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ASYMPTOTIC EXPANSIONS FOR THE *CUSUM* PROCEDURE IN A CHANGE POINT PROBLEM

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Abstract

Let $\{\xi_n\}$ be a sequence of i.i.d random variables, $N = \min\{n \geq 1 : \xi_1 + \ldots + \xi_n \notin [0, b)\}$. Asymptotic expansions at $b \to \infty$ are obtained for the distribution and expectation of S_N under Cramér type conditions. These results are applied to the study of asymptotic properties of the CUSUM procedure in a change point problem.

Key words and phrases: change point problem, CUSUM procedure, random walk with two barriers, factorization method.

1. INTRODUCTION

Suppose that we observe sequentially the independent random variables $x_1, x_2, \ldots$, in a situation where the distribution function of x_i is known to equal F_1 for $i = 1, \ldots, m-1$, and F_2 if $i \geq m$. The change point m is unknown. (We put $m = 1$ if all the observations have distribution F_2, and $m = \infty$ if no changes in distribution occur, i.e., $\mathbf{P}(x_i \leq y) = F_1(y)$ for all $i \geq 1$).

The problem is to detect the change in distribution as soon as possible. It is typical of many applications (quality control, radiolocation, hydroacoustics).

There is a considerable number of papers concerning the topic (see, e.g., [1] – [10]; see also [11] for more detailed references). In the cited papers some methods of detection were proposed and studied.

Let t be the stopping time when alarm is called according to a specific algorithm of detection. It is natural to try to make the difference $t - m$ minimal in some sense when a change in distribution has occurred and $t \geq m$. On the other hand, the value of t must be large enough when there is no change. This requirements will be formalized below by the introduction of the quantities $\tau_1(t)$ and $\tau_2(t)$.

The properties of the so-called CUSUM (cumulative sum) procedure are studied here. This procedure had been introduced in [1]; it was considered also in a number of other papers. We remind here its main features. Suppose that the distributions F_i are mutually absolutely continuous and let f_i be the density of F_i with respect to a σ-finite measure. Put

$$\eta_k = \ln \frac{f_2(x_k)}{f_1(x_k)}, \quad Y_n = \eta_1 + \ldots + \eta_n, \ n \geq 1. \tag{1}$$

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The initial segment $Y_1, \ldots, Y_{m-1}$ of the random walk has negative drift by the inequality

$$\int f_1 \ln\left(\frac{f_2}{f_1}\right) \le \int f_1 \left(\frac{f_2}{f_1} - 1\right) = 0$$

(equality is possible here only if $f_1 \equiv f_2$). At the same time, the trajectory starting at time m has positive drift. This fact is the base of the method. We decide to give alarm at time T_b, where

$$T_b = \min\left\{ n \ge 1 : \quad Y_n - \min_{1 \le k \le n} Y_k \ge b \right\}.$$

The positive number b is chosen in advance. The quality of a stopping time t in this problem can be characterized by the quantities

$$\tau_1(t) = \mathrm{E}_\infty t, \quad \tau_2(t) = \mathrm{E}_1 t,$$

(see [1], [4], [8]), where the subscript in the notation E_i stipulates that the calculation should be done under condition $m = i$. It was remarked in [4] that $\tau_2(t)$ coincides with the maximal possible mean value of the delay in reacting to a change in distribution. It turns out [8] that $\tau_2(t)$ attains its minimal value over the class of all stopping times t such that $\tau_1(t) \ge \tau > 0$ when $t = T_b$ under an appropriate choice of $b = b(\tau)$. In this case [4] for $\tau \to \infty$

$$b \sim \ln \tau, \quad \tau_2(T_b) \sim \frac{\ln \tau}{\mathrm{E}_1 \eta_1}.$$

The stopping time T_b can be defined in an equivalent way using a random walk with delay at the origin. To do so, we move the origin of our coordinate system to the minimum point of the random walk trajectory every time when the minimum is attained. Put

$$Y_0' = 0, \quad Y_{n+1}' = \max\left\{0,\ Y_n' + \eta_{n+1}\right\}.$$

Then $T_b = \min\left\{n \ge 1 :\ Y_n' \ge b\right\}$.

We shall follow two directions of study in the sequel.

1. (Direct problem). The finding of $\tau_1(T_b), \tau_2(T_b)$ given b.

2. (Converse problem). Finding the value of b such that $\tau_1(T_b) = \tau$ and calculating $\tau_2(T_b)$ as a function of τ in this case.

As will be shown below, the solution of these two problems reduces to finding the probability that a random walk starting from zero achieves the set $[b, \infty)$ earlier than the set $(-\infty, 0)$ and to calculating the expectation of the stopping time T_b. The solution of the latter problem is available in an explicit form only for some particular situations (see [12]). Therefore, most publications in this field deal with approximation formulae in different ways. One of the approaches consists in the study of asymptotics of the above-mentioned quantities in the triangular array setting where the contribution of an individual jump tends to become infinitesimal (see, e.g., [13]). Another approach deals with numerical solutions of the finite-interval integral equations satisfied by the quantities of interest. This latter approach can provide satisfactory accuracy, but the calculations may be most cumbersome. In this connection, a third direction has become widely spread; it consists in deriving for $\tau_i(T_b)$, $b \to \infty$, asymptotic approximations whose main terms can be calculated in a relatively simple way. The result of Lorden [4] mentioned above belongs to this direction as well as papers [9], [10]. A further development of this approach is to obtain, in both the direct and converse problems, asymptotic expansions that provide high accuracy under wide conditions. It is clear that the useful asymptotic expansions for $b \to \infty$ obtained in this way could be those for quantities related to the first exit time of a random walk from the interval $[0, b)$ (see, e.g., [14]).

