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A RADON TYPE TRANSFORM RELATED TO THE EULER EQUATIONS FOR IDEAL FLUID

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Abstract. We study the Nadirashvili – Vladuts transform $\mathcal{N}$ that integrates second rank tensor fields f on $\mathbb{R}^n$ over hyperplanes. More precisely, for a hyperplane P and vector η parallel to P, $\mathcal{N}f(P,\eta)$ is the integral of the function $f_{ij}(x)\xi^i\eta^j$ over P, where ξ is the unit normal vector to P. We prove that, given a vector field v, the tensor field $f = v\otimes v$ belongs to the kernel of $\mathcal{N}$ if and only if there exists a function p such that (v,p) is a solution to the Euler equations. Then we study the Nadirashvili – Vladuts potential $w(x,\xi)$ determined by a solution to the Euler equations. The function w solves some 4th order PDE. We describe all solutions to the latter equation.

Keywords: Euler equations, Nadirashvili – Vladuts transform, tensor tomography.

1. Introduction

In dimensions 2 and 3, the Euler equations

1.1 (1)
$$\sum_{j=1}^{n} v_j\frac{\partial v_i}{\partial x_j}+\frac{\partial p}{\partial x_i}=0\quad (i=1,\dots,n),$$

1.2 (2)
$$\operatorname{div} v=\sum_{i=1}^{n}\frac{\partial v_i}{\partial x_i}=0$$

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describe steady flows of ideal incompressible fluid. The equations are also of some mathematical interest in an arbitrary dimension. Here $v = \big(v_1(x),\dots,v_n(x)\big)$ is a vector field on $\mathbb{R}^n$ (the fluid velocity) and p is a scalar function on $\mathbb{R}^n$ (the pressure). Only real solutions (v,p) are physically sensible. Nevertheless, all our results are valid for solutions with complex-valued functions v_i and p. We have to consider complex-valued functions and vector fields since we use the Fourier transform.

We consider only solutions (v,p) to the Euler equations (1)–(2) which are defined on the whole of $\mathbb{R}^n$, are sufficiently smooth, and satisfy some decay conditions at infinity. The reader can easily find minimal regularity and decay conditions for every statement below. To simplify the presentation, we will always assume the functions v_i $(i=1,\dots,n)$ and p to belong to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth functions fast decaying at infinity together with all derivatives (the term "smooth" is used as a synonym of "C^∞-smooth").

One can ask: Do there exist non-trivial (i.e., not identically equal to zero) solutions to the Euler equations such that v_i and p belong to $\mathcal{S}(\mathbb{R}^n)$? The answer to the question is "yes". Moreover, there exist non-trivial solutions such that v_i and p belong to the space $C_0^\infty(\mathbb{R}^n)$ of smooth compactly supported functions. In the case of any even dimension n, an example of such a solution is presented in [2, 7]. In the case of $n=3$, the existence of such a solution is proved in the breakthrough article [3] by Gavrilov, see also [1, 7]. We guess (although have not proven) such a solution exists in any odd dimension.

Let $\langle\cdot,\cdot\rangle$ be the standard dot-product on $\mathbb{R}^n$ and $|\cdot|$, the corresponding norm. Let $\mathbb{S}^{n-1}=\{\xi\in\mathbb{R}^n \mid |\xi|=1\}$ be the unit sphere. To our knowledge, the following observation belongs to Nadirasvili – Vladuts [5]. Let (v,p) be a solution to the Euler equations (1)–(2) such that the functions v_i $(i=1,\dots,n)$ and p belong to $\mathcal{S}(\mathbb{R}^n)$. Then, for every $(\xi,q)\in\mathbb{S}^{n-1}\times\mathbb{R}$ and for every vector $\eta\in\mathbb{R}^n$ satisfying $\langle\xi,\eta\rangle=0$,

1.3
$$\int\limits_{\langle\xi,x\rangle=q}\langle v(x),\xi\rangle\langle v(x),\eta\rangle\,dx=0, \tag{3}$$

where dx is the $(n-1)$-dimensional Lebesgue measure on the hyperplane $\langle\xi,x\rangle=p$. For a fixed $(\xi,q)\in\mathbb{S}^{n-1}\times\mathbb{R}$, (3) involves $n-1$ linearly independent equations since η belongs to the $(n-1)$-dimensional space $\xi^\perp=\{\eta\in\mathbb{R}^n \mid \langle\xi,\eta\rangle=0\}$. An easy proof of (3) is presented at the beginning of the next section for the sake of completeness.

