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EDGE 4-CRITICAL KOESTER GRAPH OF ORDER 28

A.A. DOBRYNIN

Abstract. A Koester graph G is a simple 4-regular plane graph formed by the superposition of a set S of circles in the plane, no two of which are tangent and no three circles have a common point. Crossing points and arcs of S correspond to vertices and edges of G, respectively. A 4–chromatic edge critical Koester graph of order 28 generated by intersection of six circles is presented. This improves an upper bound for the smallest order of such graphs. The previous upper bound was established by Gerhard Koester in 1984 by constructing a graph with 40 vertices.

Keywords: plane graph, 4-critical graph, Grötzsch–Sachs graph, Koester graph.

1. Introduction

In this paper we are concerned with simple undirected connected graphs. A graph is *k-chromatic* if its chromatic number is equal to k. A graph is called *edge 4-critical* if it is 4-chromatic and the removal of any edge decreases its chromatic number. Numerous results and problems related to critical graphs can be found in [1, 12]. Consider a graph $G = G(S)$ formed by the superposition of a set S of circles in the plane, no two of which are tangent and no three circles share a point. Crossing points and arcs of the set S correspond to vertices and edges of G, respectively. Since every two circles in the plane have exactly two crossing points, G is always a 4-regular plane graph of even order. Such regular graphs are called *Koester graphs*. To describe the mutual position of circles of S, we use the *characteristic graph* $H(S)$. It is the intersection graph of the circles: vertices of $H(S)$ correspond to members of S and two vertices are adjacent if and only if the corresponding circles

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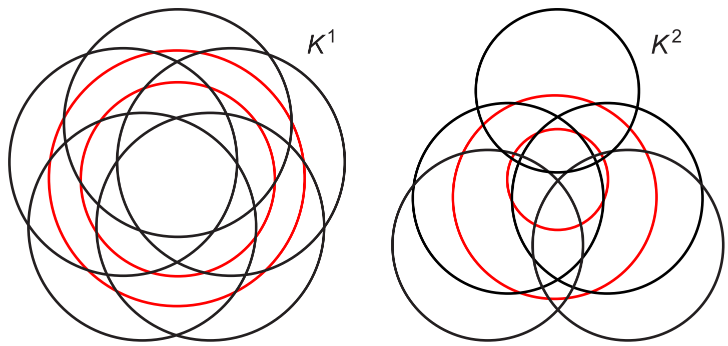

FIG. 1. Edge 4-critical Koester graphs of order 40.

intersect. We use notation $H(G)$ if a graph G is generated by S. Koester graphs are a special case of *Grötzsch–Sachs graphs* which are formed by intersections of closed curves in the plane. The first studies concerning coloring of such graphs are due to H. Grötzsch. H. Sachs actively discussed the problems arising in this area at conferences. We briefly describe some results on the coloring of Grötzsch–Sachs graphs (see [18, 9, 10, 11, 13, 14, 15, 16, 17, 19, 8]). F. Jaeger proved that $\chi(G) \leq 3$ if $\chi(H(G)) \leq 3$ [9, 10]. Examples and infinite families of 4-chromatic or edge 4-critical Grötzsch–Sachs graphs were presented in [2, 4, 5, 6, 7, 8]. In particular, Grötzsch–Sachs–Koester's conjecture stating that if $\chi(H(G)) = 4$ then $\chi(G) \leq 3$ was disproved by constructing counterexamples on 18 vertices [4]. In 1984, G. Koester found the first example of an edge 4-critical Grötzsch–Sachs graph (see graph K^1 in Fig. 1). Graph K^1 of order 40 is generated by a set of seven circles and its characteristic graph is $K_7 - e$ [13, 14, 15]. It was the only example of graphs of this kind for a long time. A new edge 4-critical Koester graph K^2 can be obtained by rearranging circles of K^1 as shown in Fig. 1 [3]. Since $H(K^2) \cong K_7 - e$, graph K^2 also has 40 vertices. In this paper, an edge 4-critical Koester graph of order 28 formed by six circles in the plain is presented. This gives a new upper bound for the smallest order of such graphs.