The so-called factorization method of studying asymptotics of the distributions related to the first exit time from an interval was developed in author's earlier papers [15]—[17]. By this method, we prove below Theorem 1 on the asymptotic representations for the distribution of the position of a random walk at the first exit time from the interval $[0, b)$. Its proof includes an identity for the characteristic function of the distribution studied; this allows one to reduce the problem to that of studying superpositions of renewal processes. We apply then some asymptotic expansions in the renewal theorem under Cramér's condition. The asymptotic expansions for the expectation of the first exit time from an interval follow from Theorem 1 and Wald's identity (Theorem 2). These results seem to be of independent interest; they can be applied in other branches of sequential analysis, storage problems, etc. Asymptotic expansions for $\tau_1(T_b)$ follow from these theorems; they form the contents of Theorem 3 and part 1 of Corollary 1. Asymptotic expansions for $\tau_2(T_b)$ are obtained in the same way (Theorem 4 and part 2 of Corollary 1). Finally, some asymptotic expansions for the converse problem are presented in Theorems 5 and 6.

Thus, in solving asymptotically both problems, the gravity centre is transferred to the calculation of coefficients in the expansions derived here. For all the problems related to the first-exit time through a straight-line boundary, the characteristic feature is that the above coefficients are inevitably some functionals of the first ladder-height distribution for a random walk. The fact implies some computational difficulties. The computational aspects of constructing expansions are discussed in § 4; some explicit formulae for their coefficients are presented in the case of Ganssian sequences.

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2. MAIN RESULTS: THE DIRECT PROBLEM

We shall consider the general scheme of a random walk generated by the sequence $\{\xi_n\}_{n=1}^{\infty}$ of i.i.d. random variables. Let us introduce the following conditions.

I. $\mathbf{E}\xi_1 < 0$.

II. The function $f(\lambda) = \mathbf{E}e^{\lambda\xi_1}$ can be analytically continued into the strip $-\gamma < \operatorname{Re}\lambda < \beta$ for some $\gamma > 0, \beta > 0$, and it is continuous on the boundary of this strip. The values of $f(\lambda)$ in the strip are those resulting from this (unique) analytic continuation.

III. Suppose that $f(\beta) \geq 1$; we require in addition that

$$\int_0^\infty y e^{\beta y} \mathbf{P}(\xi_1 \in dy) < \infty$$

whenever $f(\beta) = 1$.

Under these conditions, there exists a unique positive solution $\lambda = q$ of the equation $f(\lambda) = 1,\ \ 0 < q \le \beta$.

IV. The function $r(\lambda) = 1 - f(\lambda)$ has no zeros except $\lambda = 0$ and $\lambda = q$ in the strip $-\gamma \le \mathrm{Re}\,\lambda \le \beta$.

Put

$$S_n = \sum_{i=1}^{n} \xi_i,\ \ \eta_\pm = \inf\{n \ge 1:\ S_n \gtrless 0\},\ \ \inf\varnothing = \infty,\ \ \chi_\pm = S_{\eta_\pm},$$

$$G_1(y) = \mathbf{P}(\chi_\pm < y,\ \eta_+ < \infty),\ \ G_2(y) = \mathbf{P}(-\chi_- < y).$$

V. Denote by $G_j^0(y)$ the sum of the discrete and singular components of $G_j, j = 1, 2$; then

$$\int_0^\infty e^{\beta y}\, dG_1^0(y) < 1,\ \ \int_0^\infty e^{\gamma y}\, dG_2^0(y) < 1.$$

These conditions need comment. If we put $\xi_k = \eta_k,\ k = 1, 2, \ldots$, under conditions of the change point problem (see (1)), and if $\mathbf{E}\,|\,\eta_1\,| < \infty$ for $m = 1$ and for $m = \infty$, then conditions I—III hold automatically; here $\beta \ge 1,\ q = 1$. The requirements $\gamma > 0$ and $\beta > 1$ are, as a rule, fulfilled in concrete situations. Condition IV is introduced here to make reading easier. The method exposed below permits one to consider the case of a function $r(\lambda)$ having other zeros in addition to 0 and q (these other zeros must necessarily be complex), but all the statements become more complicated in this case. Condition V stipulates that the absolutely continuous component of the distribution of ξ_1 should be "massive" enough, and its "size" should depend on β and γ. Some sufficient conditions which imply condition V can be found in [15].

Denote

$$r_\pm(\lambda) = 1 - \mathbf{E}(e^{\lambda\chi_\pm};\ \eta_\pm < \infty). \tag{2}$$

It is well known (see, e.g., [19]) that $r_+(\lambda)\, r_-(\lambda) = r(\lambda)$. For this reason, functions (2) are usually called factorization components. Their properties have been extensively studied by now. It turns out that these functions play an important role in solving problems related to the first level–crossing times. There exists another form of factorization components:

$$r_\pm(\lambda) = \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n}\mathbf{E}(e^{\lambda S_n};\ S_n \gtrless 0)\right\}. \tag{3}$$

Under conditions I—V we will use the following notations:

$$q_1 = -\frac{r_+(0)}{q r'_+(q)},\ \ q_2 = \frac{r_-(q)}{q r'_-(0)} = -\frac{r'(q)}{r'(0)} q_1, \tag{4}$$

Let, in addition, the measures $G_\pm$ be ones defined by the relations

$$\frac{r_+(\lambda)}{(\lambda - q) r'_+(q)} = \int_0^\infty e^{\lambda y} G_+(dy),\ \ \ \mathrm{Re}\,\lambda \le \beta,$$

$$\frac{r_-(\lambda)}{\lambda r_-(0)} = \int_{-\infty}^{0} e^{\lambda y} G_-(dy), \quad \mathrm{Re}\lambda \ge -\gamma. \tag{5}$$

