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PERIODIC SOLUTIONS OF THE SPATIAL EXTENSION OF A CONDITIONALLY PERIODIC SYSTEM

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ABSTRACT. We consider a linear system of differential equations $x' = a(t)x - \mu x$ with a conditionally periodic matrix a and a parameter $\mu \in \mathbb{C}$. We prove that there exists a nonempty set $M \subset \mathbb{C}$ such that for each $\mu \in M$ the spatial periodic extension of this system, which is a system of first order partial differential equations, has a generalized (in the framework of Schwartz's theory of distributions) periodic solution.

Keywords: conditionally periodic system, quasi-periodic solution, periodic Schwartz's distribution, linear homogenous system.

0. INTRODUCTION

Consider the system of differential equations

$$x' = a(t)x - \mu x \tag{0.1_\mu}$$

with a real conditionally periodic matrix $a(t) = A(et)$, where A is a continuous ω_j -periodic in φ_j ($j=1, \dots, m$) $n \times n$ -matrix function of m variables $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbb{R}^m$, $\omega = (\omega_1, \dots, \omega_m)$, $e = (1, \dots, 1)$ is an m -vector, $t \in \mathbb{R}$, $\mu \in \mathbb{C}$, and the frequencies $\beta_i = 2\pi/\omega_i$ are rationally incommensurable.

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Consider also the spatial periodic extension of this system

$$\sum_{j=1}^m \psi_j' = A(\varphi)\psi - \mu\psi \quad (0.2_\mu)$$

and remark that if system (0.2_μ) has a continuous ω_j-periodic in φ_j (j = 1, ..., m) solution ψ, then the conditionally periodic vector x(t) = ψ(et) solves system (0.1_μ).

The idea of the periodic extension of a conditionally periodic function goes back to P. Bohl [2] who was the founder of the theory of conditionally periodic functions and who proved that each continuous conditionally periodic function has the unique periodic extension.

If m = 1, then there exists a non-empty set M ⊂ ℂ such that for each μ ∈ M system (0.1_μ) has a periodic solution.

We assume that m > 1. The problem is whether system (0.1_μ) has a conditionally periodic solution for some μ ∈ ℂ (or system (0.2_μ) has a periodic one). This is well-known to be falls, in general, even for n = 1. In this case x(t) = c exp ∫₀^t (a(t) - μ) dt is a solution of system (0.1_μ) with x(0) = c and ∫₀^t (a(t) - μ) dt may not be conditionally periodic even though μ is the mean value of a.

The aforementioned problem is related to that of existence of almost periodic solutions of a homogeneous system [1, 3, 10, 9]. It was proved in these papers that there exists a Bohr almost periodic solution [1, 3, 10] or a Besicovitch almost periodic solution [9] if the system has some additional properties (such as the Favard separation condition, for example).

We assume nothing but that the matrix A is sufficiently smooth and prove that there exists a non-empty set M ⊂ ℂ such that for any μ ∈ M there exists a periodic distribution ψ satisfying (0.2_μ). In a sequel to the present paper we will apply this fact to prove that for any μ ∈ M there exists a conditionally periodic distribution satisfying (0.1_μ).

We now present some results in [7] (see also [8]), which we are going to use. Since this paper is hardly available, we prove these results at the end of the paper.

Consider the system of integral equations

$$x(\varphi) = \int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi)x(\varphi - e\xi) d\xi + x_0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})), \quad (0.3)$$

where $\hat{\varphi} = (\varphi_2, \dots, \varphi_m)$, $\hat{e} = (1, \dots, 1)$ is an (m - 1) -vector. For any continuous function $x^0: \mathbb{R}^{m-1} \rightarrow \mathbb{R}^n$ this system has a unique continuous solution which is ω_k-periodic in φ_k if so is x⁰ (k = 2, ..., m). The vector x = x(et; φ₁₀, x⁰) is a solution of system (0.1₀) with x(eφ₁₀; φ₁₀, x⁰) = x⁰($\hat{e}\varphi_{10}$).

Side by side with (0.3) it is useful to consider the associated matrix equation

$$X(\varphi) = \int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi)X(\varphi - e\xi) d\xi + E, \quad (0.4)$$

where E is the identity matrix. By X(φ; φ₁₀) we denote the solution of this equation and X₀(φ) = X(φ; 0).

Lemma 0.1 . Let A be a continuous ω_j -periodic in φ_j ($j=1, \dots, m$) matrix function. Then matrix equation (0.4) has a unique continuous solution X which has the following properties.

(a) The matrix $X(\varphi; \varphi_{10})$ is non-singular for all $\varphi \in \mathbb{R}^m$, $\varphi_{10} \in \mathbb{R}$ and ω_j -periodic in φ_j ($j=2, \dots, m$).

(b) The solution of system (0.3) can be represented in the form

$$x(\varphi; \varphi_{10}, x^0) = X(\varphi; \varphi_{10})x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})).$$

(c) $X(\varphi_1 + \omega_1, \hat{\varphi}; \varphi_{10}) = X(\varphi; \varphi_{10})X(\varphi_{10} + \omega_1, \hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}); \varphi_{10})$; in particular, $X_0(\varphi_1 + \omega_1, \hat{\varphi}) = X_0(\varphi)X_0(\omega_1, \hat{\varphi} - \hat{e}\varphi_1)$ [here the vector φ has the form $(\varphi_1, \hat{\varphi})$].

(d) The matrix $X_0(et)$ is the normalized fundamental matrix of system (0.1₀).

Lemma 0.2. Let the matrix A be r -times continuously differentiable on \mathbb{R}^m ; then so is the matrix X_0 .

The matrix X_0 is a solution of a differential equation which we are going to get now. Put $\varphi_1 - \xi = \zeta$ in (0.4). Then

$$X_0(\varphi) = \int_{\varphi_{10}}^{\varphi_1} A(\zeta, \hat{\varphi} + \hat{e}(\zeta - \varphi_1))X_0(\zeta, \hat{\varphi} + \hat{e}(\zeta - \varphi_1)) d\zeta + E.$$

Let the matrix A be continuously differentiable; then so is X_0 . Differentiating both sides of the above system w.r.t. φ_j and summing w.r.t. j from 1 to m , we obtain

$$\sum_{j=1}^m (X_0)'_j = A(\varphi)X_0. \quad (0.5)$$

The key role in our research is played by the system

$$X_0(\omega_1, \hat{\varphi})L(\hat{\varphi} - \hat{e}\omega_1) = \lambda L(\hat{\varphi}). \quad (0.6)$$

