambda^2}) \cap L^2(I, V^2_{\Lambda^2})$ *is a corresponding to* g *solution of* (36)*. If moreover* $g \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$*, then the solution* g *is unique.*

Proof. Indeed, let $g_1, g_2 \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ are two solutions of (21) with corresponding forms $u_1, u_2 \in C(I, H^1_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ satisfying (36). Hence differential form $g = g_1 - g_2$ satisfies

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\|g\|^2_{L^2_{\Lambda^3}} + \mu\|\nabla g\|^2_{L^2_{\Lambda^2}} & = & \big\langle -(g_1^2 - g_2^2) - \big(\nabla g_1 \cdot (\varphi^2 \nabla g_1 + \Pi^2 u_1) - \\ & & \nabla g_2 \cdot (\varphi^2 \nabla g_2 + \Pi^2 u_2)\big), v\big\rangle, \\ g(\cdot,0) & = & 0. \end{array} \right. \tag{42}$$

Integrating by $t \in I_{t_0}$ we get

$$\|g(\cdot,t\|^2_{L^2_{\Lambda^3}} + 2\mu\int_0^{t_0} \|\nabla g\|^2_{L^2_{\Lambda^2}} dt \le 2\int_0^{t_0} |\langle g(g_1+g_2)+ \tag{43}$$

$$\left(\nabla g_1 \cdot (\varphi^2 \nabla g_1 + \Pi^2 u_1) - \nabla g_2 \cdot (\varphi^2 \nabla g_2 + \Pi^2 u_2)\right), v\rangle\big|\, dt,$$

We have to estimate right side of (43). First, it follows from Gagliardo-Nirenberg interpolation inequality that

$$2\int_0^{t_0} |\langle g(g_1+g_2), g\rangle|\, dt \leq \tag{44}$$

$$c\|(g_1+g_2)\|_{C(I_{t_0}, L^2_{\Lambda^3})} \int_0^{t_0} \left(\|\nabla g\|^{3/2}_{L^2_{\Lambda^3}} \|g\|^{1/2}_{L^2_{\Lambda^3}} + \|g\|^2_{L^2_{\Lambda^3}} \right) dt \leq$$

$$\mu \int_0^{t_0} \|\nabla g\|^2_{L^2_{\Lambda^3}}\, dt + c_1 \int_0^{t_0} \|g\|^2_{L^2_{\Lambda^3}}\, dt$$

with positive constants c, c_1. Next,

$$2\int_0^{t_0} \left|\left\langle \nabla g_1 \cdot \varphi^2 \nabla g_1 - \nabla g_2 \cdot \varphi^2 \nabla g_2 + \nabla g_1 \cdot \varphi^2 \nabla g_2 - \nabla g_1 \cdot \varphi^2 \nabla g_2, g\right\rangle\right| dt \leq \tag{45}$$

$$2\int_0^{t_0} \left|\left\langle \nabla g_1 \cdot \varphi^2 \nabla g, g\right\rangle\right| dt + 2\int_0^{t_0} \left|\left\langle \nabla g \cdot \varphi^2 \nabla g_2, g\right\rangle\right| dt \leq$$

$$c_1 \|g_1\|_{C(I_{t_0}, H^1_{\Lambda^3})} \int_0^{t_0} \left(\|\nabla g\|^{3/4}_{L^2_{\Lambda^3}} \|g\|^{5/4}_{L^2_{\Lambda^3}} + \|g\|^2_{L^2_{\Lambda^3}} \right) dt +$$

$$c_2 \|g_1\|_{C(I_{t_0}, H^1_{\Lambda^3})} \int_0^{t_0} \left(\|\nabla g\|^{7/4}_{L^2_{\Lambda^3}} \|g\|^{1/4}_{L^2_{\Lambda^3}} + \|g\|^2_{L^2_{\Lambda^3}} \right) dt \leq$$

$$\mu \int_0^{t_0} \|\nabla g\|^2_{L^2_{\Lambda^3}}\, dt + c \int_0^{t_0} \|g\|^2_{L^2_{\Lambda^3}}\, dt$$

with some positive constants c, c_1 and c_2. Finally,

$$2\int_0^{t_0} \left|\left\langle \nabla g_1 \cdot \Pi^2 u_1 - \nabla g_2 \cdot \Pi^2 u_2 + \nabla g_2 \cdot \Pi^2 u_1 - \nabla g_2 \cdot \Pi^2 u_1, g\right\rangle\right| dt \leq \tag{46}$$

$$2\int_0^{t_0} \left|\left\langle \nabla g \cdot \Pi^2 u_1, g\right\rangle\right| dt + 2\int_0^{t_0} \left|\left\langle \nabla g_2 \cdot \Pi^2 (u_1 - u_2), g\right\rangle\right| dt$$

