ON THE SYMPLECTIC GROUPS $C_2(2^n)$

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ABSTRACT. In this paper, we prove that symplectic groups $C_2(2^n)$, where $2^{2n}+1$ is a prime number can be uniquely determined by the order of group and the number of elements with the same order.

Keywords: Element order, Prime graph, Symplectic group.

1. **Introduction**

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of orders of elements in G. If $k \in \pi_e(G)$, then we denote the set of the number of elements of order k in G by $\mathsf{m}_k(G)$ and the set of the number of elements with the same order in G by $nse(G)$. In other words, $nse(G) = \{m_k(G) \ : \ k \in \pi_e(G)\}$. Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.

The characterization of groups by $nse(G)$ pertains to Thompson's problem ([13]) which Shi posed it in [16]. The first time, this type of characterization was studied by Shao and Shi. In [15], they proved that if S is a finite simple group with $|\pi(S)| = 4$, then S is characterizable by $nse(S)$ and $|S|$. Next, in the way the authors in ([4, 6, 7, 8, 9], proved that some of groups are characterizable by the order of groups and the number of elements with the same order. Next, groups such as Sporadic groups, S_p where p is a prime, Suzuki groups, simple K_4-groups, ${}^2G_2(q)$, where $q \pm \sqrt{3q}+1$ are prime numbers, $L_2(p)$, where p is a prime and $L_2(2^n)$ where 2^n-1 or 2^n+1 is a prime number, the Symplectic group $C_2(3^n)$,

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where $n = 2k$ $(k \geq 0)$ and $(\frac{3^{2n}+1}{2})$ is a prime number and $L_3(q)$ where $0 < q = 5k \pm 2$, $(k \in \mathbb{Z})$ and $q^2 + q + 1$ is a prime number. Following this result, in [1, 2, 14], it is proved that sporadic simple groups, projective special linear groups $PSL_2(q)$, $PGL_2(q)$, $L_2(p)$, where $2^n - 1$ or $2^n + 1$ is prime number and Suzuki groups $Sz(q)$, where $q - 1$ is a prime number can be uniquely determined by order of group and nse. In this paper, we prove that symplectic group $C_2(2^n)$, where $2^{2n} + 1$ is a prime number can be uniquely determined by the order of group and the number of elements with the same order. In fact, we prove the following theorem.

Main Theorem. Let G be a group with $|G| = |C_2(2^n)|$ and $nse(G) = nse(C_2(2^n)$, where $2^{2n} + 1$ is a prime number. Then $G \cong C_2(2^n)$.

2. Notation and Preliminaries

Lemma 2.1. [11] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(1) $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;
(2) $|H|$ divides $|K| - 1$;
(3) K is nilpotent.

Definition 2.2. *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.3. [3] *Let G be a 2-Frobenius group of even order. Then*

(1) $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;
(2) G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.

Lemma 2.4. [18] *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(1) G is a Frobenius group;
(2) G is a 2-Frobenius group;
(3) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.

Lemma 2.5. [10] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. *Let G be a finite group. Then for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 2.5, the proof is straightforward. □

Lemma 2.7. [19] *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.

(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 2.8. *Let G be a symplectic group $C_2(2^n)$, where $2^{2n} + 1$ is a prime number. Then $m_p(G) = (p-1)|G|/(4p)$ and for every $i \in \pi_e(G) - \{1, p\}$, p divides $m_i(G)$.*

Proof. Since $|G_p| = p$, we deduce that G_p is a cyclic group of order p. Thus $m_p(G) = \varphi(p)n_p(G) = (p-1)n_p(G)$. Now it is enough to show $n_p(G) = |G|/(4p)$. By [12], p is an isolated vertex of $\Gamma(G)$. Hence $|C_G(G_p)| = p$ and $|N_G(G_p)| = xp$ for a natural number x. We know that $N_G(G_p)/C_G(G_p)$ embed in $Aut(G_p)$, which implies $x \mid p - 1$. Furthermore, by Sylow Theorem, $n_p(G) = |G : N_G(G_p)|$ and $n_p(G) \equiv 1 \pmod p$. Therefore p divides $|G|/(xp) - 1$. Thus $q^2 + 1$ divides $q^4(q^4 - 1)(q^2 - 1)/(xp) - 1$. It follows that $q^2 + 1$ divides $(q^2 + 1)(q^6 - 3q^4 + 4q^2 - 4) + (4 - x)$, so we have $p \mid 4 - x$ and since $x \mid p - 1$, we obtain that $x = 4$. Let $i \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, we conclude that $p \nmid i$ and $pi \notin \pi_e(G)$. Thus G_p acts fixed point freely on the set of elements of order i by conjugation and hence $|G_p| \mid m_i(G)$. So we conclude that $p \mid m_i(G)$. □