Suppose that $L : \mathbb{R}^{m-1} \rightarrow \mathbb{C}^n$ is a continuous ω_j -periodic in φ_j ($j = 2, \dots, m$) vector function which satisfies (0.6) for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $\mu = \frac{1}{\omega_1} \ln \lambda$. We claim that

$$\psi = X_0(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1) \exp(-\mu\varphi_1)$$

is periodic, $\psi(et)$ is a conditionally periodic solution of (0.1 _{μ}), and if L is differentiable, then ψ satisfies (0.2 _{μ}). Indeed, ψ is ω_j -periodic in φ_j ($j=2, \dots, m$) since X and L are so. Let us prove that it is periodic in φ_1 . By item (c) of Lemma 0.1, we have

$$\begin{aligned} \psi(\varphi_1 + \omega_1, \hat{\varphi}) &= X_0(\varphi_1 + \omega_1, \hat{\varphi})L(\hat{\varphi} - \hat{e}(\varphi_1 + \omega_1))e^{-\mu\omega_1}e^{-\mu\varphi_1} \\ &= X_0(\varphi)X_0(\omega_1, \hat{\varphi} - \hat{e}\varphi_1)L(\hat{\varphi} - \hat{e}(\varphi_1 + \omega_1))\lambda^{-1}e^{-\mu\varphi_1} \\ &\stackrel{(0.6)}{=} X_0(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1)\lambda\lambda^{-1}e^{-\mu\varphi_1} = \psi(\varphi). \end{aligned} \quad (0.7)$$

In view of item (d) of Lemma 0.1, it is clear that $\psi(et) = X_0(et)L(\hat{0})e^{-\mu t}$ is a solution of (0.1 _{μ}). That ψ solves (0.2 _{μ}) one can check by direct calculation using (0.5) and the relation $\sum_{j=1}^m (L(\hat{\varphi} - \hat{e}\varphi_1))'_j = 0$.

In the first section of this paper, we present the basic facts about periodic distributions and show that if there exists a periodic distribution L that solves (0.6) for some $\lambda \neq 0$, then $\psi = X_0(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1) \exp(-\mu\varphi_1)$ is a generalized periodic solution of system (0.2 _{μ}).

In the second section, we prove that there exists a non-empty set $\Lambda \subset \mathbb{C}$ such that for each $\lambda \in \Lambda$ the distribution L exists. And in the third section we prove Lemma 0.1.

1. PERIODIC DISTRIBUTIONS

The definition and properties of periodic distributions, which we use, can be found in the monographs [12, 13, 6]. We introduce vector periodic distributions following [6] (see also [4, Ch.1]).

Let $P_n^q(\omega)$ be the Banach space of ω_j -periodic in φ_j ($j = 1, \dots, m$) q -times continuously differentiable functions $y: \mathbb{R}^m \rightarrow \mathbb{C}^n$ with the norm $\|y\|_{P_n^q(\omega)} = \max_{|r| \leq q} \max_{\varphi \in \mathbb{R}^m} \|D^r y(\varphi)\|$, where $\|\cdot\|$ is a norm on a finite-dimensional space, $r = (r_1, \dots, r_m)$, r_j are non-negative integers, $D^r y(\varphi) = \frac{\partial^{|r|} y(\varphi)}{\partial \varphi_1^{r_1} \dots \partial \varphi_m^{r_m}}$, and $|r| = r_1 + \dots + r_m$.

Let $P_{n_2}^q(\omega)$ be the Banach space of periodic q -times continuously differentiable $n \times n$ matrices with the norm $\|A\|_{P_{n_2}^q(\omega)} = \max_{|s| \leq n} \|a_s\|_{P_n^r(\omega)}$, where $a_s \in P_n^q(\omega)$ ($s = 1, \dots, n$) are columns of A .

Denote by $P_n^{q'}(\omega)$ the space of linear continuous maps $x: P_n^q(\omega) \rightarrow \mathbb{C}^n$, where $x = (x_1, \dots, x_n)^T$, x^T is the transpose of x and x_k are linear continuous functionals on $P_1^q(\omega)$.

Every $x \in P_n^{q'}(\omega)$ determines a linear continuous functional on $P_n^q(\omega)$ by the rule $\langle x, y \rangle_\omega = \sum_{k=1}^n \langle x_k, y_k \rangle_\omega$, where $y = (y_1, \dots, y_n)^T \in P_n^q(\omega)$ and $\langle x_k, y_k \rangle_\omega$ is the value of x_k at the point $y_k \in P_1^q(\omega)$.

The space $P_n^{q'}(\omega)$, endowed with the norm $\|x\|_{P_n^{q'}(\omega)} = \sup_{\|y\|_{P_n^q(\omega)}=1} |\langle x, y \rangle_\omega|$, is the Banach space, and it is isomorphic to the dual space of $P_n^q(\omega)$ [6, p.267].

Consider the countably normed space $P_n(\omega) = \bigcap_{q=0}^{\infty} P_n^q(\omega)$ and its dual $P_n'(\omega) = \bigcup_{q=0}^{\infty} P_n^{q'}(\omega)$. By definition, a sequence $\{y^k\}$ in $P_n(\omega)$ converges to $y \in P_n(\omega)$ if $\|y^k - y\|_{P_n^q(\omega)} \rightarrow 0$ for every non-negative integer q , and $\{x^k\}$ in $P_n'(\omega)$ converges weakly to $x \in P_n'(\omega)$ if $\langle x^k - x, y \rangle_\omega \rightarrow 0$ for every $y \in P_n(\omega)$ as $k \rightarrow \infty$. The latter is equivalent to the fact that $x^k \rightarrow x$ weakly in some $P_n^{q'}(\omega)$ [4, Ch.1].

The space $P_1'(\omega)$ is isomorphic to the space of the Schwartz periodic distributions [6, 12] that is why $x \in P_1'(\omega)$ is called an ω -periodic distribution, and therefore $x \in P_n'(\omega)$ we call an ω -periodic vector distribution [13, 6].