2. Construction of a graph

Let K be a 4-regular graph of order 28 generated by six circles in the plane as shown in Fig. 2. Since the circle set has the unique pair of non-crossing circles, the characteristic graph of K is $K_6 - e$. The circles correspond to two cycles of order 8 and four cycles of order 10 of K: disjoint red cycles $R_1 = (14, 16, 27, 21, 19, 8, 1, 5)$ and $R_2 = (11, 15, 26, 25, 24, 10, 2, 3)$, blue cycle $M = (4, 14, 18, 20, 21, 23, 22, 10, 9, 3)$, black cycle $B = (4, 11, 13, 26, 28, 23, 19, 7, 6, 5)$, green cycle $G = (1, 2, 9, 12, 25, 28, 27, 20, 17, 6)$, and orange cycle $O = (15, 16, 18, 17, 7, 8, 22, 24, 12, 13)$.

The automorphism group of K contains an involution that maps cycles R_1 and R_2 into itself, cycle B to cycle M, and cycle G to cycle O. Vertices 4, 23, 12, and 17 are fixed by the involution and the other vertices form twelve two-element orbits: (1,16), (5,14), (22,28), (8,27), (2,15), (9,13), (7,20), (3,11), (10,26), (19,21), (6,18), and (24,25). Note that vertices of edges (5,14), (19,21) of R_1 and (3,11), (24,25) of R_2 form orbits.

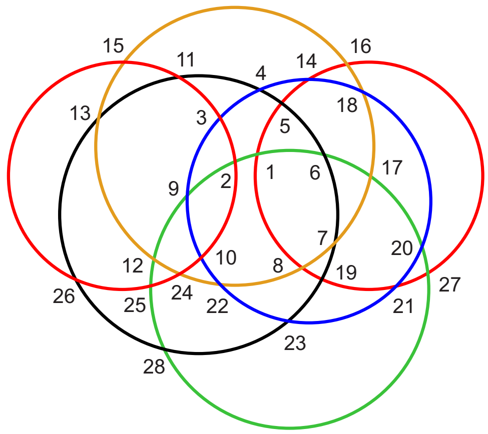

FIG. 2. Edge 4-critical Koester graph K of order 28.

Proposition 1. *Koester graph K is 4-chromatic.*

Proof. The presence of odd cycles in K implies $\chi(K) \geq 3$. Since K is a 4-regular graph, $\chi(K) \leq 4$ by Brooks' theorem. Suppose that K is a 3-chromatic graph and try to find a 3-coloring of K. First, we color vertices of some pentagonal face (5-face) in all possible ways. Further we show that it is impossible to extend any coloring of this face to the entire graph K. Take face $f_5 = (17, 18, 14, 5, 6)$ for the initial coloring (see Fig. 2). The colored vertices of the graph will be drawn as red, white and green circles. We need exactly 3 colors for vertices of a 5-face in a 3-chromatic graph. Color of one vertex of a 5-face always distinct from colors of the other vertices of the face. Let this unique color be red. The following two useful rules will be applied in the process of graph coloring.

1. Let a graph G be obtained from the simple path of order 4 by joining a new vertex with non-pendant vertices of the path. Suppose that pendant vertices of G have the same color in a proper 3-coloring. Then the vertex of degree two of G must have this color.

2. Let G consists of two triangles having one common edge. Then two vertices of degree 2 of G must have the same color in a proper 3-coloring of G.

All possible extensions of the colorings of the initial face f_5 are shown in Fig. 3. Every uncolored vertex gets a forced color at some step of the coloring procedure. The numbers of steps are depicted near graph vertices. The question mark shows an edge whose vertices cannot be properly colored. If a vertex color is determined, say, in step 6 by the vertex colors obtained at steps 2 and 5, we use record $s(2, 5) \to s(6)$. The number of the applied coloring rule will be indicated as a subscript of s. The initial 5-face can be colored by five cases. In accordance with the initial coloring of face f_5, we consider five cases.

Case 1. Let vertex 17 of face f_5 be red (see Fig. 3a). The following sequence of coloring steps uniquely defines colors of some vertices of graph K: $s(1, 2) \to s(7)$, $s(2, 4) \to s(8)$, $s(4, 5) \to s(9)$, $s(7, 8) \to s(10)$, $s(7, 10) \to s(11)$, $s(6, 11) \to s(12)$, and $s(6, 12) \to s(13)$. Since the end-vertices of edge (1,8) have the same color, we cannot get a proper 3-coloring of graph K .

