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ON SUFFICIENT CONDITIONS OF TRANSIENCE FOR A STOCHASTIC DIFFERENTIAL EQUATION WITH SWITCHING

KIRILL MOSIEVICH

Abstract. Conditions sufficient for the transience of the process have been established for the Markov diffusion model with switching and two modes, transient and ergodic, with intensities bounded away from zero. This paper shows limitations on the conditions for exponential ergodicity with a given switching system.

Keywords: diffusion, switching, variable switching intensities, transience.

1. Introduction

The problem of finding sufficient ergodicity conditions is an actual problem [5] [6], [7].Cloez and Hairer considered in their paper [5] the exponential case. Veretennikov A. Yu. reported on the polynomial case [6]. The transient case shows us the optimality of the results obtained in the ergodic case. At the present step in my work I assume the opposite, but stronger requirements, to the conditions introduced in the work of Cloez and Hairer. Let's move on to the problem statement.

On a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with an one-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\geq 0}$ on it, an one-dimensional SDE with switching is considered,

$$dX_t = b(X_t, Z_t)dt + dW_t,\ t \geq 0\ , X_0 = x,\ , Z_0 = z, \tag{1}$$

where Z_t - is a continuous-time Markov process on the state space $S = \{0, 1\}$ with (positive) intensities of respective transitions $\lambda_{01} =: \lambda_+,\ \lambda_{10} =: \lambda_-$. The process Z_t is assumed to be independent of W and

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adapted to the filtration $(\mathcal{F}_t)$, and the switching moment are exponentially. We assume that these intensities Z are constants. The process X_t is "transience"under the regime $Z_t = 1$, while under the regime $Z_t = 0$ is assumed positive recurrent. Denote

$$b(x,0) = b_-(x),\ b(x,1) = b_+(x),$$

$$b_+ := ||b_+(x)|| = \sup_x |b_+(x)|,\ b_- := ||b_-(x)|| = \sup_x |b_-(x)|.$$

2. Main results

Consider the moments of switching of the process Z:

$$0 \le \inf\left(t \ge 0 : Z_t = 0\right) := T_0 < T_1 < ... < T_n := \inf\left(t > T_{n-1}, Z_t \ne Z_{T_{n-1}}\right) < \dots$$

The existence and pathwise uniqueness of the solution (1) follows easily from [1], or [2], or [3]. Note, that this process is a Markov process with continuous trajectories. Thus [8], it is the Feller process, therefor this process is the strong Markov.

Theorem 1. *Let the drift b_t be bounded and Borel meassurable, and let $\exists r_-,\ r_+ > 0$ such, that*

$$b_-(x) \ge -r_-,\ b_+(x) \ge r_+, \forall x \in \mathbb{R}$$

and

$$\frac{r_+}{\lambda_+} > \frac{r_-}{\lambda_-}.$$

Then $X_t \stackrel{a.s.}{\to} \infty$, $t \to \infty$. More over, $\exists c_0' > 0$,

$$\liminf_{t\to\infty} \frac{X_t}{t} > c_0', \quad a.s.$$

Lemma 1. *Under the assumptions of the theorem*

$$\frac{T_{2n}}{n} \stackrel{a.s.}{\to} \frac{1}{\lambda_+} + \frac{1}{\lambda_-} := c_2$$

In particular, for $0 < c_0 < \frac{1}{\lambda_+} + \frac{1}{\lambda_-}$, $\exists c_1 > 0 :\ \mathbb{E}e^{\lambda(c_0 n - T_{2n})} < (1 - c_1\lambda)^n < 1$, $\forall \lambda \ll 1$.

The proof of Lemma 1. Note, that

$$T_{2n} = \underbrace{-0 + T_0}_{\Delta T_0} + \underbrace{-T_0 + T_1}_{\Delta T_1} + \underbrace{-T_1 + T_2}_{\Delta T_2} + \dots + \underbrace{-T_{2n-1} + T_{2n}}_{\Delta T_{2n}} = T_0 + \sum_{i=0}^{2n} \Delta T_i,$$

Then $\{\Delta T_i + \Delta T_{i+1}\}$ - i.i.d., and $\forall n \in \mathbb{N}$, $\mathbb{E}|\Delta T_i|^n < \infty$, $\mathbb{E}\Delta T_i + \Delta T_{i+1} = \frac{1}{\lambda+} + \frac{1}{\lambda-} > 0$.
By Law of large numbers

$$\frac{1}{n} T_{2n} \stackrel{\text{a.s.}}{\to} \frac{1}{\lambda+} + \frac{1}{\lambda-} =: c_2,\ n \to \infty,$$