We are going to mainly deal with the spaces of vector distributions $P_n^{q'}(\omega)$ and spaces of test functions $P_n^q(\omega)$. Suppose $x \in P_n^{q'}(\omega)$. Let us remember the definitions of

$$\begin{aligned} \text{derivative: } & \langle x'_j, y \rangle_\omega = -\langle x, y'_j \rangle_\omega, \quad \forall y \in P_n^{q+1}(\omega), \\ \text{multiplication by matrix: } & \langle Ax, y \rangle_\omega = \langle x, A^T y \rangle_\omega, \quad \forall y \in P_n^q(\omega), \\ & A \in P_{n_2}^q(\omega), \\ \text{translation by } \tau: & \langle x(\varphi - e\tau), y(\varphi) \rangle_\omega = \langle x(\varphi), y(\varphi + e\tau) \rangle_\omega, \quad \forall y \in P_n^q(\omega), \\ & \tau \in \mathbb{R}. \end{aligned}$$

It was proved in the monographs [12, p.225], [13, p.130] that every $x \in P_n'(\omega)$ has the unique Fourier-series expansion

$$x(\varphi) = \sum_{k \in \mathbb{Z}^m} c_k \exp(i(k\beta\varphi)), \quad (1.1)$$

where

$$c_k = (\langle x_1(\varphi), \exp(-i(k\beta\varphi)) \rangle_\omega, \dots, \langle x_n(\varphi), \exp(-i(k\beta\varphi)) \rangle_\omega)^T / (\omega_1 \dots \omega_m),$$

$$k = (k_1, \dots, k_m) \in \mathbb{Z}^m, \beta = (\beta_1, \dots, \beta_m), \beta_r = 2\pi/\omega_r, (k\beta\varphi) = \sum_{j=1}^m k_j \beta_j \varphi_j.$$

Besides, there exist $K > 0$ and $p \in \mathbb{N}$ such that

$$\|c_k\| \leq K \|k\|^p. \quad (1.2)$$

Inversely, if there exist $K > 0$, $p \in \mathbb{N}$ such that for any $k \in \mathbb{Z}^m$ inequality (1.2) holds, then series (1.1) converges weakly and $x \in P'_n(\omega)$ [6, pp.265,266], [12, p.225], [13, p.130].

It was shown in the previous section that if (0.6) has a continuous periodic solution L , then $\psi(\varphi) = X_0(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1) \exp(-\mu\varphi_1)$ is continuous and periodic. Now suppose that $L \in P'^r_n(\hat{\omega})$ and $X_0 \in P'^r_{n_2}(\omega)$ (for some non-negative integer r); then $\psi \in P'^r_n(\hat{\omega})$ for any $\varphi_1 \in \mathbb{R}$. Likewise (see (0.7)), it can be proved that ψ is ω_1 -periodic in φ_1 . That is why we are going to regard ψ as a distribution given by

$$\langle \psi, y \rangle_\omega = \int_0^{\omega_1} \langle \psi, y \rangle_{\hat{\omega}} d\varphi_1, \quad y \in P'_n(\omega).$$

In this proof we use a standard fact in calculus : *Let $y : [\alpha, \beta] \times K \rightarrow \mathbb{C}^n$ be uniformly continuous for given $\alpha, \beta \in \mathbb{R}$ and $K \subset \mathbb{R}^{m-1}$, let $\{\varphi_{1k}\} \subset [\alpha, \beta]$; then $\varphi_{1k} \rightarrow \varphi_{10}$ implies $y(\varphi_{1k}, \hat{\varphi}) \rightarrow y(\varphi_{10}, \hat{\varphi})$ uniformly for $\hat{\varphi} \in K$ as $k \rightarrow \infty$.*

Lemma 1.1. *Let $A \in P'^r_{n_2}(\omega)$, $L \in P'^r_n(\hat{\omega})$, and $r \geq 0$; then $\psi(\varphi) = X_0(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1) \exp(-\mu\varphi_1) \in P'^r_n(\hat{\omega})$ for any $\varphi_1 \in \mathbb{R}$ and the function $g(\varphi_1) = \langle \psi, y \rangle_{\hat{\omega}}$ is continuous w.r.t. $\varphi_1 \in [0, \omega_1]$ for every $y \in P'_n(\hat{\omega})$.*

Proof. By Lemma 0.2, it follows that $X_0 \in P'^r_{n_2}(\omega)$. Hence, $\psi \in P'^r_n(\hat{\omega})$ for each $\varphi_1 \in \mathbb{R}$. Put $X_{0\mu}(\varphi) = X_0(\varphi) \exp(-\mu\varphi_1)$ and take into account that then $\psi(\varphi) = X_{0\mu}(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1)$ and g can be written as $g(\varphi_1) = \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)y(\hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}}$. The vector $v(\varphi) = X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)y(\hat{\varphi} + \hat{e}\varphi_1)$ and its partial derivatives of $|j|$ th order ($|j| = 1, \dots, r$) are uniformly continuous on $[0, \omega_1] \times \mathbb{R}^{m-1}$. Therefore, if $[0, \omega_1] \ni \varphi_{1k} \rightarrow \varphi_{10} \in [0, \omega_1]$ as $k \rightarrow \infty$, then $v(\varphi_{1k}, \hat{\varphi})$ and $D^j v(\varphi_{1k}, \hat{\varphi})$ tends uniformly w.r.t. $\hat{\varphi} \in \mathbb{R}^{m-1}$ to $v(\varphi_{10}, \hat{\varphi})$ and $D^j v(\varphi_{10}, \hat{\varphi})$, respectively. From this, remembering that L is a continuous mapping, we conclude that $g(\varphi_{1k}) \rightarrow g(\varphi_{10})$, hence g is continuous. \square

Lemma 1.2. *Let $A \in P'^r_{n_2}(\omega)$, $L \in P'^r_n(\hat{\omega})$, and $r \geq 0$; then $\psi \in P'^r_n(\hat{\omega})$.*

Proof. First, by using the fact that the mapping $\langle \cdot, \cdot \rangle_{\hat{\omega}} : P'^r_n(\hat{\omega}) \times P'_n(\hat{\omega}) \rightarrow \mathbb{C}$ is bilinear and separately continuous, we prove that $\langle \psi, y \rangle_{\hat{\omega}}$ is continuous in $\varphi_1 \in [0, \omega_1]$ if $y \in P'_n(\hat{\omega})$.

It follows from Lemma 1.1 that the function $\psi : [0, \omega_1] \rightarrow P'^r_n(\hat{\omega})$ is continuous, i.e. $[0, \omega_1] \ni \varphi_{1k} \rightarrow \varphi_{10}$ implies

$$\langle \psi(\varphi_{1k}, \cdot), z \rangle_{\hat{\omega}} \rightarrow \langle \psi(\varphi_{10}, \cdot), z \rangle_{\hat{\omega}}, \quad (1.3)$$

for every $z \in P'_n(\hat{\omega})$.

Besides, if $y \in P'_n(\hat{\omega})$, then $\|y(\varphi_{1k}, \hat{\varphi}) - y(\varphi_{10}, \hat{\varphi})\|_{P'_n(\hat{\omega})} \rightarrow 0$ since the derivatives $D^j y(\varphi)$, $0 \leq |j| \leq r$ are uniformly continuous on \mathbb{R}^m , and hence

$$\langle x, y(\varphi_{1k}, \cdot) \rangle_{\hat{\omega}} \rightarrow \langle x, y(\varphi_{10}, \cdot) \rangle_{\hat{\omega}}, \quad (1.4)$$

for each $x \in P'^r_n(\hat{\omega})$.