The Theorem 4 implies that $u_1 = u_2$. On the other hand, integrating by parts we easily see that

$$2\int_0^{t_0} \left|\left\langle \nabla g \cdot \Pi^2 u_1, g\right\rangle\right| dt = 0,$$

and then (46) equals to zero.

Finally, using by (43) - (46) we get

$$\|g(\cdot, t\|^2_{L^2_{\Lambda^3}} \leq c \int_0^{t_0} \|g\|^2_{L^2_{\Lambda^3}}\, dt$$

with some constant $c > 0$. Then it follows from Gronwall-Perov's Lemma that $\|g(\cdot,t)\|_{L^2_{\Lambda^3}} = 0$, then the Problem (21) has unique solution. □

Theorem 5. *Let $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ with $k > 3/2$. Then for all*

$$(f, u_0) \in B^{k+1,2(s-1),s-1}_{\Lambda^2,\mathrm{for}}(X_T) \times V^{2s+k+1}_{\Lambda^2}$$

there exist a time $T_k \in (0,T]$ such that the Problem (20) has solution

$$g \in B^{k,2s,s}_{\Lambda^3,\mathrm{for}}(X_{T_k}).$$

Moreover, solution g is unique, if form u in (19) satisfied (36).

Proof. First of all, denote by

$$\Lambda_r = \begin{cases} \Lambda^3, & r \text{ is even,} \\ \Lambda^2, & r \text{ is odd.} \end{cases}$$

As before, let g_m be the Faedo-Galerkin approximations, see (23). We start with the following priory estimates.

Lemma 3. *Under hypothesis of the Theorem 5, if $(f, u_0) \in B^{k+1,0,0}_{2,\mathrm{for}}(X_T) \times V^{k+3}_{\Lambda^2}$ with some $k \in \mathbb{Z}_+$, then there exist a time $T_k \in (0,T]$ such that*

$$\|\nabla_2^{k'} g_m\|^2_{C(I_{T_k}, L^2_{\Lambda_{k'}})} + \mu \|\nabla_2^{k'+1} g_m\|^2_{L^2(I_{T_k}, L^2_{\Lambda_{k'+1}})} \le C_{k'} \tag{47}$$

for any $0 \le k' \le k+2$, where $I_{T_k} = [0, T_k]$ and the constants $C_{k'} = C_{k'}(\mu, f, u_0) > 0$ depending on k', μ and the norms $\|f\|_{B^{k+1,0,0}_{2,\mathrm{for}}(X_{T_k})}$, $\|u_0\|_{V^{k+3}_{\Lambda^2}}$ but not on m.

Proof. Indeed, if $k' = 0$ then (47) follows immediately from (34) and Gronwall-Perov's Lemma. Now, substituting g_m and $\nabla_3^{2r} g_m$ in (20) instead of g and v respectively with some $r \in \mathbb{N}$ and integrating by $t \in [0,T]$ we get

$$\|\nabla_3^r g_m(\cdot,t)\|^2_{L^2_{\Lambda_r}} + 2\mu \int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda^3}} dt = \tag{48}$$

$$\|\nabla_3^r g_m(\cdot,0)\|^2_{L^2_{\Lambda_r}} + 2\int_0^t \langle \operatorname{div} f - g_m^2 - \nabla g_m \cdot \varphi^2 \nabla g_m - \nabla g_m \cdot \Pi^2 u_m, \nabla_3^{2r} g_m \rangle dt.$$

We have to estimate the right side of (48). First,

$$2\left| \int_0^t \langle \operatorname{div} f, \nabla_3^{2r} g_m \rangle dt \right| \le 2\int_0^t \|\nabla_3^{r-1} \operatorname{div} f\|_{L^2_{\Lambda_{r-1}}} \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}} dt \le \tag{49}$$

$$\frac{4}{\mu} \int_0^t \|\nabla_3^{r-1} \operatorname{div} f\|^2_{L^2_{\Lambda_{r-1}}} dt + \frac{\mu}{4} \int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda_{r+1}}} dt.$$

