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ALGEBRAS OF BINARY FORMULAS FOR $\aleph_0$-CATEGORICAL WEAKLY CIRCULARLY MINIMAL THEORIES: PIECEWISE MONOTONIC CASE

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Abstract: Algebras of binary isolating formulas are described for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 having a non-trivial piecewise (non-strictly) monotonic function.

Keywords: weak circular minimality, algebra of binary formulas, $\aleph_0$-categorical theory, circularly ordered structure, convexity rank.

1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of a type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [1, 2]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [3]–[7].

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Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

Let $\mathcal{M} = \langle M, \leq \rangle$ be a linearly ordered set. If we connect two endpoints of $\mathcal{M}$ (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x,y,z) \to K(y,z,x))$;

(co2) $\forall x \forall y \forall z (K(x,y,z) \land K(y,x,z) \Leftrightarrow x = y \lor y = z \lor z = x)$;

(co3) $\forall x \forall y \forall z (K(x,y,z) \to \forall t [K(x,y,t) \lor K(t,y,z)])$;

(co4) $\forall x \forall y \forall z (K(x,y,z) \lor K(y,x,z))$.

Sometimes we will identify $\mathcal{M}$ and the universe M if a linear/circular order is fixed.

The notion of *weak circular minimality* was studied initially in [8]. Let $A \subseteq M$, where $\mathcal{M}$ is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a,c,b)$, $c \in A$ holds, or for any $c \in M$ with $K(b,c,a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $\mathcal{M} = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in $\mathcal{M}$. Recall [9] that such a structure $\mathcal{M}$ is called *circularly minimal* if any definable (with parameters) of M is a union of finitely many intervals and points in $\mathcal{M}$. Clearly, the weak circular minimality is a generalization of circular minimality. Notice also that any weakly o-minimal structure is weakly circular minimal. The converse, in general, fails. The study of weakly circularly minimal structures was continued in the papers [10]–[16].

Let $\mathcal{M}$ be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(\mathcal{M})$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on $\mathcal{M}$ is an arbitrary G-invariant equivalence relation on $\mathcal{M}$. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on $\mathcal{M}$.

Let $\mathcal{M}$, $\mathcal{N}$ be circularly ordered structures. The 2-*reduct* of $\mathcal{M}$ is a circularly ordered structure with the same universe of $\mathcal{M}$ and consisting of predicates for each $\emptyset$-definable relation on $\mathcal{M}$ of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure $\mathcal{M}$ is *isomorphic* to $\mathcal{N}$ *up to binarity* or *binarily isomorphic to* $\mathcal{N}$ if the 2-reduct of $\mathcal{M}$ is isomorphic to the 2-reduct of $\mathcal{N}$.

Notation.

(1) $K_0(x,y,z) := K(x,y,z) \land y \neq x \land y \neq z \land x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure $\mathcal{M}$. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

The following definition can be used in a circular ordered structure as well.

Definition 1. [17], [18] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set A* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.
2) $RC(A) = 0$ if A is finite and non-empty.
3) $RC(A) \geq 1$ if A is infinite.
4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

Let $f : M \to M$ be an $\emptyset$-definable function with $Dom(f) = I \subseteq M$, where I is an open convex set. We say that f is *monotonic-to-right (left) on I* if it preserves (reverses) the relation K_0, i.e. for any $a, b, c \in I$ such that $K_0(a, b, c)$ we have $K_0(f(a), f(b), f(c))$ ($K_0(f(c), f(b), f(a))$). We also say that f is *piecewise monotonic-to-right (left) on M* if there exists an $\emptyset$-definable non-trivial equivalence relation $E(x, y)$ partitioning M into finitely many infinite convex classes so that f is monotonic-to-right on each E-class and f is not monotonic-to-left (right) on M/E, where by M/E we denote the set of representatives of E-classes in M.

Example. [10] Let $M := \langle M, K, E^2, f^1 \rangle$ be a circularly ordered structure, where M is a disjoint union of $\mathbb{Q}_1$, $\mathbb{Q}_2$, $\ldots$, $\mathbb{Q}_6$, where $\mathbb{Q}_i$ is a copy of the ordering of rational numbers $\mathbb{Q}$. The symbol E interprets an equivalence relation on M as follows: $E(a, b)$ iff there is $1 \leq i \leq 6$ with $a, b \in \mathbb{Q}_i$.