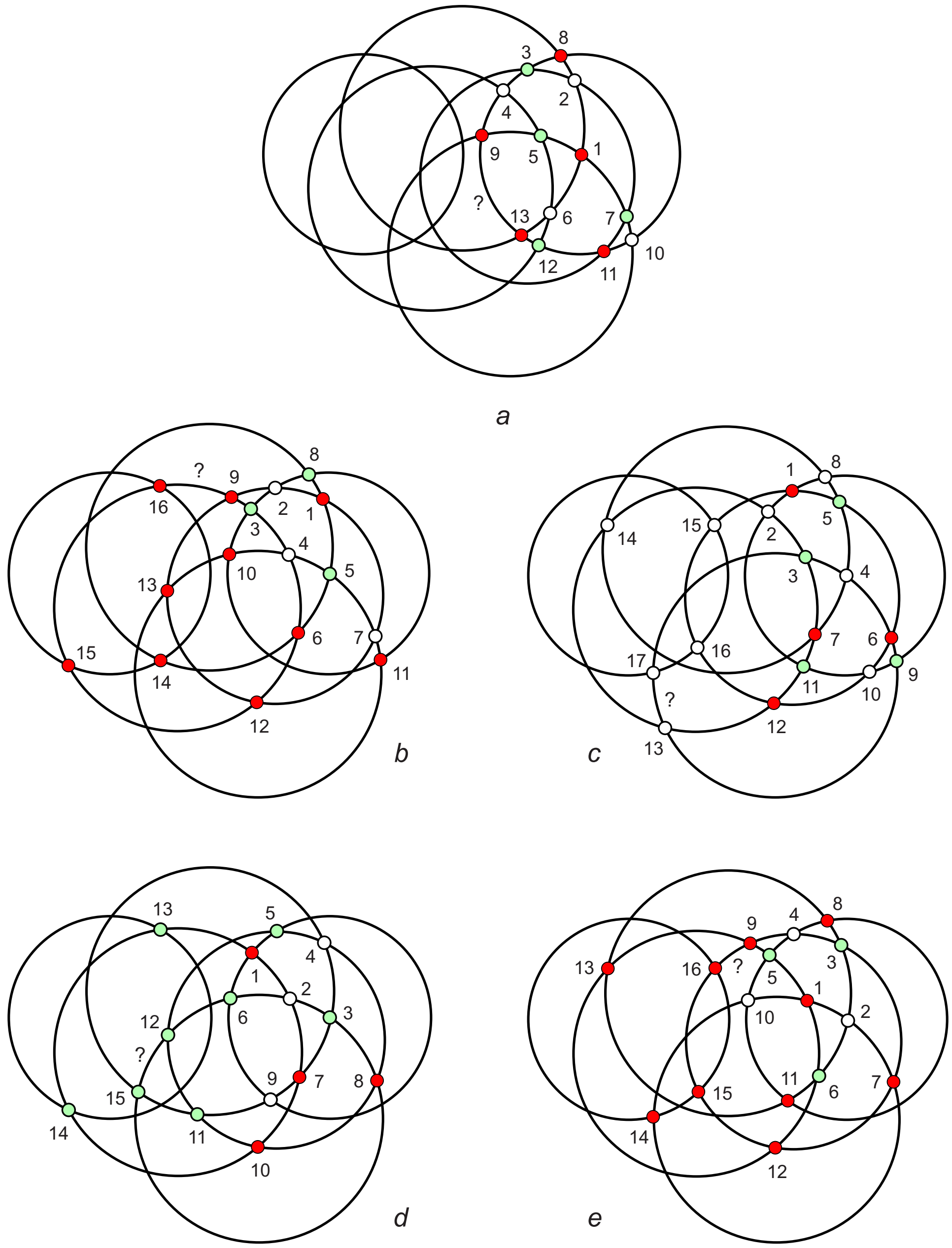

FIG. 3. Steps of 3-colorings of graph K.

Case 2. Let vertex 18 of face f_5 be red (see Fig. 3b). Then $s(4,5) \to s(6)$, $s(1,5) \to s(7)$, $s(1,2) \to s(8)$, $s(2,3) \to s(9)$, $s(3,4) \to s(10)$, $s(7,8) \to s(11)$, $s_1(6,11) \to s(12)$, $s_1(9,10) \to s(13)$, $s_1(12,13) \to s(14)$, $s_1(12,14) \to s(15)$, and $s_2(15) \to s(16)$. As a result of this partial coloring, edge (4,11) has end-vertices with the same color.