Therefore, by the bilinear mapping continuity theorem [11, Th. 2.17], in view of relations (1.3), (1.4), we get

$$\langle \psi(\varphi_{1k}, \cdot), y(\varphi_{1k}, \cdot) \rangle_{\hat{\omega}} \rightarrow \langle \psi(\varphi_{10}, \cdot), y(\varphi_{10}, \cdot) \rangle_{\hat{\omega}}, \quad y \in P_n^r(\omega).$$

This proves that $\langle \psi, y \rangle_{\hat{\omega}}$ is continuous, and hence integrable w.r.t. $\varphi_1 \in [0, \omega_1]$.

To conclude the proof, we note that

$$\left| \int_0^{\omega_1} \langle \psi, y \rangle_{\hat{\omega}} d\varphi_1 \right| \leq \omega_1 \sup_{\varphi_1 \in [0, \omega_1]} |\langle \psi, y \rangle_{\hat{\omega}}| \leq \omega_1 K \sup_{\varphi_1 \in [0, \omega_1]} \|y\|_{P_n^r(\hat{\omega})} \leq \omega_1 K \|y\|_{P_n^r(\omega)}.$$

The existence of $K = \sup_{\varphi_1 \in [0, \omega_1]} \|\psi\|_{P_n^r(\hat{\omega})}$ follows from the Principle of Uniform Boundedness. Indeed, due to Lemma 1.1, $\langle \psi, y \rangle_{\hat{\omega}}$ is bounded in $\varphi_1 \in [0, \omega_1]$ at each point $y \in P_n^r(\hat{\omega})$. Hence, the collection $\{\langle \psi, \cdot \rangle_{\hat{\omega}} : \varphi_1 \in [0, \omega_1]\}$ of the continuous linear mappings from $P_n^r(\hat{\omega})$ into \mathbb{C} is bounded. The inequality just obtained proves the lemma. \square

We say that $\psi \in P_n^{r'}(\omega)$ is a solution of system (0.2 $_{\mu}$) if

$$\left\langle \sum_{i=1}^m \psi'_i, y \right\rangle_{\omega} = \langle A\psi - \mu\psi, y \rangle_{\omega}, \quad (1.5)$$

for any $y \in P_n^{r+1}(\omega)$.

Lemma 1.3. *Let $A \in P_{n^2}^r(\omega)$, $L \in P_n^r(\hat{\omega})$ be a solution of (0.6) for some $\lambda \neq 0$, $r \geq 1$, and $\mu = \omega_1^{-1} \ln \lambda$; then the distribution $\psi(\varphi) = X_{0\mu}(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1) \in P_n^{r'}(\omega)$ is a solution of system (0.2 $_{\mu}$).*

Proof. It follows from Lemma 1.2 that $\psi \in P_n^{r'}(\omega)$, and hence $\psi'_j \in P_n^{r+1}(\omega)$, $j = 1, \dots, m$.

Let $y \in P_n^{r+1}(\omega)$. We are going to use the relations

$$\begin{aligned} & \int_0^{\omega_1} \langle L(\hat{\varphi} - \hat{e}\varphi_1), (X_{0\mu}^T(\varphi)A^T(\varphi) - \mu X_{0\mu}^T(\varphi))y(\varphi) \rangle_{\hat{\omega}} d\varphi_1 \\ &= \int_0^{\omega_1} \langle (A(\varphi)X_{0\mu}(\varphi) - \mu X_{0\mu}(\varphi))L(\hat{\varphi} - \hat{e}\varphi_1), y(\varphi) \rangle_{\hat{\omega}} d\varphi_1 \\ &= \langle A\psi - \mu\psi, y \rangle_{\omega} \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} \partial X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) / \partial \varphi_1 &= \sum_{i=1}^m (X_{0\mu}^T)_i'(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \\ &= X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)A^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) - \mu X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1). \end{aligned} \quad (1.7)$$

The latter follows from (0.5).

So, remembering that $y \in P_n^{r+1}(\omega)$, we have

$$\begin{aligned} \left\langle \sum_{i=1}^m \psi'_i, y \right\rangle_{\omega} &= -\langle X_{0\mu}(\varphi)L(\hat{\varphi} - \hat{e}\varphi_1), \sum_{i=1}^m y'_i(\varphi) \rangle_{\omega} \\ &= -\int_0^{\omega_1} \langle L(\hat{\varphi} - \hat{e}\varphi_1), X_{0\mu}^T(\varphi) \sum_{i=1}^m y'_i(\varphi) \rangle_{\hat{\omega}} d\varphi_1 \\ &= -\int_0^{\omega_1} \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) y'_1(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1 \\ &\quad - \int_0^{\omega_1} \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \sum_{i=2}^m y'_i(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1. \end{aligned}$$

Denote the first summand at the right in the above relation by I_1 , the second by I_2 , and transform I_1 .

$$\begin{aligned}
 I_1 &= - \int_0^{\omega_1} \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \partial y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) / \partial \varphi_1 \rangle_{\hat{\omega}} d\varphi_1 \\
 &+ \int_0^{\omega_1} \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \sum_{i=2}^m y'_i(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1 \\
 &= \int_0^{\omega_1} \langle L(\hat{\varphi}), \partial X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) / \partial \varphi_1 y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1 \\
 &- \int_0^{\omega_1} \langle L(\hat{\varphi}), \partial (X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)) / \partial \varphi_1 \rangle_{\hat{\omega}} d\varphi_1 \\
 &+ \int_0^{\omega_1} \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \sum_{i=2}^m y'_i(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1, \\
 I_2 &= - \int_0^{\omega_1} \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \sum_{i=2}^m y'_i(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 I_1 + I_2 &= \int_0^{\omega_1} \langle L(\hat{\varphi}), \partial X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) / \partial \varphi_1 y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1 \\
 &- \int_0^{\omega_1} \langle L(\hat{\varphi}), \partial (X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)) / \partial \varphi_1 \rangle_{\hat{\omega}} d\varphi_1.
 \end{aligned}$$

We denote the first integral in the previous relation by J_1 [and the second by J_2] and transform it using (1.7):

$$\begin{aligned}
 J_1 &= \int_0^{\omega_1} \langle L(\hat{\varphi}), (X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) A^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \\
 &- \mu X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)) y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} d\varphi_1 = \int_0^{\omega_1} \langle L(\hat{\varphi} - \hat{e}\varphi_1), (X_{0\mu}^T(\varphi) A^T(\varphi) \\
 &- \mu X_{0\mu}^T(\varphi)) y(\varphi) \rangle_{\hat{\omega}} d\varphi \stackrel{(1.6)}{=} \langle A\psi - \mu\psi, y \rangle_{\omega}.
 \end{aligned}$$