It is easily seen that these measures are absolutely continuous with respect to the Lebesgue measure. Finally, define the first exit time N from the interval $[0, b)$ by the relation

$$N = \min\{ n \ge 1 : S_n \notin [0, b) \}.$$

Theorem 1. *Let conditions* I—V *be satisfied. Then for each* $x \ge 0$ *and* $b \to \infty$

$$\mathbf{P}(S_N \ge b + x) = \frac{\rho r_-(q) e^{-qb} G_+([x, \infty))}{1 - q_1 q_2 e^{-qb}} + R_1(x, b), \tag{6}$$

$$\mathbf{P}(S_N < -x) = \rho \mathbf{P}(\chi_- < -x) - \frac{\rho r_-(q) q_1 e^{-qb} G_-((-\infty, -x))}{1 - q_1 q_2 e^{-qb}} + R_2(x, b), \tag{7}$$

where

$$\operatorname*{Var}_{(-\infty,-x)} R_2(y, b) = o\left(e^{-\beta b - \gamma x}\right) + o\left(e^{-(\gamma+q)b - \gamma x}\right) \operatorname{sgn} x + o\left(e^{-(\gamma+2q)b - \gamma x}\right),$$

$$\operatorname*{Var}_{[x,\infty)} R_1(y, b) = o\left(e^{-\beta(b+x)}\right) + o\left(e^{-(\gamma+2q)b - \beta x}\right),$$

$$\rho = (1 - \mathbf{P}(\chi_- = 0))^{-1} = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}(S_n = 0) \right\}.$$

Note immediately that the last equality follows from comparing (2) and (3). Formulae (6), (7) will be used further for $x = 0$; therefore note that

$$G_+([0, \infty)) = q_1, \quad G_-((-\infty, 0)) = 1.$$

The above follows from (4) and (5); in addition, we ascertain that $\rho \mathbf{P}(\chi_- < 0) = 1$ by putting $\lambda = 0$ in (2).

We obtain our next result calculating the asymptotic representation for $\mathbf{E} S_N$ with the help of (6), (7), and Wald's identity, $\mathbf{E} S_N = \mathbf{E}\xi_1 \, \mathbf{E} N$.

Theorem 2. *Under conditions* I—V *for* $b \to \infty$

$$\mathbf{E}N = |\mathbf{E}\xi_1|^{-1} \left\{ \rho r_-'(0) - \frac{\rho r_-(q) q_1 e^{-qb}}{1 - q_1 q_2 e^{-qb}} \left[b + \frac{1}{q} + \frac{2 r_+'(0)}{r_+(0)} - \frac{\mathbf{E}\xi_1^2}{2\mathbf{E}\xi_1} \right] \right\}$$
$$+ o\left(b e^{-\beta b}\right) + o\left(e^{-(\gamma+q)b}\right). \tag{8}$$

Let us now study random walks with delay at the origin. Put

$$Z_0 = 0, \quad Z_n = \max\{0, Z_{n-1} + \xi_n\}, \quad n \ge 1, \qquad T = \min\{n \ge 1 : Z_n \ge b\}.$$

It was noted in [1] that

$$\mathbf{E}T = \mathbf{E}N [\mathbf{P}(S_N \ge b)]^{-1}. \tag{9}$$

Indeed, consider the sequence of stopping times

$$N_1 = N, \quad N_{k+1} = \min\{n \ge 1 : \xi_{N_1 + \dots + N_k + 1} + \dots + \xi_{N_1 + \dots + N_k + n} \notin [0, b)\}, \ k \ge 1,$$

and put $\nu = 1$ if $S_N \ge b$ and $\nu = \min\{k > 1 : S_{N_1 + \dots + N_k} - S_{N_1 + \dots + N_{k-1}} \ge b\}$ in the opposite case. Then

$$T = N_1 + \dots + N_\nu, \ \mathbf{P}(\nu = k) = [\mathbf{P}(S_N < 0)]^{k-1} \mathbf{P}(S_N \ge b);$$

some standard calculations imply the relation $\mathbf{E}T = \mathbf{E}N\mathbf{E}\nu$. Substitute expansions (8) and (6) into (9) for $x = 0$; after some simple transformations we obtain the following result.

Theorem 3. *Let conditions* I—V *be satisfied. Then for* $b \to \infty$

$$\mathbf{E}T = |\mathbf{E}\xi_1|^{-1}\left\{\frac{1}{qq_1q_2}e^{qb} - b - \frac{2}{q} - \frac{2r'_+(0)}{r_+(0)} + \frac{\mathbf{E}\xi_1^2}{2\mathbf{E}\xi_1}\right\}$$
$$+ o\left(e^{(2q-\beta)b}\right) + o\left(e^{-\gamma b}\right). \tag{10}$$

Recall that always $\beta \ge q$; if $q \le \beta < 2q$, expansion (10) yields only the main term. All terms in (10) are substantial if $\beta \ge 2q$. It was shown in [17] that the quantities β, γ increase when F_1, F_2 belong to a family of distributions $\{F_\theta\}$ possessing certain regularity properties with respect to θ, and the distance between F_1 and F_2 tends to zero.

When the drift is positive, the study of asymptotics of $\mathbf{E}T$ reduces to that of the expectation $\mathbf{E}\bar{T}$, where

$$\bar{T} = \min\{n \ge 1 : \bar{Z}_n \le -b\},$$
$$\bar{Z}_0 = 0, \bar{Z}_n = \min\{0, \bar{Z}_{n-1} + \xi_n\},$$

and, as before, $\mathbf{E}\xi_1 < 0$.