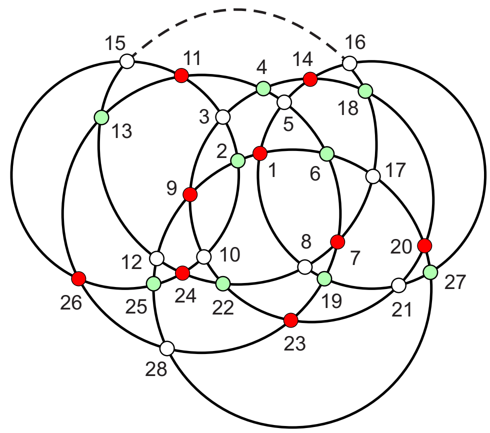

FIG. 4. 3-coloring of graph $K-(15,16)$.

Case 3. Let vertex 14 of face f_5 be red (see Fig. 3c). We have $s(4,5)\to s(6)$, $s(3,4)\to s(7)$, $s(1,5)\to s(8)$, $s(6,8)\to s(9)$, $s(6,9)\to s(10)$, $s(7,10)\to s(11)$, $s(10,11)\to s(12)$, $s(9,12)\to s(13)$, $s_1(8,13)\to s(14)$, $s_1(2,14)\to s(15)$, $s_2(15)\to s(16)$, and $s_1(14,16)\to s(17)$. End-vertices of edge (25,28) cannot be colored properly.

Case 4. Let vertex 5 of face f_5 be red (see Fig. 3d). Then $s(1,2)\to s(6)$, $s(2,3)\to s(7)$, $s(3,4)\to s(8)$, $s(6,7)\to s(9)$, $s_1(7,8)\to s(10)$, $s_1(9,10)\to s(11)$, $s_1(6,11)\to s(12)$, $s_1(5,12)\to s(13)$, $s_2(13)\to s(14)$, and $s_1(11,14)\to s(15)$. End-vertices of edge (9,12) have the same colors.

Case 5. Let vertex 6 of face f_5 be red (see Fig. 3e). Then $s(1,2)\to s(6)$, $s(2,3)\to s(7)$, $s(3,4)\to s(8)$, $s(4,5)\to s(9)$, $s(1,5)\to s(10)$, $s(6,10)\to s(11)$, $s_1(7,11)\to s(12)$, $s_1(8,9)\to s(13)$, $s_1(12,13)\to s(14)$, $s_1(12,14)\to s(15)$, and $s_2(15)\to s(16)$. Vertices of edge (3,4) are colored in the same color.

The considered cases lead to the impossibility of 3-coloring of graph K and, therefore, K is a 4-chromatic graph. □

Proposition 2. *Koester graph K is edge critical.*

Proof. To prove that every edge of K is critical, we present a 3-coloring of graph $K-e$ for every edge e of K. Because of symmetry of K, it is sufficient to consider 30 edges. Since the involution fixes two edges of cycles R_1 and R_2 of length 8, we should check 5 edges from every cycle. Then we check all edges from cycles M and O of length 10. Vertex colorings of $K-e$ for these edges e are presented in Table 1. The header of Table 1 contains the vertex numbering of K. The colors of vertices are located in the columns of the table. As an illustration, a 3-coloring of graph $K-(15,16)$ is depicted in Fig. 4 □

The order of the constructed graph K improves an upper bound for the smallest order of edge 4-critical Koester graphs. Computer search shows that there are no such graphs with the number of vertices less than 22. It would be interesting to answer to the following question: does there exist an edge 4-critical Koester graph of order 22, 24, or 26?

TABLE 1. 3-colorings of graph K.