Consider $c_+ = \frac{1}{\lambda_+}$ and $c_- = \frac{1}{\lambda_-}$.

$$\begin{aligned}
\mathbb{P}\left(\frac{T_{2n}}{n} - c_0 < -\epsilon_1\right) &= \mathbb{P}\left(-T_{2n} + c_0 n > n\epsilon_1\right) \\
&\overset{\forall\lambda\geq 0}{=} \mathbb{P}\left(e^{-\lambda T_{2n} + c_0\lambda n} > e^{\lambda\epsilon_1 n}\right) \overset{\text{Markov's inequality}}{\leq} e^{-\lambda\epsilon_1 n}\mathbb{E}e^{-\lambda T_{2n} + c_0\lambda n} \\
&\overset{c_0 = c_+ + c_-}{=} e^{-\lambda\epsilon_1 n}\mathbb{E}e^{-\lambda T_0} e^{-\lambda\left(\sum_{i=1}^n (c_+ - \Delta T_{2i-1}) + \sum_{i=1}^n (c_- - \Delta T_{2i})\right)} \\
&= e^{-\lambda\epsilon_1 n}\mathbb{E}e^{-\lambda T_0}\prod_{i=1}^n \mathbb{E}e^{\lambda(c_+ - \Delta T_1)}\prod_{i=1}^n \mathbb{E}e^{\lambda(c_- - \Delta T_2)} \\
&= e^{-\lambda\epsilon_1 n}\mathbb{E}e^{-\lambda T_0}\left(\mathbb{E}e^{\lambda(c_+ - \Delta T_1)}\mathbb{E}e^{\lambda(c_- - \Delta T_2)}\right)^n.
\end{aligned}$$

We have,

$$\begin{aligned}
\mathbb{E}\exp(\lambda(c_+ - \Delta T_1)) &= \int_0^\infty e^{\lambda(c_+ - x)}\lambda_+ e^{-\lambda_+ x}dx = \frac{\lambda_+ e^{\lambda c_+}}{\lambda + \lambda_+}\int_0^\infty (\lambda + \lambda_+)e^{-(\lambda_+ + \lambda)x}dx \\
&= \frac{\lambda_+ e^{\lambda c_+}}{(\lambda + \lambda_+)} \overset{\text{Taylor}}{=} 1 - (\frac{1}{\lambda_+} - c_+)\lambda + \bar{o}(\lambda) \overset{c_+ = \frac{1}{\lambda_+}}{=} 1 + \bar{o}(\lambda),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}e^{\lambda(c_- - \Delta T_2)} &= \cdots = 1 - (\frac{1}{\lambda_-} - c_-)\lambda + \bar{o}(\lambda) \overset{c_- = \frac{1}{\lambda_-}}{=} 1 + \bar{o}(\lambda), \\
\mathbb{E}e^{-\lambda T_0} &= \cdots = 1 - \frac{1}{\lambda_-}\lambda + \bar{o}(\lambda).
\end{aligned}$$

$$\mathbb{E}e^{\lambda(c_0 n - T_{2n})} \leq \left(1 - \frac{1}{\lambda_-}\lambda + \bar{o}(\lambda)\right)\left(1 - (\frac{1}{\lambda_+} + \frac{1}{\lambda_-} - c_0)\lambda + \bar{o}(\lambda)\right)^n < (1 - c_1\lambda)^n,$$

As a result for $\forall\epsilon_1 > 0$, $\lambda \to 0$

$$\begin{aligned}
\mathbb{E}e^{\lambda(c_0 n - T_{2n})} &\leq \left(1 - \frac{1}{\lambda_-}\lambda + \bar{o}(\lambda)\right)\left(1 - (\frac{1}{\lambda_+} + \frac{1}{\lambda_-} - c_0)\lambda + \bar{o}(\lambda)\right)^n < (1 - c_1\lambda)^n, \\
\mathbb{P}\left(\frac{T_{2n}}{n} - c_0 < -\epsilon_1\right) &\overset{c_0 = \frac{1}{\lambda_+} + \frac{1}{\lambda_-}}{\leq} \left(1 - \frac{1}{\lambda_-}\lambda + \bar{o}(\lambda)\right)\left(1 - \epsilon_1\lambda + \bar{o}(\lambda)\right)^n.
\end{aligned}$$