Theorem 4. *Let conditions* I—V *hold. Then for* $b \to \infty$

$$\mathbf{E}\bar{T} = |\mathbf{E}\xi_1|^{-1}\left\{b + \frac{2r'_+(0)}{r_+(0)} - \frac{\mathbf{E}\xi_1^2}{2\mathbf{E}\xi_1} + \frac{q_1q_2}{q}e^{-qb}\right\} + o\left(e^{-\gamma b}\right) + o\left(be^{-\beta b}\right). \tag{11}$$

Corollary 1. *Let* $\{\eta_k\}$ *be the sequence defined in* (1), *and assume that conditions* I—V *are satisfied for* $m = \infty$. *Then*

1) *for* $\tau_1(T_b)$, *expansion* (10) *is valid in which* $\xi_i = \eta_i, q = 1$, *and all the calculations are carried out under assumption* $m = \infty$;

2) *expansion* (11) *holds for* $\tau_2(T_b)$ *(provided that one puts* $\xi_i = -\eta_i, q = 1, m = 1$ *and interchanges* β *and* $\gamma + 1$*)*.

The last assertion follows from the relation

$$\mathbf{E}_\infty e^{\lambda\eta_1} = \mathbf{E}_1 e^{(\lambda-1)\eta_1},$$

since conditions I—V hold also for the distribution of the random variable $-\eta_1$ for $m = 1$ after the interval $[-\gamma, \beta]$ is replaced by $[1-\beta, 1+\gamma]$. Condition V changes slightly in the process: χ_+ ought to denote the value of the first non-negative sum and χ_- ought to be the first strictly negative one; however, this does not influence the result.

Theorem 3 and 4 allow one to calculate $\mathbf{E}\,T_b$ also in the case when all the observations $x_1, x_2, \ldots,$ are identically distributed and their distribution function F is known although it differs from F_1 and F_2. Of course, the conditions of the theorems must be satisfied.

3. THE CONVERSE THEOREMS

As stated above, the quantity $\tau_1(T_b)$ has, under conditions I—V, the form

$$\tau_1(T_b) = c_1 e^b + c_2 b + c_3 + o\left(e^{(2-\beta)b}\right) + o\left(e^{-\gamma b}\right). \tag{12}$$

The coefficients $c_i, i = 1, 2, 3$, are defined in Theorem 3; we will not specify their values here. Equality (12) can be solved asymptotically with respect to b.

Theorem 5. *Let the conditions of Corollary 1 be satisfied, and assume the number b to be such that* $\tau_1(T_b) = \tau$. *Then for* $\tau \to \infty$

$$b = \ln\tau - \ln c_1 - \frac{c_2 \ln\tau}{\tau} + \frac{c_2 \ln c_1 - c_3}{\tau} - \frac{c_2^2 \ln^2\tau}{\tau^2}$$
$$+ \left[c_1 c_2 - 2c_2 c_3 + 2c_2^2 \ln c_1\right]\frac{\ln\tau}{\tau^2} + \left[2c_2 c_3 \ln c_1 - c_3^2 - c_2^2 \ln^2 c_1\right.$$
$$\left. - c_1 c_2 \ln c_1\right]\frac{1}{\tau^2} + \ldots + o\left(\tau^{1-\beta}\right) + o\left(\tau^{-1-\gamma}\right). \tag{13}$$

Proof. It is clear that $\tau \to \infty$ for $b \to \infty$ and, vice versa, $b \to \infty$ for $\tau \to \infty$; moreover,

$$o\left(e^{(2-\beta)b}\right) + o\left(e^{-\gamma b}\right) = o\left(\tau^{2-\beta}\right) + o\left(\tau^{-\gamma}\right).$$

It follows from (12) that

$$\ln\tau = \ln\left[c_1 e^b\left(1 + \frac{c_2}{c_1} b e^{-b} + \frac{c_3}{c_1} e^{-b} + o\left(e^{(1-\beta)b}\right) + o\left(e^{-(\gamma+1)b}\right)\right)\right]$$
$$= \ln c_1 + b + \frac{c_2}{c_1} b e^{-b} + \frac{c_3}{c_1} e^{-b} + o\left(e^{(1-\beta)b}\right) + o\left(e^{-(\gamma+1)b}\right)$$
$$- \frac{1}{2}\left[\frac{c_2}{c_1} b e^{-b} + \frac{c_3}{c_1} e^{-b} + o\left(e^{(1-\beta)b}\right) + o\left(e^{-(\gamma+1)b}\right)\right]^2 + \ldots$$
$$= b + \ln c_1 + \left(\frac{c_2}{c_1} b + \frac{c_3}{c_1}\right) e^{-b} - \frac{1}{2}\left(\frac{c_2}{c_1}\right)^2 b^2 e^{-2b} - \frac{1}{2}\left(\frac{c_3}{c_1}\right)^2 b^2 e^{-2b}$$
$$- \frac{c_2 c_3}{c_1^2} b e^{-2b} + \ldots + o\left(e^{(1-\beta)b}\right) + o\left(e^{-(\gamma+1)b}\right).$$

Hence

$$b = \ln\tau - \ln c_1 - \left(\frac{c_2}{c_1} b + \frac{c_3}{c_1}\right) e^{-b} + \frac{1}{2}\left(\frac{c_2}{c_1}\right)^2 b^2 e^{-2b} + \frac{c_2 c_3}{c_1^2} b e^{-2b}$$

$$+\frac{1}{2}\left(\frac{c_3}{c_1}\right)^2 e^{-2b} + \ldots + o\left(e^{(1-\beta)b}\right) + o\left(e^{-(\gamma+1)b}\right). \tag{14}$$

In addition, the relation

$$e^{-b} = \frac{c_1}{\tau} + \frac{c_2 b e^{-b}}{\tau} + \frac{c_3 e^{-b}}{\tau} + o\left(\frac{1}{\tau} e^{(1-\beta)b}\right) + o\left(\frac{1}{\tau} e^{-(\gamma+1)b}\right) \tag{15}$$

follows from (12). Substituting in the right-hand side of (14) the expressions for b and e^{-b} given in (14) and (15), we obtain a recurrence process leading to (13). The proof is completed.

