A Note on Brøndsted's Fixed Point Theorem

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Abstract. We show that for the case of uniformly convex Banach spaces the conditions of the Brøndsted fixed point theorem can be relaxed.

1. Introduction. The main theorem

The object of this short note is a fixed point theorem by Arne Brøndsted. Let us formulate this theorem.

Let $(X, \|\cdot\|)$ be a Banach space and let $M \subset X$ be a closed set. We denote a closed unit ball as $B = \{\|x\| \le 1\}$. Assume that

$$M \cap B = \emptyset. \tag{1.1}$$

Consider a mapping $T : M \to M$ that maps each $x \in M$ in the direction of the ball: if $Tx \ne x$ then there exists $t > 1$ such that

$$x + t(Tx - x) \in B. \tag{1.2}$$

Theorem 1 ([**1**]). *In addition to the assumptions above suppose that*

$$\inf\{\|x\| \mid x \in M\} > 1. \tag{1.3}$$

Then the mapping T has a fixed point.

Observe that condition (1.3) is stronger than condition (1.1) if only $\dim X = \infty$.

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To prove theorem 1 Brøndsted endows the set M with a partial order in the following way.

If $x, y \in M$ then by definition we write $x < y$ provided either $x = y$ or there exists $t > 1$ such that

$$x + t(y - x) \in B.$$

Formula (1.2) takes the form

$$x < Tx, \quad \forall x \in M. \tag{1.4}$$

Then Brøndsted observes that this partial order is finer than one of the Caristi type [**3**] and thus by some other of his results [**2**] the fixed point exists.

Our aim is to show that for the class of uniformly convex Banach spaces X theorem 1 remains valid even in the critical case when condition (1.3) is replaced with (1.1). This fact does not follow from the original Brøndsted's method.

Recall a definition. *A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if for any $\sigma > 0$ there exists $\gamma > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \sigma$ then $\|x + y\| \le 2 - \gamma$.*

For example the space $L^p, \quad p \in (1, \infty)$ is uniformly convex particularly ℓ_p is uniformly convex; each uniformly convex Banach space is reflexive; a Hilbert space is uniformly convex, see [**4**] and references therein.

THEOREM 2. *Assume that X is a uniformly convex Banach space. If the mapping T satisfies condition (1.4) and condition (1.1) is fulfilled then T has a fixed point.*

1.1. An example. For the space X take $\ell_p, \quad 1 < p < \infty$. Introduce sets

$$M_n = \{\mathbf{x} = \{x_k\} \in \ell_p \mid x_n \ge 1 + 1/n\}, \quad M = \bigcup_{n \in \mathbb{N}} M_n.$$

It is not hard to show that the set M is closed and $M \cap B = \emptyset$. A sequence

$$\mathbf{x}_j = (0, \dots, 0, 1 + 1/j, 0, \dots), \quad (1 + 1/j \text{ stands at j-th place})$$

belongs to M and $\|\mathbf{x}_j\| \to 1$ as $j \to \infty$.

Thus the set M suits for theorem 2 but it does not suit for theorem 1.

Now take any nonempty closed set $M \subset X, \quad M \cap B = \emptyset$ in a uniformly convex Banach space X and let $f : B \to B$ be a mapping.

Construct T as follows. Take $\mathbf{x} \in M$ and let

$$\lambda_0 = \min\{\lambda \in [0,1] \mid \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in M\}, \quad \mathbf{y} = f\Big(\frac{\mathbf{x}}{\|\mathbf{x}\|}\Big).$$

It is clear $\lambda_0 > 0$. By definition put $T\mathbf{x} = \lambda_0 \mathbf{x} + (1-\lambda_0)\mathbf{y}$. We obviously obtain $\mathbf{x} < T\mathbf{x}$ and T has a fixed point.

2. Proof of theorem 2

The scheme of the proof is quite standard by itself. It is clear that a maximal element of the set M provides a fixed point. To prove that the maximal element exists we check the conditions of the Zorn lemma.

This argument and the technique developed below give possibility to conduct a direct proof of theorem 1 as well.

PROPOSITION 1. *Suppose that vectors $a, x \in X$ have the following properties*

$$\|(1-t)a + tx\| > 1, \quad \forall t \in (0,1), \quad \|x\| > 1, \quad \|a\| = 1.$$

Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that inequality $\|x\| \le 1 + \delta$ implies $\|x - a\| \le \varepsilon$.

This proposition admits a "physical" interpretation. Let x be a light source placed away from the ball $B: \quad \|x\| > 1$. According to the proposition the diameter of the light spot on the ball tends to zero as x approaches the ball: $\|x\| \to 1$.

Here the uniform convexity of the norm is essential: such a feature is not yet kept for the norm $\|(p,q)\| = \max\{|p|, |q|\}$ in $\mathbb{R}^2$.