Let $c_0 = \frac{1}{\lambda_+} + \frac{1}{\lambda_-}$, $\forall\epsilon_1 > 0\,\exists\lambda_0 := \lambda_0(\epsilon_1, c_0) : \forall\lambda \ll \lambda_0$

$$\mathbb{P}\left(\frac{T_{2n}}{n} - c_0 < -\epsilon_1\right) \leq \left(1 - \frac{1}{\lambda_-}\lambda\right)(1 - \epsilon_1\lambda)^n < \kappa_1^n < 1,$$

Similarly, $\forall\epsilon_2 > 0\,\exists\lambda_0 > 0 : \forall\lambda \ll \lambda_0$

$$\mathbb{P}\left(\frac{T_{2n}}{n} - c_0 > \epsilon_2\right) < \kappa_2^n < 1,$$

□

Lemma 2. *Under the assumptions of the theorem $\exists c_0 > 0$,*

$$\liminf_{n\to\infty} \frac{X_{T_{2n}}}{n} \overset{a.s.}{\geq} c_0.$$

The proof of Lemma 2. consider the case, when $Z_0 = 0$. The case $Z_0 = 1$ is treated similarly.
Using Ito's theorem, we take $a > 0$, whose value we will choose later.

$$\begin{aligned} d\exp(-\lambda X_t + at) &= +a\exp(-\lambda X_t + at)dt - \lambda\exp(-\lambda X_t + at)dX_t + \frac{\lambda^2}{2}\exp(-\lambda X_t + at)(dX_t)^2 \\ &= +a\exp(-\lambda X_t + at)dt - \lambda\exp(-\lambda X_t + at)(bdt + dW_t) + \frac{\lambda^2}{2}\exp(-\lambda X_t + at)dt. \end{aligned}$$

The integral form: $\mathbb{E} := \mathbb{E}_x$,

$$\begin{aligned} \mathbb{E}\exp(&- \lambda(X_{T_2} - x) + aT_2) - 1 = \mathbb{E}\int_0^{T_2} \exp(-\lambda(X_t - x) + at)(a - \lambda b_t + \frac{\lambda^2}{2})dt \\ &\leq \underbrace{(a - r_+\lambda)\mathbb{E}\int_0^{T_1} \exp(-\lambda(X_t - x) + at)dt}_{=:I_1} + \underbrace{(a + r_-\lambda)\mathbb{E}\int_{T_1}^{T_2} \exp(-\lambda(X_t - X_{T_1}) + at)dt}_{=:I_2}. \end{aligned}$$

Let $t \in [0, T_1]$. Due to the uniqueness of the solution X_t is equal to

$$\bar{X}_t - x = \int_0^t b_+(\bar{X}_s)\, ds + \int_0^t dW_s.$$

Using the comparison theorem [4], with $t \in [0, T_1]$ we get:

$$\bar{X}_t = x + \int_0^t b_+(\bar{X}_s)\, ds + \int_0^t dW_s \overset{a.s.}{\geq} x + \int_0^t r_+\, ds + \int_0^t dW_s =: \tilde{X}_t.$$

We want to get grades on $\mathbb{E}\int_0^{T_1} \exp(-\lambda(X_t - x) + at)dt$, $\mathbb{E}\int_{T_1}^{T_2} \exp(-\lambda(X_{T_2} - X_{T_1}) + at)dt$.
Using the comparison theorem, Fubini's theorem, the stochastic exponent and Z independent $\mathcal{F}^W$, we can

get,

$$
\begin{aligned}
\mathbb{E}\int_0^{T_1} \exp(-\lambda(X_t - x) + at)dt &\le \mathbb{E}\int_0^{T_1} \exp(-\lambda(\tilde{X}_t - x) + at)dt \\
&= \mathbb{E}\int_0^{T_1} \exp((a-\lambda r_+)t - \int_0^t \lambda dW_s)dt \\
&= \mathbb{E}\int_0^{T_1} \exp((a-\lambda r_+)t)\exp(-\int_0^t \lambda dW_s)dt \\
&= \mathbb{E}\int_0^{\infty} 1\{t<T_1\}\exp((a+\frac{\lambda^2}{2}-\lambda r_+)t)\underbrace{\exp(-\int_0^t \lambda dW_s - \frac{1}{2}\int_0^t \lambda^2 ds)}_{stoch.exp.}\,dt \\
&= \int_0^{\infty} \mathbb{E}1\{t<T_1\}\exp((a+\frac{\lambda^2}{2}-\lambda r_+)t)\underbrace{\exp(-\int_0^t \lambda dW_s - \frac{1}{2}\int_0^t \lambda^2 ds)}_{stoch.exp.}\,dt \\
&= \int_0^{\infty} \exp((a+\frac{\lambda^2}{2}-\lambda r_+)t)\mathbb{E}I\{t<T_1\} * 1\,dt \\
&= \int_0^{\infty} \exp((a+\frac{\lambda^2}{2}-\lambda r_+)t)\exp(-\lambda_+ t)\,dt \\
&= \frac{1}{\lambda_+ - a + \lambda r_+ - \lambda^2/2}.
\end{aligned}
$$