To transform J_2 , we use differentiation of $\langle \cdot, \cdot \rangle_{\hat{\omega}}$ w.r.t. parameter φ_1 [12, p.105]:
 $\langle L(\hat{\varphi}), \partial (X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1)) / \partial \varphi_1 \rangle_{\hat{\omega}}$
 $= \partial \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} / \partial \varphi_1.$

As a result we get

$$\begin{aligned}
 J_2 &= \int_0^{\omega_1} \partial \langle L(\hat{\varphi}), X_{0\mu}^T(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) y(\varphi_1, \hat{\varphi} + \hat{e}\varphi_1) \rangle_{\hat{\omega}} / \partial \varphi_1 d\varphi_1 \\
 &= \langle L(\hat{\varphi}), (X_{0\mu}^T(\omega_1, \hat{\varphi} + \hat{e}\omega_1) y(\omega_1, \hat{\varphi} + \hat{e}\omega_1) - X_{0\mu}^T(0, \hat{\varphi}) y(0, \hat{\varphi})) \rangle_{\hat{\omega}} \\
 &= \langle X_{0\mu}(\omega_1, \hat{\varphi}) L(\hat{\varphi} - \hat{e}\omega_1) - X_{0\mu}(0, \hat{\varphi}) L(\hat{\varphi}), y(0, \hat{\varphi}) \rangle_{\hat{\omega}}.
 \end{aligned}$$

Remembering that L satisfies (0.6), $X_{0\mu}(\omega_1, \hat{\varphi}) = X_0(\omega_1, \hat{\varphi})\lambda^{-1}$, and $X_{0\mu}(0, \hat{\varphi}) = X_0(0, \hat{\varphi}) = E$, we get $J_2 = 0$.

Summarizing all the above, we get (1.5). □

2. GENERALIZED SOLUTIONS OF SYSTEMS (0.6) AND (0.2_μ)

Consider the system

$$X_0(\omega_1, \hat{\varphi}) L(\hat{\varphi} - \alpha_s) = \lambda L(\hat{\varphi}), \tag{2.1}$$

where $\alpha_s = (p_{2s}\omega_2/q_{2s}, \dots, p_{ms}\omega_m/q_{ms})$, $p_{js} \in \mathbb{Z}$, $q_{js} \in \mathbb{N}$. Due to the incommensurability of the frequencies $\beta_i = 2\pi/\omega_i$ we can choose p_{js}, q_{js} such that $\alpha_s \rightarrow \hat{e}\omega_1$ as $s \rightarrow \infty$.

We have now to prove that, for each $s \in \mathbb{N}$, there exists a distribution L_s which solves this system for some $\lambda_s \in \mathbb{C}$ and that the sequence $\{L_s\}$ has a weak partial limit L which satisfies (0.6) for some $\lambda \in \mathbb{C}$.

Lemma 2.1. *Let $A \in P_{n^2}^0(\omega)$; then for any $s \in \mathbb{N}$ there exists a distribution $L_s \in P_n^0(\hat{\omega})$ and a complex number $\lambda_s \neq 0$ which satisfy (2.1).*

Proof. Let us endeavour to satisfy (2.1) by the distribution

$$L_s(\hat{\varphi}) = \sum_{j=0}^{\nu_s-1} f_{sj} \delta(\hat{\varphi} - j\alpha_s), \quad (2.2)$$

where ν_s is the least common multiple of q_{j_s} ($j = 2, \dots, m$), f_{sj} are constant column vectors, $\delta \in P_1^0(\hat{\omega})$ is the Dirac distribution: $\langle f_{sj} \delta(\hat{\varphi} - \hat{\varphi}^0), y \rangle_{\hat{\omega}} = f_{sj}^T y(\hat{\varphi}^0)$, $y \in P_n^0(\hat{\omega})$.

Note that in our case δ is a periodic functional since it is defined on the space of periodic functions.

Recall that L_s satisfies (2.1) if, for any $y \in P_n^0(\hat{\omega})$, the following relation holds

$$\langle X_0(\omega_1, \hat{\varphi}) L_s(\hat{\varphi} - \alpha_s), y \rangle_{\hat{\omega}} = \langle \lambda_s L_s(\hat{\varphi}), y \rangle_{\hat{\omega}}.$$

Upon substituting L_s into the left and right sides of this equation, we find

$$\begin{aligned} \langle X_0(\omega_1, \hat{\varphi}) L_s(\hat{\varphi} - \alpha_s), y(\hat{\varphi}) \rangle_{\hat{\omega}} &= \langle L_s(\hat{\varphi} - \alpha_s), (X_0(\omega_1, \hat{\varphi}))^T y(\hat{\varphi}) \rangle_{\hat{\omega}} = \\ \langle L_s(\hat{\varphi}), (X_0(\omega_1, \hat{\varphi} + \alpha_s))^T y(\hat{\varphi} + \alpha_s) \rangle_{\hat{\omega}} &= \sum_{j=0}^{\nu_s-1} \langle f_{sj} \delta(\hat{\varphi} - j\alpha_s), (X_0(\omega_1, \hat{\varphi} + \alpha_s))^T y(\hat{\varphi} + \\ \alpha_s) \rangle_{\hat{\omega}} &= \\ \sum_{j=0}^{\nu_s-1} f_{sj}^T (X_0(\omega_1, (j+1)\alpha_s))^T y((j+1)\alpha_s) &= \sum_{j=0}^{\nu_s-1} (X_0(\omega_1, (j+1)\alpha_s) f_{sj})^T y((j+1)\alpha_s), \\ \text{and } \langle \lambda_s L_s(\hat{\varphi}), y(\hat{\varphi}) \rangle_{\hat{\omega}} &= \lambda_s \sum_{j=0}^{\nu_s-1} \langle f_{sj} \delta(\hat{\varphi} - j\alpha_s), y(\hat{\varphi}) \rangle_{\hat{\omega}} = \\ \lambda_s \sum_{j=0}^{\nu_s-1} f_{sj}^T y(j\alpha_s). \end{aligned}$$

Thus

$$\sum_{j=0}^{\nu_s-1} (X_0(\omega_1, (j+1)\alpha_s) f_{sj})^T y((j+1)\alpha_s) = \lambda_s \sum_{j=0}^{\nu_s-1} f_{sj}^T y(j\alpha_s).$$