For $\xi\in\mathbb{S}^{n-1}$, let $P_\xi:\mathbb{R}^n\to\xi^\perp$ be the orthogonal projection, it is expressed by $P_\xi\eta=\eta-\langle\xi,\eta\rangle\xi$. In (3), we can replace $\eta\in\xi^\perp$ with $P_\xi\eta$ for an arbitrary $\eta\in\mathbb{R}^n$

1.4
$$\int\limits_{\langle\xi,x\rangle=q}\langle v(x),\xi\rangle\langle v(x),P_\xi\eta\rangle\,dx=0\quad\big((\xi,\eta,q)\in\mathbb{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}\big). \tag{4}$$

We will treat this equation for $|\eta|=1$ since we are going to integrate with respect to η.

Let $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)$ be the Schwartz space of (complex-valued) vector fields $v:\mathbb{R}^n\to\mathbb{C}^n$ and $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$ be the Schwartz space of functions $f:\mathbb{R}^n\to\mathbb{C}^n\otimes\mathbb{C}^n$. Elements of the latter space are called second rank (smooth fast decaying) tensor fields on $\mathbb{R}^n$. More generally, for a smooth vector bundle $E\to M$ over a smooth compact manifold, the Schwartz space $\mathcal{S}(E)$ of functions on E can be defined with the help of a finite atlas and partition of unity subordinate to the atlas. In particular, for the trivial vector bundle $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}\to\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, we have the well defined Schwartz space $\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R})$ of functions $\varphi(\xi,\eta,q)$ fast decaying in q.

The Schwartz spaces $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$ and $\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R})$ are furnished with corresponding topologies.

Introducing the tensor field $f\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$ by $f_{ij}=v_iv_j$, we write (4) as

1.5
$$\int\limits_{\langle\xi,x\rangle=q} f_{ij}(x)\xi^i(P_\xi\eta)^j\,dx=0\quad\big((\xi,\eta,q)\in\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}\big). \tag{5}$$

We use the Einstein summation rule: the summation from 1 to n is assumed over every index repeated in lower and upper positions in a monomial. To adopt our formulas to the summation rule, we use either lower or upper indices for denoting coordinates of vectors and tensors. For instance, $\xi^i=\xi_i$ in (5). There is no difference between covariant and contravariant tensors since we use Cartesian coordinates only.

D1.1 **Definition 1.** *The linear continuous operator*

1.6
$$\mathcal{N}:\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)\to\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R}) \tag{6}$$

defined by

1.7
$$(\mathcal{N}f)(\xi,\eta,q)=\int\limits_{\langle\xi,x\rangle=q} f_{ij}(x)\xi^i(P_\xi\eta)^j\,dx \tag{7}$$

will be called the Nadirashvili – Vladuts transform.

Thus, given a solution $(v,p)\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)\times\mathcal{S}(\mathbb{R}^n)$ to the Euler equations (1)–(2), the tensor field $f=v\otimes v\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$ belongs to the kernel of the operator $\mathcal{N}$. Our first main result is the converse statement.

Th1.1 **Theorem 1.** *Given a divergence-free vector field $v\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)$, the tensor field $f=v\otimes v$ satisfies $\mathcal{N}f=0$ if and only if there exists a function $p\in\mathcal{S}(\mathbb{R}^n)$ such that (v,p) is a solution to the Euler equations* (1).

By the definition (7), the operator $\mathcal{N}$ integrates $f_{ij}(x)\xi^i(P_\xi\eta)^j$ over hyperplanes. Therefore $\mathcal{N}$ is called "a Radon type transform" in the title of our article. But actually, at least in the 3D case, $\mathcal{N}$ is closely related to *the ray transform* that integrates symmetric tensor fields over lines. The relationship is encoded in some function $w(x,\xi)$ that will be called *the Nadirashvili – Vladuts potential.* It was introduced in [5]. We give an alternative definition of w in Proposition 2 below. In our opinion, Proposition 2 gives a better understanding of the relationship between the Nadirashvili – Vladuts transform and ray transform.

The Nadirashvili – Vladuts potential satisfies some 4th order PDE [5, equation (4.5)]. We write the equation in a little bit different form (see the equation (71) below) and present an alternative proof. Our second main result is Theorem 3 below which describes all solutions of the equation (71). The general solution depends on two arbitrary functions.

To author's knowledge, only one example of a solution (v,p) of the Euler equitions (1)–(2) is known so far such that the functions v_i $(i=1,2,3)$ and p belong to $\mathcal{S}(\mathbb{R}^3)$ [3]. Probably such solutions can be classified. Theorem 3 can be considered as the first step toward such a classification.

2. Proof of Theorem 1

We first prove the statement "if" of Theorem 1. Let $(v,p)\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)\times\mathcal{S}(\mathbb{R}^n)$ be a solution to the Euler equations (1)–(2). It suffices to prove (3) in the case when the hyperplane $\langle\xi,x\rangle=q$ coincides with the coordinate hyperplane $x_n=0$. Indeed, the Euler equations are invariant under a change of Cartesian coordinates while the equation (3) is independent of the coordinate choice. Given a hyperplane P, we can choose Cartesian coordinates so that P coincides with the coordinate hyperplane $x_n=0$.