Under conditions of Corollary 1, the asymptotic expansion for $\tau_2(T_b)$ has the form

$$\tau_2(T_b) = c_4 b + c_5 + c_6 e^{-b} + o\left(e^{(1-\beta)b}\right) + o\left(b e^{-(\gamma+1)b}\right). \tag{16}$$

The next result follows by superposition of (16) and (13).

Theorem 6. *Under the conditions of Theorem 5, for* $\tau \to \infty$

$$\tau_2(T_b) = c_4 \ln\tau + (c_5 - c_4 \ln c_1) - \frac{c_2 c_4 \ln\tau}{\tau} + \frac{c_1 c_6 + c_4(c_2 \ln c_1 - c_3)}{\tau}$$

$$- c_4 c_2^2 \frac{\ln^2\tau}{\tau^2} + \left\{c_4\left[c_1 c_2 - 2c_2 c_3 + 2c_2^2 \ln c_1\right] + c_1 c_2 c_6\right\}\frac{\ln\tau}{\tau^2}$$

$$+ \left\{c_4\left[2c_2 c_3 \ln c_1 - c_3^2 - c_2^2 \ln^2 c_1 - c_1 c_2 \ln c_1\right] + c_1 c_3 c_6 - c_1 c_2 c_6 \ln c_1\right\}\frac{1}{\tau^2}$$

$$+ \ldots + o\left(\tau^{1-\beta}\right) + o\left(\tau^{-1-\gamma}\ln\tau\right).$$

4. EXAMPLES

In this section we discuss the computational aspects of the above-obtained results and consider some examples.

One can find the number q (the positive solution of the equation $\mathrm{E} e^{\lambda\xi_1} = 1$) with appropriate accuracy by successive approximations in the case when an explicit formula is not available. The determination of the strip $-\gamma \le \operatorname{Re}\lambda \le \beta$ in which $\mathrm{E} e^{\lambda\xi_1} \ne 1$ everywhere except $\lambda = 0$ and $\lambda = q$ reduces to analyzing the complex solutions of this equation. The analysis presents no difficulties, but calculation of coefficients of the expansions is more complicated. All of them reduce to the calculation of values of the factorization components and their first derivatives at certain points. Explicit expressions for the factorization components are often unknown. An approximation for the above-mentioned quantities can be constructed by statistical simulation method since by (2) the values of factorization components and their derivatives at fixed points coincide with the expectations of some known functions of the first positive (or non-positive) sum. Representation (3) can be useful in many cases (see examples below).

Let us consider again the change point problem.

Example 1. Put

$$f_j(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(y-\theta_j)^2\right\}, \quad \theta_2 > \theta_1,$$

and suppose that the observations $x_1, x_2, \dots,$ are normally distributed with parameters $(\theta, 1)$, where θ is not necessarily equal to θ_1 or θ_2. Then

$$\eta_k = (\theta_2 - \theta_1)x_k - \tfrac{1}{2}\left(\theta_2^2 - \theta_1^2\right), \quad \mathrm{E}\,\eta_1 = (\theta_2 - \theta_1)\theta - \tfrac{1}{2}\left(\theta_2^2 - \theta_1^2\right),$$

$$\mathrm{E}\,e^{\lambda\eta_1} = \exp\left\{\lambda\left[(\theta_2 - \theta_1)\theta - \tfrac{1}{2}\left(\theta_2^2 - \theta_1^2\right)\right] + \frac{\lambda^2}{2}\left(\theta_2 - \theta_1\right)^2\right\}.$$

Conditions I—V are satisfied for $\theta < \frac{1}{2}\left(\theta_1 + \theta_2\right)$; we have in this case $q = \frac{u}{v}$, where $v = \theta_2 - \theta_1$, $u = \theta_1 + \theta_2 - 2\theta$. Some simple calculations show that $1 - \mathrm{E}\,e^{\lambda\eta_1} = 0$ also for $\lambda = z_k \pm iy_k$, $k = \pm 1, \pm 2, \dots,$ where

$$z_k = \frac{u}{2v} + \frac{2\pi k}{v^2 y_k}, \quad y_k = \frac{1}{2v}\left(-\frac{u^2}{2} + \left(\frac{u^4}{4} + 64\pi^2 k^2\right)^{1/2}\right)^{1/2},$$

i.e., β may be chosen equal to $z_1 - \varepsilon$ for arbitrary sufficiently small values of $\varepsilon > 0$, $-\gamma = z_{-1} + \varepsilon$. We have $z_1 = z_1(v) \to \infty$ for $\theta = \theta_1$ and $v \to 0$; $z_1(v) \to 1$ for $v \to \infty$, and $z_1(0.1) = 25.569$, $z_1(0.5) = 5.526$, $z_1(1) = 3.032$, $z_1(2) = 1.804$. This points to high accuracy of the above-obtained expansions for small v.

The coefficients of the expansions here can be calculated using relation (3) (including the case of $\theta \neq \theta_i$).

