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MAXIMUM PRINCIPLE FOR SOLUTIONS OF THE MODIFIED NEWTONIAN GRAVITATIONAL POTENTIAL EQUATION IN AN UNBOUNDED DOMAIN

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Abstract. The generalized nonlinear Poisson equation is considered, which determines the so-called modified Newtonian dynamics (MOND) in cosmology and physics of galaxies. The theorem on the asymptotic behavior of solutions of this equation for the potential at large distances is proved. Our main result is an estimate of the constraints on possible nonaxisymmetric deviations of MOND solutions from the axisymmetric potential distribution for the Newtonian case.

Keywords: maximum principle, elliptic type equation, modified gravity.

1. Introduction

Let us discuss the motivation for our research related to the current problems of modern cosmology and physics of galaxies. If a point particle of mass m is located at a point with radius vector $\mathbf{r}_0$, then it creates gravitational field for which the potential at point $\mathbf{r}$ is

$$\Phi = -Gm/|\mathbf{r} - \mathbf{r_0}|\,, \tag{1}$$

where G is the gravitational constant and $\Phi = -Gm/r$ for $\mathbf{r}_0 = 0$ (the particle is in the center of the spherical coordinate system and r is the distance from the particle to the observation point). If the mass is distributed continuously in space and is

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characterized by the volume mass density $\varrho(\mathbf{r})$, then the gravitational potential is determined by the Poisson equation in the classical nonrelativistic approximation:

$$\Delta\,\Phi = 4\pi G\varrho\,, \tag{2}$$

where $\Delta = \nabla^2$ is the Laplace operator. The potential (1) is a solution to the equation (2) for the right-hand side with $\varrho = m\,\delta^3(\mathbf{r}-\mathbf{r}_0) = m\,\delta(r)/(4\pi r^2)$ (δ is the delta function). The gravitational acceleration $\mathbf{g} = -\nabla\Phi$ determines the radial profile of the circular velocity curve for a test (so-called cold) particle in an axisymmetric potential [1]:

$$V_c(r) = \left(r_z\,\frac{\partial\Phi}{\partial r_z}\right)^{1/2}\,, \tag{3}$$

r_z is the radial coordinate in a cylindrical coordinate system.

Astronomical observational data give us the distribution of baryonic (visible = stellar + gaseous) matter in galaxies or galaxy clusters in different ranges of radiation. For example, the stellar component is the most massive in the disk of S-galaxies and follows an exponential brightness profile along the radial coordinate $I \propto \exp(-r_z/r_d)$ ($r_d = \text{const}$ is the radial scale). If we assume that the brightness is proportional to the mass density of baryonic matter along the line of sight, then a similar exponential law is valid for the density $\varrho = \varrho_0\exp(-r_z/r_d)$ as well. Such a density profile gives a falling rotation curve $V_c(r_z)$ at large distances from the center $r_z \gg 2r_d$, which contradicts the observational data on the rotation of matter (stars and gas) at the periphery of galaxies.

Figure 1 shows the typical behavior of the theoretical curve (3) and the observed rotation curves of the cold gas for typical galaxies in which the falling velocity of rotation is practically not observed. Circular velocity models are constructed assuming a constant radial mass/luminosity ratio. The difference between V_{obs} and V_c ($V_{obs} - V_c > 0$) requires an explanation, which can be based either on dark matter or modified gravity (4). Examples of NGC 2903 and our Milky Way are typical, which show that the classical Newtonian gravitational potential based on (2) without dark mass is not able to explain the rotation curves at the periphery of galaxies [7]. Results of this kind and a number of more complex arguments put forward the hypothesis of the existence of a hidden dark mass (See the review [1] and the literature there), which is now dominant in the scientific community.

The main problem with the dark mass approach is that almost 40 years of active searches for physical particles of this dark mass have not been successful. There are certain problems in formally explaining observational data using the concept of dark mass [8]. For example, we are forced to select different radial profiles of the dark halo for each galaxy separately [9].