This relation holds if the vectors f_{sj} satisfy the system

$$\begin{cases} X_0(\omega_1, \alpha_s) f_{s0} & = & \lambda_s f_{s1} \\ X_0(\omega_1, 2\alpha_s) f_{s1} & = & \lambda_s f_{s2} \\ \dots & \dots & \dots \\ X_0(\omega_1, (\nu_s - 1)\alpha_s) f_{s\nu_s-2} & = & \lambda_s f_{s\nu_s-1} \\ X_0(\omega_1, \nu_s \alpha_s) f_{s\nu_s-1} & = & \lambda_s f_{s0}. \end{cases} \quad (2.3)$$

Indeed, let $y \in P_n^0(\hat{\omega})$. Then it follows from (2.3) that

$$\begin{cases} (X_0(\omega_1, \alpha_s) f_{s0})^T y(\alpha_s) & = & \lambda_s (f_{s1})^T y(\alpha_s) \\ (X_0(\omega_1, 2\alpha_s) f_{s1})^T y(2\alpha_s) & = & \lambda_s (f_{s2})^T y(2\alpha_s) \\ \dots & \dots & \dots \\ (X_0(\omega_1, (\nu_s - 1)\alpha_s) f_{s\nu_s-2})^T y((\nu_s - 1)\alpha_s) & = & \lambda_s (f_{s\nu_s-1})^T y((\nu_s - 1)\alpha_s) \\ (X_0(\omega_1, \nu_s \alpha_s) f_{s\nu_s-1})^T y(\nu_s \alpha_s) & = & \lambda_s (f_{s0})^T y(\nu_s \alpha_s). \end{cases}$$

Putting $y(\nu_s \alpha_s) = y(\hat{0})$ in the right-hand side of the last equation and summing these equations we obtain the desired relation.

It follows from (2.3) that

$$X_0(\omega_1, \nu_s \alpha_s) X_0(\omega_1, (\nu_s - 1)\alpha_s) \dots X_0(\omega_1, \alpha_s) f_{s0} = \lambda_s^{\nu_s} f_{s0}. \quad (2.4)$$

Consider some root $G_s = (X_0(\omega_1, \nu_s \alpha_s) X_0(\omega_1, (\nu_s - 1) \alpha_s) \dots X_0(\omega_1, \alpha_s))^{\frac{1}{\nu_s}}$, some eigenvalue λ_s of G_s , and the corresponding eigenvector f_{s0} ; then λ_s and f_{s0} satisfy (2.4). Substituting λ_s and f_{s0} into (2.3), we successively find vectors $f_{sk}, k = 1, \dots, \nu_s - 1$. Hence, we get the distribution L_s which is the solution of (2.1). It can be seen from (2.2) that $L_s \in P_n^0(\hat{\omega})$ since $f_{sj} \delta(\hat{\varphi} - j \alpha_s) \in P_n^0(\hat{\omega})$. \square

Lemma 2.2. *Let $A \in P_n^0(\omega)$ and let λ_s be an eigenvalue of G_s , $s \in \mathbb{N}$; then there exists a convergent subsequence $\{\lambda_{s_k}\}$ of the sequence $\{\lambda_s\}$ with nonzero limit.*

Proof. The matrix function $X_0(\omega_1, \cdot)$ is non-singular continuous and $\hat{\omega}$ -periodic in $\hat{\varphi} \in \mathbb{R}^{m-1}$. Hence, there exist $\alpha, \beta > 0$ such that for any $x \in \mathbb{R}^n$, $\hat{\varphi} \in \mathbb{R}^{m-1}$ we have $\alpha \|x\| \leq \|X_0(\omega_1, \hat{\varphi})x\| \leq \beta \|x\|$. From these inequalities we get $\|G_s^{\nu_s}\| \leq \beta^{\nu_s}$, $\|G_s^{-\nu_s}\| \leq \alpha^{-\nu_s}$ (the matrix norm induced by the vector norm $\|\cdot\|$ is also denoted by $\|\cdot\|$). Since $\lambda_s^{\nu_s}$ and $\lambda_s^{-\nu_s}$ are eigenvalues of the matrices $G_s^{\nu_s}$ and $G_s^{-\nu_s}$ respectively, it follows that $|\lambda_s^{\nu_s}| \leq \beta^{\nu_s}$, $|\lambda_s^{-\nu_s}| \leq \alpha^{-\nu_s}$ and, finally, $\alpha \leq |\lambda_s| \leq \beta$, $s \in \mathbb{N}$. On the strength of that there exists a partial limit λ of $\{\lambda_s\}$ such that $\alpha \leq |\lambda| \leq \beta$. Hence, the desired subsequence exists. \square

Now we want to show that there exist sequences $\{\lambda_s\}$ and $\{L_s\}$ such that $\{L_s\}$ has a weak limit L which is a solution of (0.6) with $\lambda = \lim \lambda_s$.

Consider the Fourier-series of L_s :

$$L_s(\hat{\varphi}) = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{s\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi})),$$

$$t_{s\hat{k}} = (\omega_2 \dots \omega_m)^{-1} \sum_{j=0}^{\nu_s-1} f_{sj} \exp(-ij(\hat{k}\hat{\beta}\alpha_s)), \quad (\hat{k}\hat{\beta}\alpha_s) = \sum_{r=2}^m k_r \beta_r p_{sr} \omega_r / q_{sr}.$$

Lemma 2.3. *For each $s \in \mathbb{N}$ the mapping $\hat{k} \rightarrow t_{s\hat{k}}$ is ν_s -periodic w.r.t. k_r ($r = 2, \dots, m$).*

Proof. Recall that $\hat{k} = (k_2, \dots, k_m)$, $\beta_r = 2\pi/\omega_r$, ν_s is the least common multiple of q_{rs} , and $\alpha_s = (p_{2s}\omega_2/q_{2s}, \dots, p_{ms}\omega_m/q_{ms})$. Since $\nu_s/q_{rs} \in \mathbb{Z}$ ($r = 2, \dots, m$), we have

$$\begin{aligned} t_{s(k_2, \dots, k_r + \nu_s, \dots, k_m)} &= \sum_{j=0}^{\nu_s-1} f_{sj} \exp(-ij(\hat{k}\hat{\beta}\alpha_s + \nu_s(2\pi/\omega_r)p_{rs}\omega_r/q_{rs})) \\ &= \sum_{j=0}^{\nu_s-1} f_{sj} \exp(-ij(\hat{k}\hat{\beta}\alpha_s)) = t_{s\hat{k}}. \end{aligned}$$