Let us apply the above considerations to the expansions for $\tau_i(T_b)$ contained in Corollary 1. It was stated in [15] that in our case

$$q_1 = q_2 = \frac{2}{(\theta_2 - \theta_1)^2}\exp\left\{-2\sum_{n=1}^{\infty}\frac{1}{n}\Phi\left(\frac{1}{2}(\theta_1 - \theta_2)\sqrt{n}\right)\right\},$$

where Φ is the standard normal distribution function. Denote the expectation by $a = -\mathrm{E}_\infty \eta_1 = \frac{(\theta_2 - \theta_1)^2}{2}$; then $\mathrm{D}_\infty \eta_1 = 2a$, and we obtain from (3) the following relations:

$$-\frac{r_+'(0)}{r_+(0)} = \sum_{n=1}^{\infty}\frac{1}{2n\sqrt{\pi a n}}\int_0^{\infty} y\exp\left\{-\frac{(y + na)^2}{4na}\right\}dy$$

$$= \sqrt{\frac{a}{\pi}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}e^{-na/4} - a\sum_{n=1}^{\infty}\left(1 - \Phi\left(\sqrt{\frac{na}{2}}\right)\right)$$

(here $r(\lambda) = 1 - \mathrm{E}_\infty e^{\lambda\eta_1}$). To find $\frac{r_-'(1)}{r_-(1)}$, note that for $m = 1$ the distribution of the random variable $-\eta_1$ is identical to that of η_1 for $m = \infty$. Hence, $\tilde{r}(\lambda) \equiv 1 - \mathrm{E}_1 e^{-\lambda\eta_1} = r(\lambda)$. On the other hand,

$$1 - \mathrm{E}_1 e^{-\lambda\eta_1} = r(1 - \lambda) = r_+(1 - \lambda)r_-(1 - \lambda) = r_+(\lambda)r_-(\lambda),$$

i.e., $r_{\pm}(\lambda) = r_{\mp}(1-\lambda)$ and

$$\frac{r'_-(1)}{r_-(1)} = -\frac{r'_+(0)}{r_+(0)}.$$

Example 2. Let

$$f_j(y) = \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left\{ -\frac{y^2}{2\sigma_j^2} \right\}, \quad \sigma_1^2 < \sigma_2^2,$$

then $\eta_k = \alpha x_k^2 + \frac{1}{2}\ln\theta$, where $\alpha = \dfrac{\sigma_2^2 - \sigma_1^2}{2\sigma_2^2\sigma_1^2}$, $\theta = \dfrac{\sigma_1^2}{\sigma_2^2}$. Suppose that the observations have normal distribution with parameters $(0, \sigma^2)$, then

$$\mathrm{E}\, e^{\lambda\eta_1} = \theta^{\lambda/2}(1 - 2\alpha\lambda\sigma^2)^{-1/2}.$$

Let σ be such that $\mathrm{E}\,\eta_1 \equiv \alpha\sigma^2 + \frac{1}{2}\ln\theta < 0$. Now we investigate the question whether the equation $1 - 2\alpha\lambda\sigma^2 - \theta^\lambda = 0$ has complex roots. Denoting $z = \theta^{\mathrm{Re}\,\lambda}$, $t = -\,\mathrm{Im}\,\lambda \ln\theta$, $a = -2\alpha\sigma^2/\ln\theta$, we obtain the system

$$\begin{cases} z\cos t = 1 + a\ln z \\ z\sin t = at \end{cases}$$

and after that the equation

$$a\,t\,\mathrm{ctg}\,t = 1 + a\ln\frac{at}{\sin t}$$

which has an increasing sequence of roots $t_k \in (2k\pi, 2k\pi + \pi/2)$, $k \geq 1$. Since $z_k \equiv (\ln\theta)^{-1}\ln\left(a\,t_k/\sin t_k\right) < 0$, we can choose $-\gamma = z_1 + \varepsilon$ for an arbitrary small number $\varepsilon > 0$ and set $\beta = (2\alpha\sigma^2)^{-1} - \varepsilon$. Let $\sigma = \sigma_1$. It is easy to see that $\beta \to \infty$, $\gamma \to \infty$ for $\sigma_1^2/\sigma_2^2 \to 1$. Let us calculate the quantities in the expansions for $\tau_i(T_b)$. Put $1 - r(\lambda) = \mathrm{E}_\infty e^{\lambda\eta_1}$ as above. Then the relation

$$r_+(0) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{P}_\infty(Y_n > 0) \right\} = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} G_n(na_1, \infty) \right\}$$

follows from (3); we nse notations

$$G_n(A, B) = \int_A^B \frac{1}{2^{n/2}\Gamma(n/2)} y^{n/2-1} e^{-y/2}\,dy, \quad a_i = -\frac{\ln\theta}{2\alpha\sigma_i^2}.$$

Put $\hat{r}(\lambda) = 1 - \mathrm{E}_1 e^{\lambda\eta_1}$, then $\hat{r}_{\pm}(\lambda - 1) = r_{\pm}(\lambda)$ and

$$-r'_+(1) = -\hat{r}'_+(0) = -\frac{\hat{r}'(0)}{\hat{r}_-(0)} = \mathrm{E}_1\eta_1 \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{P}(Y_n < 0) \right\}$$

$$= \left(\alpha\sigma_2^2 + \frac{\ln\theta}{2}\right) \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} G_n(0, na_2) \right\}.$$

We derive in a similar way the expressions

$$r_-(1) = \hat{r}_-(0) \exp\left\{ - \sum_{n=1}^{\infty} \frac{1}{n} G_n(0, na_2) \right\},$$

$$r'_-(0) = \frac{r'(0)}{r_+(0)} = -\left(\alpha\sigma_1^2 + \frac{\ln\theta}{2}\right) \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} G_n(na_1, \infty) \right\}.$$