In the early 1980s, M. Milgrom proposed an alternative hypothesis to the dark mass [10]. The basic idea behind this approach is to reject the Newtonian Potential (1) at large distances in order to be consistent with observational data. This requires a transition to a mathematical model of the potential, which is different from the classical case of (2) (See works [10, 11]) and such models are called Modified Newtonian Dynamics (MOND).

The generalization of (2) is the following equation [12]:

$$\nabla\,\left[\mu(\xi)\,\nabla(\tilde{\Phi}(\mathbf{r}))\right] = 4\pi G\varrho\,, \tag{4}$$

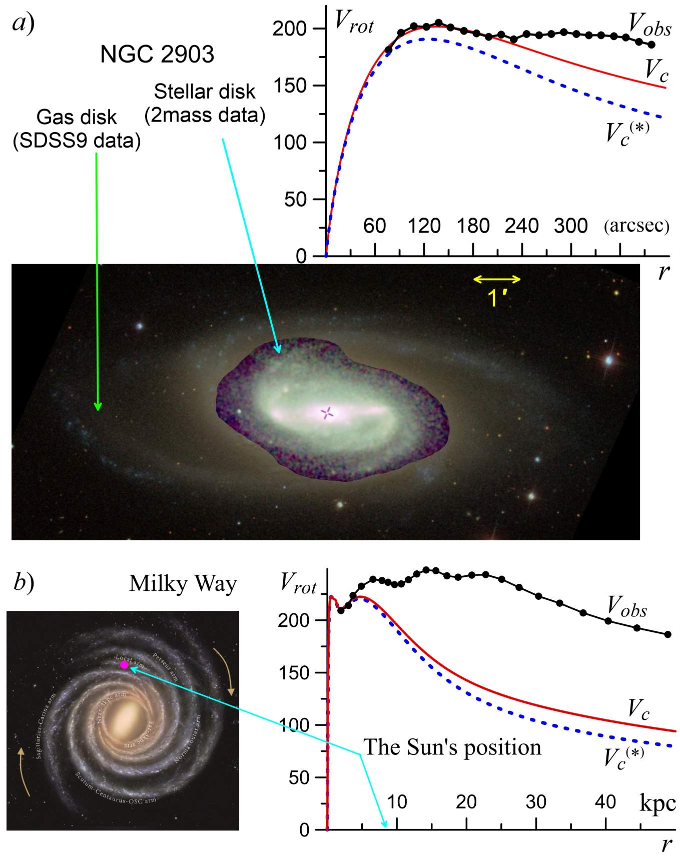

Рис. 1. Rotation curves in galactic disks: the solid black line and circles are the observed rotation curve (V_{obs}), the solid red line is the circular velocity V_c, calculated from the observed distribution of baryonic matter in the disk (stars + gas) without a dark halo, the dashed blue line is the circular velocity of the stellar component $V_c^{(*)}$ (stellar disk + stellar bulge), the distance from the center r is measured in arsec, $[V_{rot}]$ =km·sec^{-1}. *a*) Galaxy NGC 2903 [2], a synthetic image of a galaxy consists of data 2mass for the stellar component and Sloan Digital Sky Survey (SDSS9) by database HyperLEDA [3], $r = 400$ arcsec corresponds to about 17.3 kpc. *b*) Our Milky Way Galaxy, rotation curve from work [4], model by [5], the conceptual picture of the our Galaxy [6].

where $\tilde{\Phi}$ is the MOND-potential, the function $\mu = \xi/\sqrt{1+\xi^2}$ provides a limiting transition from MOND to Newtonian dynamics, $\xi = |\tilde{\mathbf{g}}|/a_0$, $\tilde{\mathbf{g}} = -\nabla\tilde{\Phi}$ is the acceleration in MOND. The parameter $a_0 \sim 10^{-10}$ m·sec^{-2} defines the critical acceleration that separates Newtonian dynamics and MOND, since for $\xi \to \infty$ we have a transition to classical newtonian dynamics and the case $\mu = 1$, $\tilde{\Phi} = \Phi$, $\tilde{\mathbf{g}} = \mathbf{g}$ corresponds to the equation (2). Estimates for a_0 are known very imprecisely and only in order of magnitude. Refinement is possible from a comparison of simulation data and astronomical observations.