The symbol f interprets a function on M as follows: $f(\mathbb{Q}_i) = \mathbb{Q}_{i+3}$ for each $1 \le i \le 3$, $f(\mathbb{Q}_j) = \mathbb{Q}_{j-3}$ for each $4 \le j \le 6$, and $f(q) = -q$ for all $q \in \mathbb{Q}$.

It can be proved that M is an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, f is a bijection on M so that $f^2(a) = a$ for all $a \in M$, f is monotonic-to-left on each E-class and f is monotonic-to-right on M/E, i.e. f is piecewise monotonic-to-left on M.

Lemma 1. [10] *Let M be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, f be an $\emptyset$-definable function on M. Then f cannot be piecewise monotonic-to-right on M.*

The following theorem characterizes $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures of convexity rank greater than 1 having a non-trivial piecewise (non-strictly) monotonic function up to binarity:

Theorem 1. [11] *(piecewise monotonic case) Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1 having a non-trivial piecewise (non-strictly) monotonic function so that $dcl(a) \neq \{a\}$ for some $a \in M$. Then M is isomorphic up to binarity to $M_{s,m,k} := \langle M, K, f^1, E_1^2, \ldots, E_s^2, E_{s+1}^2\rangle$, where*

- *M is a circularly ordered structure, M is densely ordered, $s \ge 1$, $k \ge 2$, k is even, k divides m, $m \ge 4$;*
- *E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints, for every $1 \le i \le s$ the relation E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints;*
- *f is a bijection on M so that $f^k(a) = a$ for any $a \in M$, for every $1 \le i \le s+1$ $f(E_i(M,a)) = E_i(M, f(a))$ and $\neg E_i(a, f(a))$, and f is piecewise monotonic-to-left on M, i.e. f is monotonic-to-left on each E_{s+1}–class and f is monotonic-to-right on M/E_{s+1}.*

In [19] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [20] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. Here we describe algebras of binary isolating formulas for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and having a non-trivial piecewise (non-strictly) monotonic function.

2 Results

Definition 2. [1] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an m-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

Example. Consider the structure $M_{1,4,2} := \langle M, K^3, f^1, E_1^2, E_2^2 \rangle$ from Theorem 1, where f is piecewise monotonic-to-left on M, E_1 is an equivalence relation partitioning M into infinitely many infinite convex classes, E_2 is an equivalence relation partitioning M into four infinite convex classes.

We assert that $Th(M_{1,4,2})$ has twelve binary isolating formulas:

$$\theta_0(x,y) := x = y, \theta_1(x,y) := K_0(x,y,f(x)) \wedge E_1(x,y),$$
$$\theta_2(x,y) := K_0(x,y,f(x)) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$
$$\theta_3(x,y) := K_0(x,y,f(x)) \wedge \neg E_2(x,y) \wedge \neg E_2(f(x),y),$$
$$\theta_4(x,y) := K_0(x,y,f(x)) \wedge \neg E_1(f(x),y) \wedge E_2(f(x),y),$$
$$\theta_5(x,y) := K_0(x,y,f(x)) \wedge E_1(f(x),y),$$
$$\theta_6(x,y) := f(x) = y, \theta_7(x,y) := K_0(f(x),y,x) \wedge E_1(f(x),y),$$
$$\theta_8(x,y) := K_0(f(x),y,x) \wedge \neg E_1(f(x),y) \wedge E_2(f(x),y),$$
$$\theta_9(x,y) := K_0(f(x),y,x) \wedge \neg E_2(f(x),y) \wedge \neg E_2(x,y),$$
$$\theta_{10}(x,y) := K_0(f(x),y,x) \wedge \neg E_1(x,y) \wedge E_2(x,y),$$
$$\theta_{11}(x,y) := K_0(f(x),y,x) \wedge E_1(x,y),$$

and

$$K_0(\theta_i(a,M), \theta_{i+1}(a,M), \theta_{i+2}(a,M)), \text{ where } 0 \le i \le 9,$$
$$K_0(\theta_{10}(a,M), \theta_{11}(a,M), \theta_0(a,M)), \quad K_0(\theta_{11}(a,M), \theta_0(a,M), \theta_1(a,M))$$

hold for any $a \in M$.