3. Proof of the Main Theorem

In this section, we prove the main theorem by the following lemmas. We denote the symplectic group $C_2(2^n)$, and prime number $2^{2n} + 1$ by C and p, respectively. Recall that G is a group with $|G| = |C|$ and $nse(G) = nse(C)$.

Lemma 3.1. *$m_2(G) = m_2(C)$, $m_p(G) = m_p(C)$, $n_p(G) = n_p(C)$, p is an isolated vertex of $\Gamma(G)$ and $p \mid m_k(G)$ for every $k \in \pi_e(G) - \{1, p\}$.*

Proof. By Lemma 2.6, for every $1 \neq r \in \pi_e(G)$, $i = 2$ if and only if $m_r(G)$ is odd. Thus we deduce that $m_2(G) = m_2(C)$. According to Lemma

2.6, $(m_p(G), p) = 1$. Thus $p \nmid m_p(G)$ and hence Lemma 2.8 implies that $m_p(G) \in \{m_1(C), m_2(C), m_p(C)\}$. Moreover, $m_p(G)$ is even, so we conclude that $m_p(G) = m_p(C)$. Since G_p and C_p are cyclic groups of order p and $m_p(G) = m_p(C)$, we deduce that $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(C) = m_p(C)$, so $n_p(G) = n_p(C)$. Now we prove that p is an isolated vertex of Γ(G). On opposite, there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $m_{tp}(G) = \varphi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$ and since $n_p(G) = n_p(R)$, it follows that $m_{tp}(G) = (t-1)(p-1)|C|k/(4p)$. If $m_{tp}(G) = m_p(C)$, then $t = 2$ and $k = 1$. Furthermore, Lemma 2.5 yields $p \mid m_2(G) + m_{2p}(G)$ and since $m_2(G) = m_2(C)$ and $p \mid m_2(C)$, we have $p \mid m_{2p}(G)$ which is a contradiction. So Lemma 2.8 implies that $p \mid m_{tp}(G)$. Hence $p \mid t-1$ and since $m_{tp}(G) < |G|$, we deduce that $p-1 \leq 5$. But this is impossible because $q^2 \leq 5$ and $q = 2^n$ Let $k \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, $p \nmid k$ and $pk \notin \pi_e(G)$. Thus G_p acts fixed point freely on the set of elements of order k by conjugation and hence $|G_p| \mid m_k(G)$. So we conclude that $p \mid m_k(G)$. □

Lemma 3.2. *The group G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K|-1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence $q^2+1 \mid q^4(q^4-1)(q^2-1)/(q^2+1) - 1$. So we conclude that $(q^2+1) \mid ((q^2+1)(q^6-3q^4+4q^2-4)+4$ Thus $p \mid 4$ which is impossible.

We now show that G is not a 2-Frobenius group. Let G be a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H, respectively. Set $|G/K| = x$. Since p is an isolated vertex of $\Gamma(G)$, we have $|K/H| = p$ and $|H| = |G|/(xp)$. By Lemma 2.3, $|G/K|$ divides $|Aut(K/H)|$. Thus $x \mid p-1$ and since $(p-1, q-1) = 1$, so $(q^2, q-1) = 1$, now since $|G/K||(p-1)$, we deduce that $q-1|H$. The group H is nilpotent. Therefore $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complement K/H, where $t = q-1$. So $|K/H|$ divides $|H_t| - 1$. It implies that $q^2+1 \leq q-2 \leq q$, but this is a contradiction. □

Lemma 3.3. *The group G is isomorphic to the group C.*

Proof. By Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. Now Lemma 3.2 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. Now according to classification theorem of finite simple groups, K/H is an alternating group, sporadic group or simple group of Lie types. So we consider the following cases:

Step 1. If $K/H \cong A_n$, where $n = p$,$p+1$, $p+2$, $n \geq 5$ is a prime number. Now by [18] $\pi(A_n) = p$,$p-2$. First, assume $q^2+1 = p$, we know that $|C_2(q)| = q^4(q^4-1)(q^2-1)$, so $q^2 = p-1$ it follows that $|C_2(q)| = (p-1)^2(p-1)^2-1)(p-2)$. Thus $|C_2(q)| = (p-1)^2(p-2)^2).p$. Since $(p-3, |C_2(q)|)|2$, so we have contradiction. Now if $q^2+1 = p-2$ then $q^2 = p-3$ it follows that $|C_2(q)| = ((p-3)^2((p-3)^2-1)(p-4)$. So $|C_2(q)| = ((p-3)^2((p-4)^2)(p-2)$. Since $p \nmid |C_2(q)|$ we have a contradiction.

Step 2. If $K/H \cong A_r(q')$, then we consider the following cases:

Case 1. If $K/H \cong A_{p'-1}(q')$, where $(p', q') \neq (3,2), (3,4)$, then by we have $\pi(A_{p'-1}(q')) = q'^{p'} - 1/(q'-1)(p'-1)$, on the other hand we know $|A_{p'-1}(q')| = \frac{q'^{p'(p'-1)/2}\prod_{i=1}^{p'-1}(q'^{i+1}-1)}{(p',q'-1)}$. Now, we conside $q^2+1 = q'^{p'} - 1/(q'-1)(p'-1)$, so $q^2 \leq q'^{p'}$. It follows that $q^4 \leq q'^{2p'}$. On the other hand we have $q'^{p'(p'-1)} \leq q^4$, which this is a contradiction.

Case 2. If $K/H \cong A_1(q')$ where $4 \mid q'+1$, then by [18] $\pi(A_1(q') = q'$, $q'-1/2$. Now assume $q^2+1 = q'$, on the other hand $|A_1(q')| \mid |G|$ thus $q'(q'^2-1)/2|q^4(q^4-1)(q^2-1)$ then $(q^2+1)(q^2+1)^2)-1)/2 \mid q^{10}-q^8-q^6+q^4$. Since that $|A_1(q')| \nmid |G|$ so that is a contradiction. Now $q^2+1 = (q'-1)/2$ then $q^2 = (q'-3)/2$. It follows that $2q^2 = q'-3$ so $q' = 2q^2+3$. Since$|A_1(q')| \nmid |G|$ that is a contradiction.

Case3. If $K/H \cong A_1(q')$ where $4 \mid q'-1$, then by [18] $\pi(A_1(q') =$ q', $q'+1)/2$. Now suppose that $q^2+1 = q'$. On the other hand $|A_1(q')| \mid |G|$ thus $q'(q'^2-1)/2 \mid q^4(q^4-1)(q^2-1)$. It follows that $(q^2+1)(q^2+1)^2)-1)/2 \mid q^{10}-q^8-q^6+q^4$. Now, since $|A_1(q')| \nmid |G|$ so this is a contradiction. Now if $q^2+1 = (q'+1)/2$ then $2q^2+2 = q'+1$ so $q' = 2q^2+1$. But $|A_1(q')| \nmid |G|$ that is a contradiction.

Case4. If $K/H \cong A_1(q')$ where $4 \mid q'$, then by[12] $\pi(A_1(q') = q' \pm 1$. Now if $q^2+1 = q'+1$ then $q' = q^2$. Since that $q'(q'^2-1) \mid q^4-1$ so