In this paper, we will focus on some mathematical results of the analysis of solutions to the generalized Poisson equation (4), leaving aside the physical meaning of MOND.

2. Radially symmetric solutions

We write the equation (4) as follows

$$Q[u] \equiv \operatorname{div}\left(\frac{|\nabla u|\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x), \tag{5}$$

where u is the dimensionless potential, $g(x)$ is a continuous function defined in $\mathbf{R}^n$ and equal to zero outside some bounded domain D. Put $R_0 = \sup\limits_{x\in D}|x|$ and choose arbitrary $R_2 > R_1 \geq R_0$.

Note that the equation (5) can be written in expanded form

$$\sum_{i,j=1}^{n} (f_{x_i} f_{x_j} + |\nabla f|^2(1+|\nabla f|^2)\delta_{ij}) f_{x_i x_j} = g(x)|\nabla f|(1+|\nabla f|^2)^{3/2}, \tag{6}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. It is not difficult to see that the equation is elliptic wherever $\nabla f \neq 0$.

Lemma 1. *Let $f(x) \in C^2(\Omega)\cap C(\overline{\Omega})$ be a solution to the equation (5), $\Omega \subset \mathbf{R}^n \setminus D$ is a limited area. Then*

$$\min_{x\in\partial\Omega} f(x) \leq f(x) \leq \max_{x\in\partial\Omega} f(x)$$

for all $x \in \Omega$.

Proof. Suppose the statement is not true. Let, for example, at some point $x_0 \in \Omega$ such that $f(x_0) > \max\limits_{\partial\Omega} f(x)$. Let c be chosen so that $\max\limits_{\partial\Omega} f(x) < c < f(x_0)$. Let us denote by Ω_0 such a connected component of the set $\{x \in \Omega : f(x) > c\}$, which contains the point x_0. It is clear that $\overline{\Omega_0} \subset \Omega$. Note now that for any locally Lipschitz function $\varphi(x)$ in Ω such that $\varphi = 0$ on the boundary $\partial\Omega$,

$$\int\limits_{\Omega} \frac{|\nabla f|\langle\nabla f, \nabla\varphi\rangle}{\sqrt{1+|\nabla f|^2}}dx = 0.$$

Putting here

$$\varphi(x) = \begin{cases} f(x) - c, & x \in \Omega_0 \\ 0, & x \in \Omega\setminus\Omega_0, \end{cases}$$

we get equality

$$\int\limits_{\Omega_0} \frac{|\nabla f|^3 dx}{\sqrt{1+|\nabla f|^2}} = 0.$$

Thus, $f \equiv c$ is in Ω_0, which contradicts the choice of c. □

It is not difficult to make sure that the function

$$u(|x|) = \pm\int\limits_{R_0}^{|x|} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}}\right)^{1/2} ds + C_2, \tag{7}$$

satisfy the equation (5) for any constants $C_1 \geq 0, C_2$ and $|x| \geq R_0$.

Lemma 2. *For arbitrary numbers M_1, M_2, there are C_1, C_2 such that a solution $u(|x|)$ of the form (7) satisfies the conditions*

$$u(R_1) = M_1, \quad u(R_2) = M_2.$$

Proof. Suppose that the inequality $M_2 \geq M_1$ is true. The unknown constants C_1 and C_2 satisfy the system of equations

$$\int\limits_{R_0}^{R_1} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds + C_2 = M_1,$$

$$\int\limits_{R_0}^{R_2} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds + C_2 = M_2.$$

From this we arrive at one equation

$$\int\limits_{R_1}^{R_2} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds = M_2 - M_1.$$

Since the inequality $M_2 \geq M_1$ is true, it is obvious that there exists $C_1 \geq 0$ for which this equality holds. In the case when $M_2 < M_1$, in the solution (7) it is necessary to take the minus sign. □

A ball of radius $R > 0$ centered at the origin will be denoted by B_R and let $S_R = \partial B_R$.