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y) \text{ where } 0 \le k \le 11.$$

It easy to check that for the algebra $\mathfrak{P}_{M_{1,4,2}}$ the Cayley table has the following form:

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11\}$
1	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{5\}$	$\{5,6,7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11,0,1\}$
2	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\ldots$	$\{9\}$	$\{10,11,0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\ldots$	$\{9\}$	$\{9\}$	$\{9\}$	$\{9\}$	$\{9\}$	$\ldots$	$\{3\}$	$\{3\}$
4	$\{4\}$	$\{4\}$	$\ldots$	$\{9\}$	$\ldots$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{4\}$
5	$\{5\}$	$\{5,6,7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11,0,1\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
6	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11\}$	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
7	$\{7\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11\}$	$\{11\}$	$\{11,0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5,6,7\}$
8	$\{8\}$	$\{8\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{10\}$	$\{10\}$	$\{10\}$	$\ldots$	$\{3\}$	$\{4,5,6,7,8\}$	$\{8\}$
9	$\{9\}$	$\{9\}$	$\{9\}$	$\ldots$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\ldots$	$\{9\}$	$\{9\}$
10	$\{10\}$	$\{10\}$	$\ldots$	$\{3\}$	$\ldots$	$\{8\}$	$\{8\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{10\}$	$\{10\}$
11	$\{11\}$	$\{11,0,1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5,6,7\}$	$\{7\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11\}$

By the Cayley table the algebra $\mathfrak{P}_{M_{1,4,2}}$ is not commutative.

Theorem 2. *The algebra $\mathfrak{P}_{M_{s,m,k}}$ of binary isolating formulas having a piecewise monotonic-to-left function on M has $2k(s+1)+m$ labels, is strictly $(2s+3)$-deterministic and is not commutative.*

Proof. Indeed, since $f^k(a)=a$, we have the following isolating formulas:

$$f^l(x)=y \text{ for every } 0\le l\le k-1.$$

Since for every $1\le i\le s$ the relation E_i is an equivalence relation partitioning every E_{i+1}–class into infinitely many infinite convex E_i–subclasses without endpoints so that the induced order on E_i–subclasses is dense without endpoints, we obtain the following binary isolating formulas:

$$K_0(f^l(x),y,f^{l+1}(x))\wedge E_1(f^l(x),y), \text{ where } 0\le l\le k-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_j(f^l(x),y)\wedge E_{j+1}(f^l(x),y),$$
$$\text{where } 0\le l\le k-1, 1\le j\le s-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_s(f^l(x),y)\wedge\neg E_s(f^{l+1}(x),y), \text{ where } 0\le l\le k-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_j(f^{l+1}(x),y)\wedge E_{j+1}(f^{l+1}(x),y),$$
$$\text{where } 0\le l\le k-1, 1\le j\le s-1,$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge E_1(f^{l+1}(x),y), \text{ where } 0\le l\le k-1.$$

Since in this structure there exists additionally an equivalence relation $E_{s+1}(x,y)$ partitioning M into m infinite convex classes, additionally the following binary isolating formulas appear:

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_s(f^l(x),y)\wedge E_{s+1}(f^l(x),y),$$

$$K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_s(f^{l+1}(x),y)\wedge E_{s+1}(f^{l+1}(x),y),$$

where $0\le l\le k-1$.

Also, the formulas $\theta^{l,i}(x,y)$ containing the conjunctive term $K_0(f^l(x), y, f^{l+1}(x))$ and extracting the i-th E_{s+1}-class to the right of E_{s+1}-class containing $f^l(x)$ for some $1\le i\le m/k-1$ (here also $0\le l\le k-1$) will be binary isolating formulas. For example, the formula $\theta^{l,1}(x,y)$ has the following form:

$$\theta^{l,1}(x,y):=K_0(f^l(x),y,f^{l+1}(x))\wedge\neg E_{s+1}(f^l(x),y)\wedge$$
$$\forall t[K_0(f^l(x),t,y)\wedge\neg E_{s+1}(t,y)\to E_{s+1}(f^l(x),t)].$$

Thus, we obtain $k+k+2k(s-1)+k+2k+k(m/k-1)=2k(s+1)+m$ binary isolating formulas.