$q'^2 \leq q'(q'^2-1) \leq q^4-1 < q^4$ so $q'^2 < q'^2$, where it is impossible. Now if $q^2+1 = q'-1$ then $q^2+2 = q'$. But $|A_1(q')| \nmid |G|$ that is a contradiction.
Step3. If $K/H \cong B_r(q')$ then we consider the following cases:
Case1. If $K/H \cong B_r(q')$ where $r = 2^m \geq 4$ then by[18] we have $\pi(B_r(q') = \frac{q'^r+1}{2}$ and also $|B_r(q')| = \frac{q'^{r^2}\prod_{i=1}^{r}(q'^{2i}-1)}{(2,q'-1)}$. Now we consider $q^2+1 = \frac{q'^r+1}{2}$ so $q^2 = \frac{q'^r-1}{2}$, in conclude $q^4 = (\frac{q'^r-1}{2})^2 \leq q'^{2r}$. Now, since that $q'^{r^2} \leq q^4$, so $q'^{r^2} \leq q'^{2r}$ in result $r \leq 2$ that is a contradiction, because $r \geq 4$.
Case2. If $K/H \cong B'_p(3)$ then by[18] $\pi(B'_p(3)) = \frac{3^{p'}-1}{2}$ and also $|B_{p'}(3))| = \frac{3^{p'^2}\prod_{i=1}^{p'}(3^{2i}-1}{2})$. Now we consider $q^2+1 = \frac{3^{p'}-1}{2}$ so $q^2 = \frac{3^{p'}-3}{2}$, in conclude $q^4 = (\frac{3^{p'}-3}{2})^2 \leq 3^{2p'}$. On the other hand, we have $3^{p'^2} \leq q^4$, so $3^{p'^2} \leq 3^{2p'}$. Next, $p' \leq 2$, which is a contradiction.

Step 4. If $K/H \cong {}^2A_3(2)$, then by[12] $\pi({}^2A_3(2)) = 5$. Now we consider $q^2+1 = 5$ so $q^2 = 4$ as $2^{2n} = 4$. It follows that $n = 1$ that is a contradiction. Now if $K/H \cong {}^2A_5(2)$, then by [12] $\pi({}^2A_3(2)) = 7, 11$. Next, we assume $q^2+1 = 7$ so $q^2 = 6$ that is a contradiction, also $q^2+1 = 11$ so $q^2 = 10$ where this is impossible.

Step5. If $K/H \cong D_r(q')$, then we can see easily the following cases we have a contradiction. If $K/H \cong D_{p'}(q')$, $p' \geq 5$ and $q' = 2$, 3 or5 we consider the following cases:
Case1. If $K/H \cong D_{p'}(2)$, then by [12]$\pi(D_{p'}(2) = 2^{p'}-1$, now we consider $q^2+1 = 2^{p'}-1$ then $q^2 = 2^{p'}-2$ so we have $q^4 = (2^{p'}-2)^2 \leq 2^{2p'}$, the other hand since $|D_{p'}(2))| = 2^{p'(p'-1)}(2^{p'}-1)\prod_{i=1}^{p'-1}(2^{2i}-1) \mid q^4(q^4-1)(q^2-1)$, now since $2^{p'(p'-1)} \leq q^4$ so $2'^{p'(p'-1)} < 2^{2p'}$ in conclude $p' < 3$ that is a contradiction.
Case 2. If $K/H \cong {}^2D_{p'+1}(2)$, where $p' = 2^{m'-1}$, $m' \geq 2$ then by[12], $\pi({}^2D_{p'+1}(2)) = 2^{p'}+1$ or $2^{p'+1}+1$. Now if $q^2+1 = 2^{p'+1}$ so $q^2 = 2^{p'}$ in result $2^{2n} = 2^{p'}$ in result $p' = 2n$, that is a contradiction, because p' is a odd and prime. For other case we have a contradiction, similarily.
Case 3. If $K/H \cong {}^2D_{p'}(3)$, where $p' = 2^{n'}+1$, $n' \geq 2$ then by [18] $\pi({}^2D_{p'}(3) = (3^{p'}+1)/4$ or $(3^{p'-1}+1)/2$. Now if $q^2+1 = (3^{p'}+1)/4$ so $q^2 = \frac{3^{p'}-3}{4}$ then $q^4 = (\frac{3^{p'}-3}{4})^2 < 3^{2p'}$. Now since $|{}^2D_{p'}(3)| = \frac{3^{p'(p'-1)}(3^{p'}+1)\prod_{i=1}^{p'-1}(3^{2i}-1)}{(4,3^{p'}+1)}$. In the way, $3^{p'(p'-1)} < 3^{p'(p'-1)}(3^{p'+1}) \leq q^4 -$

$1 < q^4$, so $3^{p'(p'-1)} < 3^{2p'}$ in conclude $p' < 3$ that is a contradiction. Now if $q^2+1 = \frac{3^{p'-1}+1}{2}$ in result $2^{2n} = \frac{3^{p'-1}+1}{2}$. So $2^{2n+1} = 3^{p'-1}-1$ it follows that $p'-1 = 2$, $2n+1 = 3$. So $p' = 3$,$n = 1$ that is a contradiction. For $K/H \cong {}^2D_{p'}(3)$, where $p' \neq 2^{n'}+1$,$p' \geq 5$. Also$K/H \cong {}^2D_{p'}(3)$, where $p' \neq 2^{m+1}$, $m \geq 2$, we have a contradiction, similarily.