Let $f(x)$ be an arbitrary solution to the equation (5) given in $\mathbf{R}^n$. We put

$$M(t) = \max_{|x|=t} f(x), \quad m(t) = \min_{|x|=t} f(x).$$

Lemma 3. *For any $R_2 > R_1 \geq R_0$, the inequality*

$$f(x) \leq M(R_1) + \int\limits_{R_1}^{|x|} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds \tag{8}$$

for all x, $R_1 \leq |x| \leq R_2$, where the constant C_1 is defined by the equality

$$\int\limits_{R_1}^{R_2} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds = |M(R_2) - M(R_1)|.$$

Proof. The inequality (8) obviously holds if $M(R_2) \leq M(R_1)$ (by Lemma 1). Suppose $M(R_2) > M(R_1)$. Consider the solution $u(|x|)$, whose existence follows from the lemma 2. In this way,

$$u(|x|) = M(R_1) + \int\limits_{R_1}^{|x|} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds,$$

where the constant $C_1 > 0$ is defined by the equality

$$\int_{R_1}^{R_2} \left(\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \right)^{1/2} ds = M(R_2) - M(R_1).$$

It is not hard to see that the inequalities hold

$$f|_{S_{R_1}} \le M(R_1) = u|_{S_{R_1}}, \quad f|_{S_{R_2}} \le M(R_2) = u|_{S_{R_2}}.$$

Next, using the comparison theorem for elliptic equations [13], Theorem 10.1, we arrive at the inequality (8). Note that this theorem is applicable, since the operator Q is elliptic on the function u. □

Consider the behavior of the constant C_1 for $R_2 \to \infty$. It's clear that

$$\frac{C_1^2 + C_1\sqrt{C_1^2 + 4s^{2n-2}}}{2s^{2n-2}} \ge \frac{C_1}{s^{n-1}}.$$

Therefore

for $n = 3$,

$$C_1 \le \left(\frac{M(R_2) - M(R_1)}{\ln R_2/R_1} \right)^2, \tag{9}$$

for $n = 2$,

$$C_1 \le \left(\frac{M(R_2) - M(R_1)}{2(\sqrt{R_2} - \sqrt{R_1})} \right)^2. \tag{10}$$

3. Main results

Theorem 1. *Let $f(x)$ be a bounded solution of the equation (5) in $\mathbf{R}^n, n \le 3$. Then the inequalities hold*

$$\min_{x \in S_{R_0}} f(x) \le f(x) \le \max_{x \in S_{R_0}} f(x)$$

for all $x \in \mathbf{R}^n \setminus B_{R_0}$.

Proof. It is enough for us to prove the right inequality (the left inequality is obtained by replacing f with $-f$). Using the estimates (9), (10) and passing to the limit as $R_2 \to \infty$ in the inequality (8), we obtain

$$f(x) \le M(R_1) \ \ \forall x \in \mathbf{R}^n \setminus B_{R_1} \ \forall R_1 \ge R_0.$$

This implies the required inequality. □

Let us investigate the question of the existence of the limit of the solution f for $|x| \to \infty$. Let $n = 3$. Consider a function $\varphi(x)$ satisfying the conditions

$$\varphi(x) = 1 \text{ for } |x| \le R_1, \quad \varphi(x) = 0 \text{ for } |x| \ge R_2.$$

Then

$$\int_{B_{R_2} \setminus D} \frac{|\nabla f| \langle \nabla f, \nabla \varphi \rangle}{\sqrt{1 + |\nabla f|^2}} = \int_{\partial D} \frac{\varphi |\nabla f| \langle \nabla f, \nu \rangle}{\sqrt{1 + |\nabla f|^2}} \equiv J.$$

Replacing φ by $f\varphi^3$ and applying Young's inequality, for an arbitrary $\varepsilon > 0$ we obtain the inequality

$$\left(1-\frac{2\varepsilon^{3/2}}{3}\right)\int\limits_{B_{R_1}\setminus D}\frac{|\nabla f|^3}{\sqrt{1+|\nabla f|^2}} \le \frac{\max\limits_{B_{R_2}}|f|}{\varepsilon^3}\int\limits_{B_{R_2}\setminus B_{R_1}}|\nabla\varphi|^3 + J.$$