Therefore,

$$q_1 = \left(\alpha\sigma_2^2 + \frac{\ln\theta}{2}\right)^{-1} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} [G_n(na_2, na_1) - 1] \right\},$$

$$q_2 = -q_1 \left(\alpha\sigma_2^2 + \frac{\ln\theta}{2}\right)\left(\alpha\sigma_1^2 + \frac{\ln\theta}{2}\right)^{-1},$$

$$-\frac{r'_+(0)}{r_+(0)} = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} y \, \mathbf{P}_{\infty}(Y_n \in dy)$$

$$= \sum_{n=1}^{\infty} \left[\alpha\sigma_1^2 G_{n+2}(na_1, \infty) + \frac{\ln\theta}{2} G_n(na_1, \infty) \right],$$

$$-\frac{r'_-(1)}{r_-(1)} = -\frac{\hat{r}'_-(0)}{\hat{r}_-(0)} = \sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{0} y \, \mathbf{P}_1(Y_n \in dy)$$

$$= \sum_{n=1}^{\infty} \left[\alpha\sigma_2^2 G_{n+2}(0, na_2) + \frac{\ln\theta}{2} G_n(0, na_2) \right].$$

The calculations for the case of exponential distributions F_i are simpler (see [15]).

The results obtained here can be used also when the explicit form of the distributions F_1 and F_2 is unknown.

Let, e.g., the observations x_i be normally distributed with zero mean and unknown variance σ^2; assume that the latter is known to satisfy the inequalities $\sigma^2 \le \sigma_1^2$ before the change in distribution and $\sigma^2 \ge \sigma_2^2$ after that (with $\sigma_1^2 < \sigma_2^2$).We shall construct process (1) under these conditions using the extreme points σ_1^2 and σ_2^2 as in Example 2 above. Then the expectations satisfy the inequality $\mathbf{E}_{\sigma} T_b \ge \mathbf{E}_{\sigma_1} T_b$ for $\sigma \le \sigma_1$, $m = \infty$, and $\mathbf{E}_{\sigma} T_b \le \mathbf{E}_{\sigma_2} T_b$ for $\sigma \ge \sigma_2$, $m = 1$ (the expectation symbols are indexed here by the parameter values). The inequalities follow from a fairly simple argument. If the random variable T is generated by the sequence $\{\xi_k\}$ as in Section 2, T' is similarly determined by $\{\xi'_k\}$, and $\xi'_k = A\xi_k + B$, then $T' \le T$ with probability one for $A \ge 1$, $B \ge 0$, and $T' \ge T$ for $0 < A \le 1$, $B \le 0$. We have in our case

$$\eta_k = \alpha\sigma^2\left(\frac{x_k}{\sigma}\right)^2 + \frac{1}{2}\ln\theta = \frac{\sigma^2}{\sigma_i^2}\left[\alpha\sigma_i^2\left(\frac{x_k}{\sigma}\right)^2 + \frac{1}{2}\ln\theta\right] - \left(\frac{\sigma^2}{\sigma_i^2} - 1\right)\frac{\ln\theta}{2}.$$

The situation when it is the mean value of the normal distribution that changes can be considered in a similar way.

5. PROOFS OF THE THEOREMS

We still have to prove the basic result, Theorem 1, and present its modification which implies Theorem 4. Let the conditions of Theorem 1 be satisfied. Consider the functions

$$Q_1(\lambda) = \mathrm{E}(e^{\lambda S_N}; S_N < 0), \quad Q_2(\lambda) = \mathrm{E}(e^{\lambda S_N}; S_N \geq b).$$

They satisfy the identities (see [12])

$$Q_1(\lambda) = Ae(\lambda) - AQ_2(\lambda),$$
$$Q_2(\lambda) = Be(\lambda) - BQ_1(\lambda), \tag{17}$$

where $e(\lambda) \equiv 1$. The operators A and B are defined in the following way. For any function $g(\lambda)$ which can be represented, for some $\lambda \in (0, q)$, as the integral

$$g(\lambda) = \int_{-\infty}^{\infty} e^{\lambda y}\, dG(y), \quad \int_{-\infty}^{\infty} e^{\lambda y}\, |dG(y)| < \infty, \tag{18}$$

we put

$$Ag(\lambda) = r_-(\lambda)[r_-^{-1}(\lambda)g(\lambda)]^{(-\infty,0)},$$
$$Bg(\lambda) = r_+(\lambda)[r_+^{-1}(\lambda)g(\lambda)]^{[b,\infty)}, \tag{19}$$

where we use the notation

$$\left[\int_{-\infty}^{\infty} e^{\lambda y}\, dG(y)\right]^D = \int_D e^{\lambda y}\, dG(y), \quad D \subset \mathrm{R}.$$

The functions $r_\pm^{-1}(\lambda)$ allow one to obtain representations of type (18) in the strip $0 < \mathrm{Re}\,\lambda < q$ under conditions I—V (see [17]). Moreover, the factorization equality $r(\lambda) = r_+(\lambda)r_-(\lambda)$ holds for $-\gamma \leq \mathrm{Re}\,\lambda \leq \beta$, and $r_+(q) = r_-(0) = 0$. Therefore

$$\int_{+0}^{\infty} e^{qy}\mathrm{P}(\chi_+ \in dy, \eta_+ < \infty) = 1,$$

and we can introduce a random variable ζ whose distribution is $\mathrm{P}(\zeta \in dy) = e^{qy}\mathrm{P}(\chi_+ \in dy, \eta_+ < \infty)$. Denote $\varphi(\lambda) = \mathrm{E}e^{\lambda\zeta}$. Then

$$r_+^{-1}(\lambda) = (1 - \varphi(\lambda - q))^{-1} = \int_0^{\infty} e^{(\lambda - q)y}\, dR(y), \tag{20}$$

where R is the renewal function corresponding to the random variable ζ. We can use further the theorem on the asymptotics of the renewal function [18] which asserts under our conditions that, for $y \to \infty$,

$$\Delta(y) \equiv R(y) - \frac{y}{\mathbf{E}\zeta} - \frac{\mathbf{E}\zeta^2}{2\mathbf{E}\zeta} = o\left(e^{(q-\beta)y}\right), \quad \operatorname*{Var}_{[y,\infty)} \Delta(t) = o\left(e^{(q-\beta)y}\right). \quad (21)$$