Step 6. If $K/H \cong G_2(q')$, then we consider the following cases:

Case 1. If $K/H \cong G_2(q')$ where $2 \leq q' \equiv 1(\mod 3)$ then by[18] $\pi(G_2(q') = q'^2 - q' + 1$ also $|G_2(q')| = q'^6(q'^3-1)(q'^2-1)(q'+1) \mid |q^4 - 1$. Now we consider$q^2+1 = q'^2 - q' + 1$ so $q^2 = q'^2 - q'$. It follows that $q^4 = q'^2 - q'^2 \leq q'^4$. But $q'^6 < q'^6(q'^3-1)(q'^2-1)(q'+1) \leq q^4 - 1 < q^4$ thus $q'^6 < q'^4$ that is a contradiction.

Case 2. If $K/H \cong G_2(q')$ where $3 \mid q'$, then by[18] $\pi(G_2(q') = q'^2 \pm q' + 1$ also $|G_2(q')| = (q'^6(q'^3-1)(q'^2-1)(q'+1) \mid |q^4 - 1$. Now we consider$q^2 + 1 = q'^2 + q' + 1$ so $q^2 = q'^2 + q'$. So $q^4 = q'^2 + q'^2 \leq (q'^3-1)^2 < q'^6$ now since $q'^6 < q'^6(q'^3 - 1(q'^2-1)(q'+1) \leq q^4 - 1 < q^4$. But $q'^6 < q'^6$ that is a contradiction.

Case 3. If $K/H \cong G_2(q')$ where where $2 < q' \equiv -1(\mod 3)$, then by [18] $\pi(G_2(q') = q'^2 + q' + 1$ also $|G_2(q')| = (q'^6(q'^3-1)(q'^2-1)(q'+1) \mid |q^4 - 1$. Now we consider$q^2 + 1 = q'^2 - q' + 1$ so $q^2 = q'^2 + q'$. As a result $q^4 = (q'^2 + q')^2 \leq (q'^3 - 1)^2 < (q'^6$. Now, since $q'^6 < q'^6(q'^3-1)(q'^2-1)(q'+1) \leq q^4 - 1 < q^4$ thus $q'^6 < q'^6$ that is a contradiction.

Step 7. If $K/H \cong E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, ${}^2F_4(2)'$, then by [18, 12] $\pi(E_7(2)) = 127$, $\pi(E_7(3)) = 1093$, $\pi({}^2E_6(2)) = 19$, $\pi({}^2F_4(2)' = 13$. Now, we consider $q^2 + 1 = 127, 1093, 19, 13$, we can see easily these equation have not any solution in natural number $\mathbb{N}$ so we have a contradiction.

Step 8. If $K/H \cong {}^3D_4(q')$ then by [18] we have $\pi({}^3D_4(q') = q'^4 - q'^2 + 1$, also $|{}^3D_4(q')| = q'^{12}(q'^6-1)(q'^2-1)(q'^4+q'^2+1) \mid q^4 - 1$. Now we consider $q^2 + 1 = q'^4 - q'^2 + 1$ so $q^2 = q'^4 - q'^2$ so $q^4 = q'^4 - q'^{2^2} < q'^8$. Since that $q'^{12} < q'^{12}(q'^6-1)(q'^2-1)(q'^4+q'^2+1) \leq q^4 - 1 < q^4$ thus $q'^{12} < q'^8$ that is a contradiction.

Step 9. If $K/H \cong F_4(q')$ where that q' is power of prime number then by [12]$\pi(F_4(q') = q'^4 - q'^2 + 1$, then we consider $q^2 + 1 = q'^4 - q'^2 + 1$ so $q^2 = q'^4 - q'^2$, since $|F_4(q')| \nmid |G|$ that is a contradiction. Now if $K/H \cong F_4(q')$, where $2 \mid q'$ and $q' > 2$ then we can see easily a contradiction.