\square

Lemma 2.4. *For any $\hat{j} \in \mathbb{Z}^{m-1}$, side by side with L_s , λ_s , system (2.1) has the solution $L_s^{\hat{j}}(\hat{\varphi}) = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{s\hat{k}-\hat{j}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ for $\lambda_s^{\hat{j}} = \lambda_s \exp(-i(\hat{j}\hat{\beta}\alpha_s))$.*

Proof. Making a change $\hat{k} - \hat{j} = \hat{r}$ in the Fourier-series of $L_s^{\hat{j}}$, we obtain $L_s^{\hat{j}} = L_s \exp(i(\hat{\beta}\hat{j}\hat{\varphi}))$. Then, putting $L_s = L_s^{\hat{j}} \exp(-i(\hat{\beta}\hat{j}\hat{\varphi}))$ into (2.1), we conclude that the lemma is true. \square

Lemma 2.5. *Let, for any $\hat{k} \in \mathbb{Z}^{m-1}$, there exists $t_{\hat{k}} = \lim_{s \rightarrow \infty} t_{s\hat{k}}$, and let there exists $C > 0$ such that for all $s \in \mathbb{N}$, $\hat{k} \in \mathbb{Z}^{m-1}$ inequality $\|t_{s\hat{k}}\| \leq C$ holds. Then*

$L_s \rightarrow L = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ and $L_s(\hat{\varphi} - \alpha_s) \rightarrow L(\hat{\varphi} - e\omega_1)$ weakly in $P_n^r(\hat{\omega})$ for $r \geq [\frac{m-1}{2}] + 1$ ($[t]$ stands for the integral part of t).

Proof. Let $y \in P_n^r(\hat{\omega})$, y_k be the Fourier constants of y , and $r \geq [\frac{m-1}{2}] + 1$; then $\sum_{\hat{k} \in \mathbb{Z}^{m-1}} \|y_{\hat{k}}\|$ converges [5]. We say that $\langle L_s, y \rangle_{\hat{\omega}} = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{s\hat{k}}^T y_{-\hat{k}}$ and $\langle L, y \rangle_{\hat{\omega}} = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{\hat{k}}^T y_{-\hat{k}}$. The proofs of these relations are similar since the boundedness of $\{t_{s\hat{k}}\}$ implies $\|t_{\hat{k}}\| \leq C$ for $\hat{k} \in \mathbb{Z}^{m-1}$, so we prove the first one. It holds if $y = y^h = \sum_{\|\hat{k}\| \leq h} y_{\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ is a trigonometrical polynomial [13, p.132], besides, the series $\sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{s\hat{k}}^T y_{-\hat{k}}$ converges because $|t_{s\hat{k}}^T y_{-\hat{k}}| \leq C\|y_{-k}\|$. Therefore, we can take the limit in $\langle L_s, y^h \rangle_{\hat{\omega}} = \sum_{\|\hat{k}\| \leq h} t_{s\hat{k}}^T y_{-\hat{k}}$ as $h \rightarrow \infty$ and get the desired relation.

To prove the weak convergence of $\{L_s\}$ to $L = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} t_{\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$, we consider $\langle L_s - L, y \rangle_{\hat{\omega}} = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} (t_{s\hat{k}} - t_{\hat{k}})^T y_{-\hat{k}}$. For given $\varepsilon > 0$ we find M_1, M_2 such that $\sum_{\|\hat{k}\| > M_1} \|y_{\hat{k}}\| < \varepsilon/(4C)$ and $\sum_{\|\hat{k}\| \leq M_1} \|t_k - t_{ks}\| < \varepsilon/(2C_1)$ for $s > M_2$, where $C_1 = \max\{\|y_{\hat{k}}\| : \|\hat{k}\| < M_1\}$, then

$$\left| \sum_{\hat{k} \in \mathbb{Z}^{m-1}} (t_{s\hat{k}} - t_{\hat{k}})^T y_{-\hat{k}} \right| < C_1 \sum_{\|\hat{k}\| \leq M_1} \|t_k - t_{ks}\| + 2C \sum_{\|\hat{k}\| > M_1} \|y_{\hat{k}}\| < \varepsilon$$

for $s > M_2$. Hence $L_s \rightarrow L$ weakly as $s \rightarrow \infty$. By the same argument we have

$$\begin{aligned} \langle L_s(\hat{\varphi} - \alpha_s) - L(\hat{\varphi} - e\omega_1), y \rangle_{\hat{\omega}} &= \\ \sum_{\hat{k} \in \mathbb{Z}^{m-1}} (t_{s\hat{k}} \exp(-i(\hat{k}\hat{\beta}\alpha_s)) - t_{\hat{k}} \exp(-i(\hat{k}\hat{\beta}\omega_1)))^T y_{-\hat{k}} &\rightarrow 0 \end{aligned}$$

as $s \rightarrow \infty$.

Owing to the weak* sequential completeness of $P_n^r(\hat{\omega})$, we obtain $L \in P_n^r(\hat{\omega})$ since $L_s \in P_n^{r_0}(\hat{\omega}) \subset P_n^r(\hat{\omega})$. \square

Theorem 2.1. *Let $A \in P_{n_2}^{r_0}(\omega)$, $r_0 = [\frac{m-1}{2}] + 1$; then there exist a distribution $L \in P_n^{r_0}(\hat{\omega})$ and $\lambda \neq 0$ satisfying (0.6).*

Proof. We claim that there exists a sequence $\{L_s\}$ such that for Fourier constants of L_s the inequalities

$$\|t_{s\hat{0}}\| = 1 \geq \|t_{s\hat{k}}\|, \hat{k} \neq \hat{0}, \quad (2.5)$$

are true for any $s \in \mathbb{N}$. Indeed, due to Lemma 2.3, for each $s \in \mathbb{N}$, there exists $t_{s\hat{k}_s}$ such that $\|t_{s\hat{k}_s}\| \geq \|t_{s\hat{k}}\|$, $\hat{k}_s \neq \hat{k}$. Owing to Lemma 2.4, the distribution $L_s^{-\hat{k}_s} / \|t_{s\hat{k}_s}\| = \sum_{\hat{k} \in \mathbb{Z}^{m-1}} \tilde{t}_{s\hat{k}} \exp(i(\hat{k}\hat{\beta}\hat{\varphi}))$ satisfies (2.1) for $\lambda = \lambda_s \exp(i(\hat{k}_s\hat{\beta}\alpha_s))$; then $\tilde{t}_{s\hat{k}} = t_{s\hat{k} + \hat{k}_s} / \|t_{s\hat{k}_s}\|$ satisfy (2.5).