Proof of proposition 1. Assume the opposite: there exist $\varepsilon > 0$ and sequences a_n, x_n,

$$\|x_n\| > 1, \quad \|a_n\| = 1, \quad \|x_n\| \to 1, \quad \|(1-t)a_n + tx_n\| > 1 \tag{2.1}$$

such that

$$\|x_n - a_n\| > \varepsilon.$$

Consequently, for all sufficiently large n the estimate

$$\|x_n - a_n\| = \Big\|x_n - \frac{x_n}{\|x_n\|} + \frac{x_n}{\|x_n\|} - a_n\Big\|$$
$$\le \alpha_n + \Big\|a_n - \frac{x_n}{\|x_n\|}\Big\|, \quad \alpha_n = \|x_n\|\Big(1 - \frac{1}{\|x_n\|}\Big) \to 0$$

implies

$$\Big\|a_n - \frac{x_n}{\|x_n\|}\Big\| \ge \varepsilon/2.$$

Plugging $t = 1/2$ in (2.1) we obtain

$$\|a_n + x_n\| > 2. \tag{2.2}$$

The inequality

$$\Big\|a_n+\frac{x_n}{\|x_n\|}\Big\|>2-\alpha_n$$

follows from (2.2) in the same way as above.

This contradicts the hypothesis of uniform convexity of the space X.

The proposition is proved.

Let $C\subset M$ be a chain then put $\rho=\inf\{\|u\|\mid u\in C\},\quad \rho\ge 1$.

The inclusion $x\in C$ implies that $\|x\|>1$ provided $\rho=1$ and $\|x\|\ge\rho$ provided $\rho>1$.

For any $x\in C$ define a set

$$K_x(\rho)=\{y\in M\mid \|y\|\ge\rho,\quad y>x\}.$$

The sets $K_x(\rho)$ are nonvoid: $x\in K_x(\rho)$ and

$$x_1<x_2\Longrightarrow K_{x_2}(\rho)\subset K_{x_1}(\rho). \tag{2.3}$$

Lemma 1. *The sets $K_x(\rho)$ are closed.*

Proof of lemma 1. Indeed, let a convergent sequence $\{y_k\}$ belong to $K_x(\rho)$ and $y_k\to y\in M$. This means that there are sequences $\{\beta_k\}\subset(0,1)$ and $\{a_k\}\subset X,\quad \|a_k\|=1$ such that

$$y_k=\beta_k a_k+(1-\beta_k)x.$$

The sequence $\{\beta_k\}$ contains a convergent subsequence; we keep the same notation for this subsequence: $\beta_k\to\beta$.

If $\beta=0$ then $\|\beta_k a_k\|\to 0$ and $y=x\in K_\rho(x)$. If $\beta\ne 0$ then

$$a_k=\frac{1}{\beta_k}y_k+\Big(1-\frac{1}{\beta_k}\Big)x\to a,\quad \|a\|=1$$

and

$$y=\beta a+(1-\beta)x.$$

Observe that since $y\in M$ the parameter β cannot be equal to 1.

The lemma is proved.

Lemma 2. *Suppose that $z\in K_x(\rho),\quad x\in C$.*
If $\rho>1$ then

$$\|z-x\|\le(\|x\|-\rho)\frac{\rho+1}{\rho-1}.$$

If $\rho=1$ then for any $\varepsilon>0$ there exists $\delta>0$ such that

$$\|x\|\le 1+\delta\Longrightarrow\|z-x\|\le\varepsilon.$$

Proof of lemma 2.

The case $\rho > 1$. Indeed, the formulas

$$x + t(z - x) = a, \quad \|a\| = 1, \quad t > 1, \quad \|x\|, \|z\| \geq \rho > 1 \tag{2.4}$$

imply $z = (a + (t-1)x)/t$ and

$$\rho \leq \|z\| \leq \frac{1}{t} + \frac{t-1}{t}\|x\|, \quad \frac{1}{t} \leq \frac{\|x\| - \rho}{\|x\| - 1}.$$

Use (2.4) again:

$$\|z - x\| = \frac{1}{t}\|a - x\| \leq \frac{1}{t}(1 + \|x\|).$$

The case $\rho = 1$. The condition of the lemma $z \in K_x(1)$ means

$$z = \tau a + (1 - \tau)x, \quad \|x\|, \|z\| > 1, \quad \|a\| = 1, \quad \tau \in (0, 1).$$

Therefore the assertion of the lemma follows from proposition 1 and the formulas

$$z - x = \tau(a - x), \quad \|z - x\| < \|x - a\|.$$

The lemma is proved.

LEMMA 3. *For any $\varepsilon > 0$ there exists $\tilde{x} \in C$ such that*

$$C \ni x > \tilde{x} \Longrightarrow \operatorname{diam} K_x(\rho) < \varepsilon.$$

Proof of lemma 3. Consider the case $\rho > 1$. By definition of the number ρ, for any $\varepsilon > 0$ there is an element $\tilde{x} \in C$ such that

$$\|\tilde{x}\| \leq \varepsilon + \rho.$$

Take any elements $z_1, z_2 \in K_{\tilde{x}}$ and apply lemma 2 for each summand from the right side of the inequality

$$\|z_1 - z_2\| \leq \|z_1 - \tilde{x}\| + \|z_2 - \tilde{x}\|.$$

Observe also that formula (2.3) implies

$$\tilde{x} < x \in C \Longrightarrow \operatorname{diam} K_x(\rho) \leq \operatorname{diam} K_{\tilde{x}}(\rho).$$

The case $\rho = 1$ is processed analogously be means of lemma 2.

The lemma is proved.

Therefore we have a nested family of the closed sets $K_x(\rho)$ which diameters tend to zero. By well-known theorem their intersection is not empty and consists of a single point:

$$\bigcap_{x \in C} K_x(\rho) = \{m\}.$$

The point $m \in M$ is an upper bound for C. Indeed, for any $x \in C$ we have $m \in K_x(\rho)$ and thus $x < m$.

The theorem is proved.

References

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