Further,

$$2\left| \int_0^t \langle g_m^2, \nabla_3^{2r} g_m \rangle dt \right| \le 2\int_0^t \|\nabla_3^{r-1}(g_m^2)\|_{L^2_{\Lambda_{r-1}}} \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}} dt. \tag{50}$$

Let $r \ge 2$, using by Hölder and Gagliardo-Nirenberg inequalities we get

$$\|\nabla_3^{r-1}(g_m^2)\|_{L^2_{\Lambda_{r-1}}} \le \sum_{|\alpha|+|\beta|=r-1} c_{\alpha\beta} \|\partial^\alpha g_m\|_{L^4_{\Lambda^3}} \|\partial^\beta g_m\|_{L^4_{\Lambda^3}} \le \tag{51}$$

$$\sum_{|\alpha|+|\beta|=r-1} c_{\alpha\beta}\left(\left(\left(\|\nabla_3^{|\alpha|+1}g_m\|_{L^2_{\Lambda_{|\alpha|+1}}}+\|\nabla_3^{|\alpha|}g_m\|_{L^2_{\Lambda_{|\alpha|}}}\right)^{3/4}\|\nabla_3^{|\alpha|}g_m\|^{1/4}_{L^2_{\Lambda_{|\alpha|}}}+\right.\right.$$

$$\left.+\|\nabla_3^{|\alpha|}g_m\|_{L^2_{\Lambda_{|\alpha|}}}\right)\left(\left(\|\nabla_3^{|\beta|+1}g_m\|_{L^2_{\Lambda_{|\beta|+1}}}+\|\nabla_3^{|\beta|}g_m\|_{L^2_{\Lambda_{|\beta|}}}\right)^{3/4}\|\nabla_3^{|\beta|}g_m\|^{1/4}_{L^2_{\Lambda_{|\beta|}}}+\right.$$

$$\left.+\|\nabla_3^{|\beta|}g_m\|_{L^2_{\Lambda_{|\beta|}}}\right)\le c\left(\|g_m\|^2_{H^{r-1}_{\Lambda^3}}+\|g_m\|^{5/4}_{H^{r-1}_{\Lambda^3}}\|\nabla_3^r g_m\|^{3/4}_{L^2_{\Lambda_r}}\right)$$

with some positive constants c and $c_{\alpha\beta}$. For the exception case $r=1$ the last inequality take the form

$$\|\nabla_3(g_m^2)\|_{L^2_{\Lambda_{r-1}}}\le c\left(\|g_m\|^2_{L^2_{\Lambda^3}}+\|g_m\|^{1/2}_{L^2_{\Lambda^3}}\|\nabla_3 g_m\|^{3/2}_{L^2_{\Lambda^2}}\right) \tag{52}$$

because of in (51) arise a case when $|\alpha|=|\beta|=r-1=0$. It follows from (50), (51) and Young's inequality that

$$2\left|\int_0^t\langle g_m^2,\nabla_3^{2r}g_m\rangle dt\right|\le\frac{\mu}{4}\int_0^t\|\nabla_3^{r+1}g_m\|^2_{L^2_{\Lambda_{r+1}}}dt+ \tag{53}$$

$$c\|g_m\|^4_{C(I,H^{r-1}_{\Lambda^3})}+c\|g_m\|^{5/2}_{C(I,H^{r-1}_{\Lambda^3})}\int_0^t\|\nabla_3^r g_m\|^{3/2}_{L^2_{\Lambda_r}}dt$$

for $r\ge 2$ and

$$2\left|\int_0^t\langle g_m^2,\nabla_3^{2}g_m\rangle dt\right|\le\frac{\mu}{4}\int_0^t\|\nabla_3^{2}g_m\|^2_{L^2_{\Lambda^3}}dt+ \tag{54}$$

$$c\|g_m\|^4_{C(I,L^2_{\Lambda^2})}+c\|g_m\|_{C(I,L^2_{\Lambda^2})}\int_0^t\|\nabla_3 g_m\|^{3}_{L^2_{\Lambda^2}}dt$$

for $r=1$ with some constant $c>0$.