The formulas

$$\exists t[f^l(x)=t\wedge K_0(f^l(t),y,f^{l+1}(t))\wedge E_1(f^l(t),y)]$$

and

$$\exists t[K_0(f^l(x),t,f^{l+1}(x))\wedge E_1(f^l(x),t)\wedge f^l(t)=y],$$

where $0\le l\le k-1$, uniquely determine the formula

$$K_0(f^{2l\text{mod } k}(x),y,f^{2l+1\text{mod } k}(x))\wedge E_1(f^{2l\text{mod } k}(x),y).$$

The formula

$$\exists t[K_0(x,t,f(x))\wedge E_1(x,t)\wedge K_0(f(t),y,f^2(t))\wedge E_1(f(t),y)]$$

is compatible with the following formulas:

$$K_0(x,y,f(x)) \wedge E_1(f(x),y),$$
$$f(x)=y,$$
$$K_0(f(x),y,f^2(x)) \wedge E_1(f(x),y).$$

While the formula

$$\exists t[K_0(f(x),t,f^2(x)) \wedge E_1(f(x),t) \wedge K_0(t,y,f(t)) \wedge E_1(t,y)]$$

uniquely determines the formula $K_0(f(x),y,f^2(x)) \wedge E_1(f(x),y)$. Consequently, the algebra $\mathfrak{P}_{M_{s,m,k}}$ is not commutative.

And in general we consider the formula

$$\exists t[K_0(f^{l_1}(x),t,f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x),t)\wedge$$
$$K_0(f^{l_2}(t),y,f^{l_2+1}(t)) \wedge E_1(f^{l_2}(t),y)],$$

where $0 \leq l_1, l_2 \leq k-1$. Such a formula for even l_2 uniquely determines the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y,f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y),$$

and for odd l_2 it is compatible with the following three formulas:

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y,f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y),$$
$$f^{l_1+l_2 (\mathrm{mod}\ k)}(x)=y,$$
$$K_0(f^{l_1+l_2-1 (\mathrm{mod}\ k)}(x),y,f^{l_1+l_2 (\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y).$$

Further consider the formula

$$\exists t[K_0(f^{l_1}(x),t,f^{l_1+1}(x)) \wedge E_1(f^{l_1}(x),t) \wedge K_0(f^{l_2}(t),y,f^{l_2+1}(t))$$
$$\wedge \neg E_j(f^{l_2}(t),y) \wedge E_{j+1}(f^{l_2}(t),y)],$$

where $0 \leq l_1, l_2 \leq k-1$, $1 \leq j \leq s$. It uniquely determines the formula

$$K_0(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y,f^{l_1+l_2+1 (\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y)$$
$$\wedge E_{j+1}(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y).$$

On the other hand, the formula

$$\exists t[K_0(f^{l_2}(x),t,f^{l_2+1}(x)) \wedge \neg E_j(f^{l_2}(x),t) \wedge E_{j+1}(f^{l_2}(x),t)$$
$$\wedge K_0(f^{l_1}(t),y,f^{l_1+1}(t)) \wedge E_1(f^{l_1}(t),y)]$$

uniquely determines the formula

$$K_0(f^{l_1+l_2-1 (\mathrm{mod}\ k)}(x),y,f^{l_1+l_2 (\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y)$$
$$\wedge E_{j+1}(f^{l_1+l_2 (\mathrm{mod}\ k)}(x),y).$$

Consider now the formulas $\theta^{l_1,i}(x,y)$ and $\theta^{l_2,j}(x,y)$ for arbitrary $0 \leq l_1, l_2 \leq k-1$, $1 \leq i,j \leq m/k-1$. If $i+j\ (\mathrm{mod}\ m/k) \neq 0$, it easy to check that the formulas

$$\exists t[\theta^{l_1,i}(x,t) \wedge \theta^{l_2,j}(t,y)] \text{ and } \exists t[\theta^{l_2,j}(x,t) \wedge \theta^{l_1,i}(t,y)]$$

uniquely determine the formula $\theta^{l_1+l_2(\mathrm{mod}\ k), i+j(\mathrm{mod}\ m/k)}(x,y)$.