Step 10. If $K/H \cong {}^2F_4(q')$ where $q' = 2^{2t+1} > 2$ then by [12]we have $\pi({}^2F_4(q') = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$, also $|{}^2F_4(q')| = q'^{12}(q'^4-1)(q'^3+1)(q'^2+1)(q'-1)(q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1) \mid q^4 - 1$.In the way, we consider $q^2+1 = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$ so $q^2 = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'}$ in result we have $q^4 = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'^2} < q'^{10}$, on the other hand we have $q'^{12} < q'^{12}(q'^4-1)(q'^3+1)(q'^2+1)(q'-1)q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1 \leq q^4 - 1 < q^4$, so $q'^{12} < q'^{10}$ that is a contradiction. For groups $K/H \cong {}^2G_2(q')$ where $q' = 3^{2t+1} > 3$ and also $K/H \cong {}^2E_6(q')$, we can see easily we have a contradiction. Similarily.

Step 11. If $K/H \cong E_6(q')$ then by [18], $\pi(E_6(q') = \frac{q'^6+q'^3+1}{(3,q'-1)}$. Now first we consider $(3, q'-1) = 1$ then $q^2+1 = q'^6 + q'^3 + 1$ so $q^2 = q'^6 + q'^3$ now since $|E_6(q')| \nmid |G|$ that is a contradiction. Now if $(3, q'-1) = 3$ we have a contradiction, similarily. If $K/H \cong E_6(q')$ where $3 \nmid q'-1$, then we have a contradiction, similarily.

Step 12. If $K/H \cong E_8(q')$ then by[18], $\pi(E_8(q') = \{q'^8 + q'^7 - q'^5 - q'^4 - q'^3 + q' + 1, (q'^8 - q'^7 + q'^5 - q'^4 - q'^3 - q' + 1, q'^8 - q'^6 + q'^4 - q'^2) + 1, q^{,})^8 - q^{,})^4) + 1\}$. For example we consider $q^2+1 = q'^8 + q'^7 - q'^5 - q'^4 - q'^3 + q' + 1$ then since $|E_8(q')| \nmid |G|$ so we have a contradiction. For other cases we have a contradiction, similarily:

Step 13. If K/H be isomorphic sporadic groups, then we can see easily equation $q^2+1 \neq 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71$ has not any solution in natural number $\mathbb{N}$, so we have a contradiction. Next, we consider the following cases:

Thus $K/H \cong C_r(q')$. Now, we consider two the following cases:

Case 1. If $K/H \cong C_{p'}(q')$ where $q' = 2$ or 3.

Case 2. If $K/H \cong C_r(q')$ where $r = 2^t \geq 2$;

First, we show case1 is impossible. For this purpose, by [18] $\pi(C_{p'}(q') = q'^{p'-1}/(2, q'-1)$ and also $|C_{p'}(q')| = \frac{q'^{p'^2}}{(2,q'-1)} \prod_{i=1}^{p'}(q'^{2i}-1) \mid q^4(q^4-1)(q^2-1)$. in the way, we consider $q^2+1 = q'^{p'-1/(2,q'-1)}$. Now, assume $q' = 2$ then $q^2 = 2^{p'-2}$. It follows that $q^4 = 2^{(p'-2)^2} \leq 2^{2p'}$. Since that $q'^{\frac{p'^2-1}{(2,q'-1)}} < q'^{p'^2} - 1 \leq q^4 - 1 < q^4$ so $2^{p'^2} < 2^{2p'}$, so $p' < 2$, where this is impossible. For $q' = 3$, we can see easily we have a contradiction. So we deduce $K/H \cong C_r(q')$ where $r = 2^t \geq 2$. First, if $r = 2$ then by [12] $\pi(C_2(q') = q'^2 + 1$. On the other hand $|C_2(q')| = q^4(q^4-1)(q^2-1)/(q^2+1)$. Now, since that $K/H \cong C_2(q')$ so $|K/H| = |C_2(q')|$. On the other hand $p \in \pi(K/H)$ thus $p = p'$ where that $p' \in \pi(C_2(q')$. Now, we consider $q^2+1 = q'^2+1$ so $q = q'$. Now, since $|K/H| = |C| = |G|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, so $H = 1$, $G = K \cong C$. Thus the proof is completed. $\square$

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