Next,

$$2\left|\int_0^t\langle\nabla g_m\cdot\varphi^2\,\nabla g_m,\nabla_3^{2r}g_m\rangle dt\right|\le \tag{55}$$

$$2\int_0^t\|\nabla_3^{r-1}\left(\nabla g_m\cdot\varphi^2\,\nabla g_m\right)\|_{L^2_{\Lambda_{r-1}}}\|\nabla_3^{r+1}g_m\|_{L^2_{\Lambda_{r+1}}}dt.$$

Analogous by (51) we have

$$\|\nabla_3^{r-1}\left(\nabla g_m\cdot\varphi^2\,\nabla g_m\right)\|_{L^2_{\Lambda_{r-1}}}\le \tag{56}$$

$$c\left(\|g_m\|_{H^{r-1}_{\Lambda^3}}\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}}+\|g_m\|^{1/4}_{H^{r-1}_{\Lambda^3}}\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}}\|\nabla_3^r g_m\|^{3/4}_{L^2_{\Lambda_r}}+\right.$$

$$\left.\|\nabla_3^r g_m\|^{1/4}_{L^2_{\Lambda_r}}\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}}\|\nabla_3^{r+1}g_m\|^{3/4}_{L^2_{\Lambda_r}}\right)$$

with $r\in\mathbb{N}$ and some constant $c>0$. Theorem 1 imply that $\|\varphi^2 g_m\|_{H^{r+1}_{\Lambda^3}}\le c\|g_m\|_{H^{r-1}_{\Lambda^3}}$ with some positive constant c, then

$$2\left|\int_0^t\langle\nabla g_m\cdot\varphi^2\,\nabla g_m,\nabla_3^{2r}g_m\rangle dt\right|\le\frac{\mu}{4}\int_0^t\|\nabla_3^{r+1}g_m\|^2_{L^2_{\Lambda_{r+1}}}dt+ \tag{57}$$

$$c\|g_m\|^4_{C(I,H^{r-1}_{\Lambda^3})} + c\|g_m\|^{5/2}_{C(I,H^{r-1}_{\Lambda^3})} \int_0^t \|\nabla_3^r g_m\|^{3/2}_{L^2_{\Lambda^r}} dt +$$

$$c\|g_m\|^{10}_{C(I,H^{r-1}_{\Lambda^3})} \int_0^t \|\nabla_3^r g_m\|^2_{L^2_{\Lambda^r}} dt$$

with $c > 0$.

Finally,

$$2\left|\int_0^t \langle \nabla g_m \Pi^2 u_m, \nabla_3^{2r} g_m\rangle dt\right| \leq \tag{58}$$

$$2\int_0^t \|\nabla_3^{r+1} g_m\|_{L^2_{\Lambda_{r+1}}} \|\nabla_3^{r-1}\left(\nabla g_m \cdot \Pi^2 u_m\right)\|_{L^2_{\Lambda_{r-1}}} dt,$$

and we have again

$$\|\nabla_3^{r-1}\left(\nabla g_m \cdot \Pi^2 u_m\right)\|_{L^2_{\Lambda_{r-1}}} \leq \tag{59}$$

$$c\Bigg(\|g_m\|_{H^{r-1}_{\Lambda^3}} \|\Pi^2 u_m\|_{H^r_{\Lambda^2}} + \|g_m\|^{1/4}_{H^{r-1}_{\Lambda^3}} \|\Pi^2 u_m\|_{H^r_{\Lambda^2}} \|\nabla_3^r g_m\|^{3/4}_{L^2_{\Lambda_r}} +$$

$$\|\nabla_3^r g_m\|^{1/4}_{L^2_{\Lambda_r}} \|\Pi^2 u_m\|_{H^r_{\Lambda^3}} \|\nabla_3^{r+1} g_m\|^{3/4}_{L^2_{\Lambda_r}}\Bigg)$$

with positive constant c. Operator Π^2 is bounded in $L^2_{\Lambda^2}$ by the Hodge Theorem 1. On the other hand Theorem 4 yields that the sequence $\{u_m\}$ is bounded in $L^2_{\Lambda^2}$ (see (39)), then $\|\Pi^2 u_m\|_{H^r_{\Lambda^2}} \leq c\|g_m\|_{H^{r-1}_{\Lambda^3}}$ and we get

$$2\left|\int_0^t \langle \nabla g_m \Pi^2 u_m, \nabla_3^{2r} g_m\rangle dt\right| \leq \frac{\mu}{4}\int_0^t \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda_{r+1}}} dt + \tag{60}$$

$$c\|g_m\|^4_{C(I,H^{r-1}_{\Lambda^3})} + c\|g_m\|^{5/2}_{C(I,H^{r-1}_{\Lambda^3})} \int_0^t \|\nabla_3^r g_m\|^{3/2}_{L^2_{\Lambda_r}} dt +$$

$$c\|g_m\|^{10}_{C(I,H^{r-1}_{\Lambda^3})} \int_0^t \|\nabla_3^r g_m\|^2_{L^2_{\Lambda_r}} dt$$

with $c > 0$.