Similarly, considering $t \in [T_1, T_2]$. Due to the uniqueness of the solution X_t is equal to

$$
\hat{X}_t = X_{T_1} + \int_{T_1}^t b_-(\hat{X}_s)\,ds + \int_{T_1}^t dW_s, \quad t \ge T_1.
$$

Using the comparison theorem [4], with $t \in [T_1, T_2]$ we get:

$$
\hat{X}_t = X_{T_1} + \int_{T_1}^t b_-(\hat{X}_s)\,ds + \int_{T_1}^t dW_s \overset{a.s.}{\ge} X_{T_1} + \int_{T_1}^t -r_-\,ds + \int_{T_1}^t dW_s =: \tilde{X}_t.
$$

We're estimating the $\mathbb{E}_{\mathcal{F}_{T_1}}(\int_{T_1}^{T_2} \exp(-\lambda(X_t - x) + at)dt := \mathbb{E}\left(\int_{T_1}^{T_2} \exp(-\lambda(X_t - x) + at)dt | X_{T_1} = x,\, T_1 = \tau\right)$:

$$
\begin{aligned}
\mathbb{E}_{\mathcal{F}_{T_1}}\int_{T_1}^{T_2} \exp(-\lambda(X_t - x) + at)dt &\le \mathbb{E}_{\mathcal{F}_{T_1}}\int_{T_1}^{T_2} \exp(-\lambda(\tilde{X}_t - x) + at)dt \\
&= \exp(aT_1)\mathbb{E}_{\mathcal{F}_{T_1}}\int_{T_1}^{T_2} \exp(-\lambda\tilde{X}_t + a(t-T_1))dt \\
&\overset{\text{time change}}{=} \exp(-\lambda X_{T_1} + aT_1)\mathbb{E}_{\mathcal{F}_{T_1}}\int_0^{T} \exp(at + \lambda r_- t - \lambda\tilde{W}_t)dt \\
&\overset{\text{similarly}}{=} \exp(-\lambda X_{T_1} + aT_1)\frac{1}{\lambda_- - a - \lambda r_- - \lambda^2/2}.
\end{aligned}
$$

Then,

$$
\mathbb{E}\left(\mathbb{E}_{\mathcal{F}_{T_1}}\int_{T_1}^{T_2} \exp(-\lambda(X_t - x) + at)dt\right) \le \frac{1}{\lambda_- - a - \lambda r_- - \lambda^2/2}\left(1 + \frac{a - r_+\lambda}{\lambda_+ - a + \lambda r_+ - \lambda^2/2}\right).
$$

Thus,

$$
I_2 \le \frac{a + r_-\lambda}{\lambda_- - a - \lambda r_- - \lambda^2/2}\left(1 + \frac{a - r_+\lambda}{\lambda_+ - a + \lambda r_+ - \lambda^2/2}\right),
$$

and

$$I_1 \le \frac{a - r_+\lambda}{\lambda_+ - a + \lambda r_+ - \lambda^2/2}.$$

As a result for, for sufficiently smaller $a, \lambda << 1$, for example, $a = \hat{a}\lambda$, $\lim_{n\to\infty} \lambda = 0$, $0 < \hat{a} < \frac{\lambda_- r_+ - \lambda_+ r_-}{\lambda_+ + \lambda_-}$.

$$\mathbb{E}_x e^{-\lambda X_{T_2} + aT_2} \le 1 + I_1 + I_2$$

$$\le 1 + \underbrace{\frac{a - r_+\lambda}{\lambda_+ - a + \lambda r_+ - \lambda^2/2}}_{(1)} + \underbrace{\frac{a + r_-\lambda}{\lambda_- - a - \lambda r_- - \lambda^2/2}}_{(2)} \left(1 + \underbrace{\frac{a - r_+\lambda}{\lambda_+ - a + \lambda r_+ - \lambda^2/2}}_{(1)} \right)$$