N	e	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	(1, 5)	1	2	1	3	1	2	1	3	3	1	2	2	3	2	1	3	3	1	2	2	3	2	1	3	1	2	1	3
2	(1, 8)	1	2	1	3	2	3	2	1	3	1	2	2	3	1	1	3	1	2	3	3	2	2	1	3	1	2	1	3
3	(2, 3)	1	2	2	3	2	3	1	2	3	1	1	1	2	1	3	2	2	3	3	1	2	3	1	2	3	1	3	2
4	(2,10)	1	2	1	2	3	2	3	2	3	2	3	2	1	1	2	3	1	2	1	3	2	1	3	3	1	3	1	2
5	(3, 4)	1	2	1	1	3	2	1	3	3	1	2	2	3	2	1	3	3	1	2	2	3	2	1	3	1	2	1	3
6	(3, 9)	1	2	1	2	3	2	3	2	1	3	3	3	2	1	1	3	1	2	1	3	2	1	3	2	1	3	1	2
7	(3,11)	1	2	1	3	2	3	1	2	3	1	1	1	2	1	3	2	2	3	3	1	2	3	1	2	3	1	3	2
8	(4,14)	1	2	1	2	3	2	1	3	3	1	3	2	1	2	2	3	3	1	2	2	3	2	1	3	1	3	1	2
9	(5,14)	1	2	1	3	2	3	1	2	3	1	2	1	3	2	1	3	2	1	3	3	1	3	2	2	3	2	2	1
10	(7, 8)	1	2	1	3	2	3	2	2	3	1	2	1	3	1	1	2	1	3	3	2	1	3	2	2	3	2	3	1
11	(7,17)	1	2	1	3	2	3	1	3	3	1	2	2	3	1	1	2	1	3	2	2	3	2	1	3	1	2	1	3
12	(8,19)	1	2	1	3	2	3	2	3	3	1	2	2	3	1	1	3	1	2	3	3	2	2	1	3	1	2	1	3
13	(8,22)	1	2	1	2	3	2	3	2	3	1	3	2	1	1	2	3	1	2	1	3	2	2	3	3	1	3	1	2
14	(9,10)	1	2	3	1	3	2	1	3	1	1	2	2	3	2	1	3	3	1	2	2	3	2	1	3	1	2	1	3
15	(10,22)	1	2	1	3	2	3	2	3	3	1	2	2	3	1	1	2	1	3	1	2	3	1	2	3	1	2	1	3
16	(10,24)	1	2	3	1	3	2	1	3	1	3	2	2	3	2	1	3	3	1	2	2	3	2	1	3	1	2	1	3
17	(12,13)	1	2	3	2	3	2	1	3	1	3	1	3	3	1	2	3	3	2	2	1	3	2	1	1	2	1	2	3
18	(12,24)	1	2	3	2	3	2	3	2	1	3	1	2	3	1	2	3	1	2	1	3	2	1	3	2	3	1	1	2
19	(13,15)	1	2	1	3	2	3	2	3	3	1	2	2	1	1	1	3	1	2	1	3	2	2	3	3	1	3	1	2
20	(14,18)	1	2	1	3	2	3	1	2	3	1	2	1	3	1	1	3	2	1	3	3	1	3	2	2	3	2	2	1
21	(15,16)	1	2	3	2	3	2	1	3	1	3	1	3	2	1	3	3	3	2	2	1	3	2	1	1	2	1	2	3
22	(16,18)	1	2	1	3	2	3	1	3	3	1	2	2	3	1	1	3	2	3	2	1	3	2	1	3	1	2	2	3
23	(17,18)	1	2	1	3	2	3	1	3	3	1	2	2	3	1	1	3	2	2	2	1	3	2	1	3	1	2	2	3
24	(18,20)	1	2	1	2	3	2	1	3	3	1	3	2	1	1	2	3	3	2	2	2	3	2	1	3	1	3	1	2
25	(19,21)	1	2	3	1	2	3	2	3	1	3	2	3	1	3	3	1	1	2	1	3	1	1	2	2	1	2	2	3
26	(20,21)	1	2	1	3	2	3	1	2	3	1	2	1	3	1	1	2	2	3	3	1	1	3	2	2	3	2	3	1
27	(21,23)	1	2	1	3	2	3	1	2	3	1	2	1	3	1	1	2	2	3	3	1	2	3	2	2	3	2	3	1
28	(22,23)	1	2	1	3	2	3	2	3	3	1	2	2	3	1	1	2	1	3	1	2	3	2	2	3	1	2	1	3
29	(22,24)	1	2	3	2	3	2	3	2	1	3	1	2	3	1	2	3	1	2	1	3	2	1	3	1	3	1	1	2
30	(24,25)	1	2	3	2	3	2	3	2	1	3	1	3	2	1	3	2	1	3	1	2	3	1	2	2	2	1	1	3

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Andrey Alekseevich Dobrynin
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: dobr@math.nsc.ru