If $i+j \pmod{m/k} = 0$, these formulas are compatible with the following $2s+3$ formulas:

$$f^{l_1+l_2+1(\mathrm{mod}\ k)}(x)=y,$$

$$K_0(f^{l_1+l_2(\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1(\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y),$$

$$K_0(f^{l_1+l_2(\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+1(\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y), \quad 1 \le j \le s,$$

$$K_0(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+2(\mathrm{mod}\ k)}(x)) \wedge E_1(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y),$$

$$K_0(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y, f^{l_1+l_2+2(\mathrm{mod}\ k)}(x)) \wedge \neg E_j(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y)$$

$$\wedge E_{j+1}(f^{l_1+l_2+1(\mathrm{mod}\ k)}(x), y), \quad 1 \le j \le s.$$

Thus, the algebra $\mathfrak{P}_{M_{s,m,k}}$ is strictly $(2s+3)$-deterministic. $\square$

References

[1] S.V. Sudoplatov, *Classification of countable models of complete theories*, Novosibirsk: NSTU, 2018.

[2] I.V. Shulepov, S.V. Sudoplatov, *Algebras of distributions for isolating formulas of a complete theory*, Siberian Electronic Mathematical Reports, **11** (2014), 380–407.

[3] S.V. Sudoplatov, *Algebras of distributions for semi-isolating formulas of a complete theory*, Siberian Electronic Mathematical Reports, **11** (2014), 408–433.

[4] D.Yu. Emelyanov, B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of distributions for binary formulas in countably categorical weakly o-minimal structures*, Algerba and Logic, **56**:1 (2017), 13–36.

[5] K.A. Baikalova, D.Yu. Emelyanov, B.Sh. Kulpeshov, E.A. Palyutin, S.V. Sudoplatov, *On algebras of distributions of binary isolating formulas for theories of abelian groups and their ordered enrichments*, Russian Mathematics, **62**:4 (2018), 1–12.

[6] D.Yu. Emelyanov, B.Sh. Kulpeshov, S.V. Sudoplatov, *On algebras of distributions for binary formulas for quite o-minimal theories*, Algebra and Logic, **57**:6 (2019), 429–444.

[7] A.B. Altayeva, B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of distributions of binary isolating formulas for almost ω-categorical weakly o-minimal theories*, Algebra and Logic, **60**:4 (2021), 241–262.

[8] B.Sh. Kulpeshov, H.D. Macpherson, *Minimality conditions on circularly ordered structures*, Mathematical Logic Quarterly, **51**:4 (2005), 377–399.

[9] H.D. Macpherson, C. Steinhorn, *On variants of o-minimality*, Annals of Pure and Applied Logic, **79** (1996), 165–209.

[10] B.Sh. Kulpeshov, *On $\aleph_0$-categorical weakly circularly minimal structures*, Mathematical Logic Quarterly, **52**:6 (2006), 555–574.

[11] B.Sh. Kulpeshov, *Definable functions in the $\aleph_0$-categorical weakly circularly minimal structures*, Siberian Mathematical Journal, **50**:2 (2009), 282–301.

[12] B.Sh. Kulpeshov, *On indiscernibility of a set in circularly ordered structures*, Siberian Electronic Mathematical Reports, **12** (2015), 255–266.

[13] B.Sh. Kulpeshov, V.V. Verbovskiy, *On weakly circularly minimal groups*, Mathematical Logic Quarterly, **61:**1-2 (2015), 82–90.

[14] B.Sh. Kulpeshov, A.B. Altayeva, *Binary formulas in countably categorical weakly circularly minimal structures*, Algebra and Logic, **55**:3 (2016), 226–241.

[15] B.Sh. Kulpeshov, *On almost binarity in weakly circularly minimal structures*, Eurasian Mathematical Journal, **7**:2 (2016), 38–49.
[16] A.B. Altayeva, B.Sh. Kulpeshov, *Almost binarity of countably categorical weakly circularly minimal structures*, Mathematical Notes, **110**:6 (2021), 813–829.
[17] B.Sh. Kulpeshov, *Weakly o-minimal structures and some of their properties*, The Journal of Symbolic Logic, **63**:4 (1998), 1511–1528.
[18] B.Sh. Kulpeshov, *A criterion for binarity of almost ω-categorical weakly o-minimal theories*, Siberian Mathematical Journal, **62**:2 (2021), 1063–1075.
[19] D.Yu. Emelyanov, B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of binary formulas for compositions of theories*, Algebra and Logic, **59**:4 (2020), 295–312.
[20] B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of binary formulas for weakly circularly minimal theories with non-trivial definable closure*, Lobachevskii Journal of Mathematics, **43**:12 (2022), 3532–3540.

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