$$\overset{\text{Taylor's formula}}{=} 1 + \underbrace{\left[(\hat{a} - r_+)\lambda_+^{-1} + (\hat{a} + r_-)\lambda_-^{-1}) \right]}_{=-c_2} \lambda + \overline{o}(\lambda).$$

$$\begin{aligned}
(1) \quad &= \quad \frac{(a - r_+)\lambda_+^{-1}\lambda}{1 - \left[\frac{a - r_+}{\lambda_+} + \frac{1}{2\lambda_+}\lambda \right] \lambda} = (a - r_+)\lambda_+^{-1}\lambda \left(1 + \left[\frac{a - r_+}{\lambda_+} + \frac{1}{2\lambda_+}\lambda \right] \lambda + \overline{o}(\lambda) \right) \\
&= \quad (a - r_+)\lambda_+^{-1}\lambda + \overline{o}(\lambda). \\
(2) \quad &= \quad \frac{(a + r_-)\lambda_-^{-1}\lambda}{1 - \left[\frac{a + r_-}{\lambda_-} + \frac{1}{2\lambda_-}\lambda \right] \lambda} = (a + r_-)\lambda_-^{-1}\lambda + \overline{o}(\lambda).
\end{aligned}$$

Note that $c_2 > 0$ in the assumptions of the theorem, and $a = \hat{a}\lambda > 0$, $0 < \hat{a} < \frac{\lambda_- r_+ - \lambda_+ r_-}{\lambda_+ + \lambda_-}$.
As a result we have,

$$\mathbb{E}_x e^{-\lambda(X_{T_2} - x) + aT_2} \le 1 - c_2\lambda + o(\lambda).$$

We similarly estimate X_{T_4},

$$\mathbb{E}_x e^{-\lambda(X_{T_4} - x) + aT_4} = \mathbb{E} e^{-\lambda(X_{T_2} - x) + aT_2} \times \underbrace{E\left(e^{-\lambda(X_{T_4} - X_{T_2}) + a(T_4 - T_2)} | \mathcal{F}_{T_2} \right)}_{\le 1 - c_2\lambda} \le (1 - c_2\lambda)^2.$$

By induction,

$$E_x e^{-\lambda(X_{T_{2n}} - x) + aT_{2n}} \le (1 - c_2\lambda)^n.$$

We obtain exponential estimates on $X_{T_{2n}}$, $n > 1$

$$\mathbb{P}_x \left(\frac{X_{T_{2n}} - x}{n} - c_0 < -\epsilon \right) = \mathbb{P}_x \left(-(X_{T_{2n}} - x) + c_0 n > n\epsilon \right)$$

$$\overset{\forall \lambda \ge 0}{=} \mathbb{P}_x \left(e^{-\lambda(X_{T_{2n}} - x) + c_0\lambda n)} > e^{\lambda\epsilon n} \right) \overset{\text{Markov's inequality}}{\le} e^{-\lambda\epsilon n} \mathbb{E} \left(e^{-\lambda(X_{T_{2n}} - x) + c_0\lambda n} \right).$$

Using the results of the lemma 1 and the Cauchy–Schwarz inequality, we have for sufficiently large n: $0 < c_0 < \min(\frac{1}{\lambda_+} + \frac{1}{\lambda_-}, \frac{\lambda_- r_+ - \lambda_+ r_-}{\lambda_+ + \lambda_-})$, $\forall \epsilon > 0 \quad \exists \lambda := \lambda(\epsilon, c_0)$,

$$\begin{aligned}
\mathbb{P}_x\left(\frac{X_{T_{2n}} - x}{n} - c_0 < -\epsilon\right) &\overset{\forall \lambda > 0}{\leq} e^{-\lambda \epsilon n}\mathbb{E}\left(e^{-\lambda(X_{T_{2n}} - x) + c_0 \lambda n}\right) \\
&\overset{\text{Cauchy–Schwarz inequality}}{\leq} e^{-\lambda \epsilon n}\left(\mathbb{E}e^{-2\lambda(X_{T_{2n}} - x) + 2c_0\lambda T_{2n}}\mathbb{E}e^{2\lambda(c_0 n - T_{2n})}\right)^{1/2} \\
&\leq (1 - \epsilon_1\lambda + \overline{o}(\lambda))^n\left((1 - c_2\lambda + \overline{o}(\lambda))^n(1 - c_1\lambda + \overline{o}(\lambda))^n\right)^{\frac{1}{2}} \\
&\overset{c := min(c_1, c_2)}{\leq} (1 - c\lambda)^n < 1.
\end{aligned}$$