For any function g satisfying (18) and possessing the property $g(\lambda) = [g(\lambda)]^{(-\infty,0]}$ we deduce the relation

$$Bg(\lambda) = r_+(\lambda)\left[\int_0^{\infty} e^{(\lambda-q)y}\left(\frac{dy}{\mathbf{E}\zeta} + d\Delta(y)\right)\int_{-\infty}^{+0} e^{\lambda y} dG(y)\right]^{[b,\infty)}$$

$$= \frac{r_+(\lambda)e^{(\lambda-q)b}g(q)}{r_+(q)(\lambda-q)} + r_+(\lambda)\int_b^{\infty} e^{(\lambda-q)y} d_y \int_{-\infty}^{+0} e^{qt}\Delta(y-t)\,dG(t). \quad (22)$$

We obtain in a similar way also the equality

$$Ag(\lambda) = \frac{r_-(\lambda)g(0)}{\lambda r_-(0)} + r_-(\lambda)\int_{-\infty}^{0} e^{\lambda y} d_y \int_0^{\infty} \Delta_1(y-t)\,dG(t) \quad (23)$$

for any function g satisfying (18) and possessing the property

$$g(\lambda) = [g(\lambda)]^{[0,\infty)};$$

here $\operatorname{Var}_{(-\infty,-x]} \Delta_1(y) = o(e^{-\gamma x})$, $x \to \infty$. Moreover, from the definition of the operator A it follows that

$$Ae(\lambda) = 1 - \rho r_-(\lambda), \quad (24)$$

where $\rho = (1 - \mathbf{P}(\chi_- = 0))^{-1}$.

Using representations (22)—(24) in (17) we obtain identities for $Q_1(\lambda)$ and $Q_2(\lambda)$, but the first of them will contain the unknown quantity $Q_2(0)$ in its right-hand side, and the second one will contain $Q_1(q)$. To find these quantities put $\lambda = q$ in the first identity and $\lambda = 0$ in the second one. Solving the resulting system of equations we obtain the following asymptotic representations for the functions Q_i:

$$Q_1(\lambda) = 1 - \rho r_-(\lambda) - \frac{r_-(\lambda)}{\lambda r_-(0)}\left[\frac{\rho r_-(q)q_1 e^{-qb}}{1 - q_1 q_2 e^{-qb}} + o\left(e^{-\beta b}\right) + o\left(e^{-(2q+\gamma)b}\right)\right]$$

$$+ r_-(\lambda)\int_{-\infty}^{0} e^{\lambda y} dR_1(y), \quad (25)$$

$$Q_2(\lambda) = \frac{r_+(\lambda)e^{(\lambda-q)b}\rho r_-(q)}{(\lambda-q)r_+(q)(1-q_1 q_2 e^{-qb})}\left[1 + o\left(e^{-(q+\gamma)b}\right) + o\left(e^{-\beta b}\right)\right]$$

$$+ r_+(\lambda) \int_b^\infty e^{(\lambda - q)y} dR_2(y), \tag{26}$$

where

$$\operatorname*{Var}_{(-\infty,-x]} R_1(y) = o\left(e^{-\gamma x - (\gamma + q)b}\right), \quad \operatorname*{Var}_{[x,\infty)} R_2(y) = o\left(e^{(q-\beta)x}\right)$$

for $x \to \infty$. Inverting the main terms of (25) and (26) with respect to λ we obtain the asymptotic representations (6) and (7). Theorem 1 is proved.

Proof of Theorem 4 is carried out similarly with evident modifications caused by the replacement of the interval $[0, b)$ with the interval $(-b, 0]$. We present here only the asymptotic representations for Laplace—Stieltjes transforms. Denote

$$\tilde{N} = \min\{n \ge 1 : S_n \notin (-b, 0]\}.$$

Then under the conditions of Theorem 4 we have, for $b \to \infty$, the relations

$$\mathrm{E}(e^{\lambda S_{\tilde{N}}}; S_{\tilde{N}} > 0) = 1 - r_+(\lambda) + r_+(\lambda) \int_{+0}^\infty e^{(\lambda - q)y} dR_3(y)$$

$$- \frac{r_+(\lambda) r_+(0) q_2 e^{-qb}}{(\lambda - q) r_+'(q)(1 - q_1 q_2 e^{-qb})} \left[1 + o(e^{-\gamma b}) + o(e^{-\beta b})\right],$$

$$\mathrm{E}(e^{\lambda S_{\tilde{N}}}; S_{\tilde{N}} \le -b) = r_-(\lambda) \int_{-\infty}^{-b+0} e^{\lambda y} dR_4(y)$$

$$+ \frac{r_-(\lambda) r_+(0) e^{-\lambda b}}{\lambda r_-'(0)(1 - q_1 q_2 e^{-qb})} \left[1 + o\left(e^{-(\gamma + q)b}\right) + o\left(e^{-\beta b}\right)\right],$$

as well as

$$\operatorname*{Var}_{[x,\infty)} R_3(y) = o\left(e^{-(\beta - q)x - \beta b}\right), \quad \operatorname*{Var}_{(-\infty,-x-b]} R_4(y) = o\left(e^{-\gamma(x+b)}\right).$$

Asymptotic expansions for the distribution of $S_{\tilde{N}}$ and its expectation $\mathrm{E}\tilde{N}$ follow immediately from these representations; they lead then to one for $\mathrm{E}\tilde{T}$.

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