It follows from (48) - (60) and Gronwall-Perov's Lemma that if $(f, u_0) \in B^{k+1,0,0}_{2,\mathrm{for}}(X_T) \times V^{k+3}_{\Lambda^2}$ and the norm $\|g_m\|_{C(I,H^{r-1}_{\Lambda^3})}$ is bounded for some $r \in \mathbb{N}$, $r \leq k+2$, then there exist a time $t_r \in (0, t_0]$ and a positive constant C_r, depending on the norms $\|f\|_{B^{r+1,0,0}_{2,\mathrm{for}}(X_{T_k})}$ and $\|u_0\|_{V^{r+3}_{\Lambda^2}}$, such that

$$\|\nabla_3^r g_m(\cdot, t)\|^2_{L^2_{\Lambda_r}} + \mu\int_0^{t_r} \|\nabla_3^{r+1} g_m\|^2_{L^2_{\Lambda^3}} dt \leq C_r(\mu, f, u_0). \tag{61}$$

Using by (61) consistently for $r = 1, \ldots, k+2$ we get family of times t_r. Denote $T_k = \min\limits_{r \leq k+2} t_r$, then (61) yields that for any $k \in \mathbb{Z}_+$ there exist a time T_k such that (3) is fulfilled. □

Theorem 3 imply that there exist a solution $g \in C(I, L^2_{\Lambda^3}) \cap L^2(I, H^1_{\Lambda^3})$ of (21). On the other hand, it follows from Lemma 3 that for each $(f, u_0) \in B^{k+1,2(s-1),s-1}_{\Lambda^2,\text{for}}(X_T) \times V^{2s+k+1}_{\Lambda^2}$ there exist a time $T_k \in (0, T]$ and a subsequence $\{g_{m'} = \operatorname{div} u_{m'}\}$ such that $\{g_{m'}\}$ converges weakly in $L^2(I_{T_k}, L^2_{\Lambda^3})$ and *-weakly in $L^\infty(I_{T_k}, H^{k+2}_{\Lambda^3}) \cap L^2(I, H^{k+3}_{\Lambda^3})$ to an element g, then $g \in B^{k,2s,s}_{\Lambda^3,\text{for}}(X_{T_k})$. Moreover, the uniqueness of g immediately follows from Corollary 1.

□

Theorem 6. *Let $s \in \mathbb{N}$ and $k \in \mathbb{Z}_+$ with $k \geq 2$. Then for all*

$$(f, u_0) \in B^{k+1,2(s-1),s-1}_{\Lambda^2,\text{for}}(X_T) \times V^{2s+k+1}_{\Lambda^2}$$

there exist a time $T^* \in (0, T]$ such that the Problem (5) has unique solution

$$(u, p) \in B^{k+1,2s,s}_{\Lambda^2,\text{vel}}(X_{T_k}) \times B^{k+2,2(s-1),s-1}_{\Lambda^2,\text{pre}}(X_{T_k}).$$

Proof. Indeed, apply the projection P^2 (see Lemma 1 above) to the equation (5) we have

$$\begin{cases} \partial_t u + \mu\Delta_2 u + P^2 N^2(u) = P^2 f & \text{in } X \times (0, T), \\ u(x, 0) = u_0 & \text{in } X, \end{cases} \tag{62}$$

then the form p actually has to satisfy the equation

$$\operatorname{rot} p = (I - P^2)(f - N^2(u)) \quad \text{in } X \times (0, T). \tag{63}$$

Multiplying (62) by $v \in V^k_{\Lambda^3}$ we get the Problem (36), then the existence and regularity of solution u follows immediately from the Theorems 4 and 5. On the other hand, it follows from Lemma 2 that there exist unique differential form $p \in B^{k+2,2(s-1),s-1}_{\Lambda^2,\text{pre}}(X_{T_k})$, satisfying (63).

□

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