For sufficiently smaller $0 < \lambda \ll 1$, $\lambda := \lambda(\epsilon)$,

$$\sum_{n=1}^{\infty} \mathbb{P}_x\left(-(X_{T_{2n}} - x) + nc_0 > n\epsilon\right) < \infty.$$

Then from the resulting inequality using the Borel-Cantelli lemma follows, that

$$\mathbb{P}_x(-(X_{T_{2n}} - x) + nc_0 \leq \epsilon n \,\text{endlessly}) = 0.$$

Thus, $\forall \epsilon > 0$ for a.s. ω, $\exists N_0 = N_0(\omega)$:

$$X_{T_{2n}} - x \geq nc_0 - \epsilon n \quad \forall n \geq N_0,$$

or

$$\liminf_{n\to\infty} \frac{X_{T_{2n}}}{n} \overset{\text{a.s.}}{\geq} c_0.$$

Similarly, the case when $Z_0 = 1$ is onsidered..
Result: $\liminf_{n\to\infty} \frac{X_{T_{2n}}}{n} \overset{\text{a.s.}}{\geq} c_0$. □

The proof of the theorem 1. Consider the moments of switching:

$$\begin{aligned}
\overline{n}(t) &:= \inf\{s \geq t : \exists n : s = T_{2n}\}, \\
n(t) &:= \inf\{s \geq t : \exists n : s = T_{2n+1}\}, \\
\underline{n}(t) &:= \sup\{s \leq t : \exists n : s = T_{2n}\}.
\end{aligned}$$

Consider

$$X_t = X_{\underline{n}(t)} + (X_t - X_{\underline{n}(t)}).$$

By the lemma 1 and 2 we get,

$$\liminf_{t\to\infty} \frac{X_{\underline{n}(t)}}{t} = \liminf_{t\to\infty} \frac{X_{\underline{n}(t)}}{\underline{n}(t)}\frac{\underline{n}(t)}{t} \overset{\text{a.s.}}{\geq} c_1 > 0$$

We want to obtain the $X_t - X_{\underline{n}(t)}$.

$$
\begin{aligned}
\frac{X_t - X_{\underline{n}(t)}}{t} &= \frac{\int\limits_{\underline{n}(t)}^{t} b(X_s, Z_s)\, ds + \int\limits_{\underline{n}(t)}^{t} dW_s}{t} \\
&\overset{\text{a.s.}}{=} \frac{\int\limits_{\underline{n}(t)}^{n(t)\wedge t} b(X_s, Z_s)\, ds + \int\limits_{n(t)\wedge t}^{t} b(X_s, Z_s)\, ds + \int\limits_{\underline{n}(t)}^{t} dW_s}{t} \\
&\overset{\text{a.s.}}{\geq} \frac{(n(t)\wedge t - \underline{n}(t))r_+ - (t - n(t)\wedge t)r_- + W_t - W_{\underline{n}(t)}}{t} \\
&= \frac{(r_+ + r_-)n(t)\wedge t - \underline{n}(t)r_+ - tr_- + W_t - W_{\underline{n}(t)}}{t}.
\end{aligned}
$$

Using the Law of the iterated logarithm, note that

$$
\begin{aligned}
\liminf_{t\to\infty}(r_+ + r_-)\frac{n(t)\wedge t}{t} &\overset{\text{a.s.}}{\geq} (r_+ + r_-)\liminf_{t\to\infty}\frac{\underline{n}(t)}{t} \\
&\overset{\text{a.s.}}{\geq} (r_+ + r_-)\liminf_{t\to\infty}\frac{\underline{n}(t)}{\overline{n}(t)} \overset{\text{a.s.}}{\geq} (r_+ + r_-) \\
\liminf_{t\to\infty}\frac{\underline{n}(t)r_+}{t} &\overset{\text{a.s.}}{\leq} \liminf_{t\to\infty}\frac{tr_+}{t} \overset{\text{a.s.}}{\leq} r_+, \\
\liminf_{t\to\infty}\frac{tr_-}{t} &= r_-, \\
\liminf_{t\to\infty}\frac{W_t - W_{\underline{n}(t)}}{t} &\overset{\text{a.s.}}{=} 0.
\end{aligned}
$$

In result,

$$
\liminf_{t\to\infty}\frac{X_t - X_{\underline{n}(t)}}{t} \overset{\text{a.s.}}{\geq} 0
$$

□

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Kirill Mosievich Viktorovich
Moscow State University,
1 Leninskie gory Str.,
Moscow, Russia
Email address: kirillmosievich@gmail.com