We believe

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| < R_1 \\ \dfrac{\ln|x|/R_2}{\ln R_1/R_2} & \text{for } R_1 \le |x| < R_2 \\ 0 & \text{for } |x| \ge R_2 \end{cases}$$

and then

$$\left(1-\frac{2\varepsilon^{3/2}}{3}\right)\int\limits_{B_{R_1}\setminus D}\frac{|\nabla f|^3}{\sqrt{1+|\nabla f|^2}} \le \frac{\max\limits_{B_{R_2}}|f|}{\varepsilon^3}\frac{4\pi}{\ln^2 R_2/R_1} + J\,.$$

Suppose the solution f is bounded in $\mathbf{R}^3$. Passing to the limit as $R_2 \to \infty$, we obtain

$$\int\limits_{B_{R_1}\setminus D}\frac{|\nabla f|^3}{\sqrt{1+|\nabla f|^2}} \le \frac{J}{1-\dfrac{2\varepsilon^{3/2}}{3}} \quad \forall\ R_1 \ge R_0.$$

Thus,

$$\int\limits_{\mathbf{R}^3\setminus D}\frac{|\nabla f|^3}{\sqrt{1+|\nabla f|^2}} < \infty.$$

If we assume that the gradient ∇f is bounded, then we get

$$\int\limits_{\mathbf{R}^3\setminus D}|\nabla f|^3 < \infty. \tag{11}$$

Theorem 2. *Let f be a bounded solution to the equation (5) such that*

$$\sup_{\mathbf{R}^3}|\nabla f| < +\infty.$$

Then there is

$$\lim_{|x|\to\infty} f(x).$$

Proof. It follows from the conditions of the theorem that the inequality (11). Next, we need the results of [14] or [15] (Theorem 3.2.2). In this works, in particular, the inequality

$$\pi\int\limits_0^R\frac{\operatorname{osc}\{f:S_r\}}{r}dr \le c_1\int\limits_{B_R}|\nabla f|^3dx,$$

where c_1 is an absolute constant. It follows that for any $R_0 \le R_1 < R_2$ there is $\overline{r} \in (R_1, R_2)$ such that

$$\operatorname{osc}\{f:S_{\overline{r}}\} \le \frac{c_1}{\pi\ln R_2/R_1}\int\limits_{B_{R_2}}|\nabla f|^3dx\,.$$

Since the integral on the right-hand side of this inequality remains bounded for $R_2 \to \infty$, this inequality implies that there is a sequence $R_k \to \infty$ along which

$$\mathrm{osc}\{f : S_{R_k}\} \to 0.$$

The maximum principle lemma 1 implies that the sequences

$$M(R_k) = \max_{x \in S_{R_k}} f(x), \quad m(R_k) \max_{x \in S_{R_k}} f(x)$$

are monotonic. Hence they have limits. By virtue of what was proved above, these limits coincide. From the inequalities

$$f(x) \geq \min\{m_{R_k}, m_{R_{k+1}}\}, \ R_k \leq |x| \leq R_{k+1},$$

$$f(x) \leq \max\{M_{R_k}, M_{R_{k+1}}\}, \ R_k \leq |x| \leq R_{k+1},$$

it follows that there is a limit of the solution $f(x)$ for $|x| \to \infty$. $\square$

We note that the finiteness of the energy type integral (11) is often a condition under which various results on the asymptotic behavior of solutions of elliptic equations are established [16, 17].

4. Conclusion

The nonlinear Poisson equation is considered, generalizing the classical Poisson equation for the potential, which is determined by a given distribution of sources, for example, the mass for the gravitational field or the electric charge for the electric field. Our efforts are aimed at studying some of the mathematical properties of such a nonlinear Poisson equation, on which the so-called hypothesis MOND (Modified Newtonian dynamics) is based as an alternative theory of gravity, capable of explaining some observational data without invoking the concept of dark mass (DM), since the nature of DM is unknown despite significant efforts to detect it.

We have proved a theorem on the asymptotic behavior of solutions at large distances from the center of the potential. This result gives restrictions on the nonaxisymmetric behavior of the potential. The degree of nonaxisymmetry at large radii decreases as $\sim E/\ln(r)$, where the constant E is determined by the integral of the expression $|\nabla u|^3$ over an infinite volume.

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