Let the Fourier constants of L_s satisfy (2.5). Taking a subsequence if necessary, we may assume in view of Lemma 2.2 that $\lambda_s \rightarrow \lambda \neq 0$. It follows from (2.5) that for each $\hat{k} \in \mathbb{Z}^{m-1}$ the sequence $\{t_{s\hat{k}}\}$ is bounded. Hence, there exists a sequence $\{s_p\}$ of positive integers such that $\lim_{p \rightarrow \infty} t_{s_p \hat{k}} = t_k$ exists and $\|t_k\| \leq 1$ for each

$\hat{k} \in \mathbb{Z}^{m-1}$. By Lemma 2.5, we get $L_{s_p} \rightarrow L$ weakly in $P_n^{r_0}(\hat{\omega})$. Besides, by Lemma 0.2, $X_0(\omega_1, \cdot) \in P_n^{r_0}(\hat{\omega})$ therefore $X_0(\omega_1, \cdot)L_{s_p} \rightarrow X_0(\omega_1, \cdot)L$ weakly in $P_n^{r_0}(\hat{\omega})$. Then taking limit in (2.1), we get $X_0(\omega_1, \hat{\varphi})L(\hat{\varphi} - \hat{e}\varphi_1) = \lambda L(\hat{\varphi})$, moreover, $L \neq 0$ since $t_{\hat{\varphi}} \neq 0$. \square

Consider the set of sequences $\{\lambda_s\}$, where λ_s is an eigenvalue of G_s and denote by Λ the set of partial limits of these sequences.

Theorem 2.2. *Let $\lambda \in \Lambda$, $\mu = \omega_1^{-1} \ln \lambda$, and $A \in P_n^{r_0}(\omega)$; then there exists a periodic distribution $\psi \in P_n^{r_0}(\omega)$ that satisfies (0.2 $_{\mu}$).*

Proof. If $A \in P_n^{r_0}(\omega)$, then, by Theorem 2.1, there exists $L \in P_n^{r_0}(\hat{\omega})$ satisfying (0.6). Therefore, Theorem 2.2 follows from Lemma 1.3. \square

3. PROOF OF LEMMA 0.1

Proof. System (0.5) has the unique continuous solution

$$\begin{aligned} X(\varphi; \varphi_{10}) &= E + \int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi) d\xi \\ &+ \sum_{k=2}^{\infty} \int_0^{\varphi_1 - \varphi_{10}} \int_0^{\varphi_1 - \varphi_{10} - \xi_k} \dots \int_0^{\varphi_1 - \varphi_{10} - \xi_2 - \dots - \xi_k} A(\varphi - e\xi_k) A(\varphi - e(\xi_k + \xi_{k-1})) \dots \\ &A\left(\varphi - e \sum_{i=1}^k \xi_i\right) d\xi_1 \dots d\xi_k \end{aligned} \quad (3.1)$$

which one can get by the method of successive approximations. This series converges uniformly w.r.t. $\varphi_1 \in [\alpha, \beta]$, $\hat{\varphi} \in \mathbb{R}^{m-1}$, where $\alpha, \beta \in \mathbb{R}$ are arbitrary, because it has the majorant

$$\sum_{s=0}^{\infty} \|A\|_{P_n^{r_0}(\omega)}^s |\varphi_1 - \varphi_{10}|^s / s!$$

This solution is ω_j -periodic in φ_j ($j = 2, \dots, m$) since so is the right part of (3.1). Besides,

$$\begin{aligned} X(\varphi + et; \varphi_{10}) &= \int_0^{\varphi_1 + t - \varphi_{10}} A(\varphi + e(t - \xi)) X(\varphi + e(t - \xi); \varphi_{10}) d\xi + E \\ &= [t - \xi = \zeta] = \int_{\varphi_{10} - \varphi_1}^t A(\varphi + e\zeta) X(\varphi + e\zeta; \varphi_{10}) d\zeta + E. \end{aligned}$$

Therefore,

$$dX(\varphi + et; \varphi_{10})/dt = A(\varphi + et)X(\varphi + et; \varphi_{10}) \quad (3.2)$$

and $X(\varphi_{10}, \hat{\varphi}; \varphi_{10}) = E$. Consequently, the matrix $X(\varphi + et, \varphi_{10})$, being a fundamental matrix of the system

$$x' = A(\varphi + et)x,$$

is non-singular for $\varphi \in \mathbb{R}^m$ and $\varphi_{10} \in \mathbb{R}$. In particular, $X_0(et)$ is the normalized fundamental matrix of system (0.1 $_0$).

The fact that the function $x(\varphi; \varphi_{10}, x^0) = X(\varphi; \varphi_{10})x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))$ is a solution of system (0.3) we prove by substituting it into this system. Taking into

account that $x(\varphi - e\xi; \varphi_{10}, x^0) = X(\varphi - e\xi; \varphi_{10})x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))$ we get

$$\begin{aligned} X(\varphi; \varphi_{10})x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})) &= \int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi)X(\varphi - e\xi; \varphi_{10})x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})) d\xi \\ + x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})) &= \left(\int_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi)X(\varphi - e\xi; \varphi_{10}) d\xi + E \right) x^0(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})), \end{aligned}$$

which is true since $X(\varphi; \varphi_{10})$ satisfies (0.4).

To prove (c) note that $X(t + \omega_1, \hat{\varphi} + \hat{e}t; \varphi_{10})$ solves matrix equation (3.2). Hence,

$$X(t + \omega_1, \hat{\varphi} + \hat{e}t; \varphi_{10}) = X(t, \hat{\varphi} + \hat{e}t; \varphi_{10})Q. \quad (3.3)$$

Putting $t = \varphi_{10}$ in this relation we get $Q = X(\varphi_{10} + \omega_1, \hat{\varphi} + \hat{e}\varphi_{10}; \varphi_{10})$. Then substituting φ_1 for t and $\hat{\psi}$ for $\hat{\varphi} + \hat{e}\varphi_1$ in (3.3) we get $X(\varphi_1 + \omega_1, \hat{\psi}; \varphi_{10}) = X(\varphi_1, \hat{\psi}; \varphi_{10})X(\varphi_{10} + \omega_1, \hat{\psi} + \hat{e}(\varphi_{10} - \varphi_1); \varphi_{10})$. \square

Using the termwise differentiation of (3.1), Lemma 0.2 can be proved in a standard way, so the proof is omitted.

It is noteworthy that if $\lambda_0 \in \Lambda$ and $\mu_0 = \omega_1^{-1} \ln \lambda_0$, then some system in the hull of (0.1_{μ_0}) has a bounded solution. It follows from this that μ_0 belongs to the Sacker-Sell spectrum of this system and λ_0 belongs to the spectrum of the monodromy operator of this system. Besides, system (0.1_{μ_0}) has a generalized conditionally periodic solution. But these will be the objects of another paper.

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