In virtue of the incompressibility equation (3) the Euler equations (1) can be written in the divergent form

2.1 (8)
$$\sum_{j=1}^{n}\frac{\partial(v_iv_j)}{\partial x_j}+\frac{\partial p}{\partial x_i}=0\quad(i=1,\dots,n),\tag{8}$$

Distinguishing the last summand of the sum, we rewrite (8) in the form

$$\sum_{j=1}^{n-1}\frac{\partial(v_iv_j)}{\partial x_j}+\frac{\partial(v_iv_n)}{\partial x_n}+\frac{\partial p}{\partial x_i}=0\quad(i=1,\dots,n-1).$$

Integrating this equation with respect to $x_1,\dots,x_{n-1}$, we obtain

$$\frac{d}{dx_n}\int\limits_{\mathbb{R}^{n-1}}v_i(x)v_n(x)\,dx_1\dots dx_{n-1}=0\quad(i=1,\dots,n-1).$$

The function $\varphi(x_n)=\int_{\mathbb{R}^{n-1}}v_i(x)v_n(x)\,dx_1\dots dx_{n-1}$ belongs to $\mathcal{S}(\mathbb{R})$. The derivative of such a function is identically equal to zero iff the function itself is identically equal to zero. Hence

$$\int\limits_{\mathbb{R}^{n-1}}v_i(x)v_n(x)\,dx_1\dots dx_{n-1}=0\quad(i=1,\dots,n-1).$$

This is equivalent to (3) for the hyperplane $\langle\xi,x\rangle=q$ coincident with the coordinate hyperplane $x_n=0$.

Observe that a tensor field of the form $f_{ij}=g\delta_{ij}$, where $g\in\mathcal{S}(\mathbb{R}^n)$ and (δ_{ij}) is the Kronecker tensor, belongs to the kernel of $\mathcal{N}$ because the integrand of (7) is identically equal to zero for such a field. Therefore it makes sense to consider the restriction of $\mathcal{N}$ to the subspace of trace-free tensor fields. We will do the restriction later but not right now.

To prove the "only if" statement of Theorem 1, we will first find the adjoint $\mathcal{N}^*$ of the Nadirashvili – Vladuts transform and then will compute the product $\mathcal{N}^*\mathcal{N}$.

We use the Hilbert space $L^2(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$ with the L^2-product

$$(f,g)=(f,g)_{L^2(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)}=\int\limits_{\mathbb{R}^n}f^{ij}(x)\overline{g_{ij}(x)}\,dx.$$

The Hilbert space $L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R})$ of functions is defined by

$$(\varphi,\psi)=(\varphi,\psi)_{L^2(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}\times\mathbb{R})}=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{-\infty}^{\infty}\varphi(\xi,\eta,p)\,\overline{\psi(\xi,\eta,q)}\,dqd\xi d\eta,$$

where $d\xi$ (and $d\eta$) is the standard volume form on $\mathbb{S}^{n-1}$.

By (7), for $f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^n \otimes \mathbb{C}^n)$ and $\varphi \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R})$,

$$(\mathcal{N}f, \varphi) = \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\mathbb{S}^{n-1}} \left[\int\limits_{-\infty}^{\infty} \int\limits_{\langle \xi, x\rangle = q} f_{ij}(x)\, \xi^i\, (P_\xi \eta)^j\, \overline{\varphi(\xi, \eta, q)}\, dx dq \right] d\xi d\eta.$$

After changing integration variables as $x = y + q\xi$, this becomes

$$(\mathcal{N}f, \varphi) = \int\limits_{\mathbb{R}^n} f_{ij}(y) \left[\int\limits_{\mathbb{S}^{n-1}} \int\limits_{\mathbb{S}^{n-1}} \overline{\varphi(\xi, \eta, \langle y, \xi\rangle)}\, \xi^i (P_\xi \eta)^j\, d\xi d\eta \right] dy.$$

This means that

2.2 (9)
$$(\mathcal{N}^*\varphi)_{ij}(x) = \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\mathbb{S}^{n-1}} \varphi(\xi, \eta, \langle x, \xi\rangle)\, \xi_i\, (P_\xi \eta)_j\, d\xi d\eta.$$

Next, we compute the product $\mathcal{N}^*\mathcal{N}$. By (7) and (9),

$$(\mathcal{N}^*\mathcal{N}f)_{ij}(x) = \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\langle y, \xi\rangle = \langle x, \xi\rangle} f_{k\ell}(y)\, \xi^k\, (P_\xi \eta)^\ell\, \xi_i\, (P_\xi \eta)_j\, dy d\xi d\eta.$$

After changing integration variables as $z = y - x$, this becomes

2.3 (10)
$$(\mathcal{N}^*\mathcal{N}f)_{ij}(x) = \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\xi^\perp} f_{k\ell}(x+z)\, \xi_i \xi^k \left[\int\limits_{\mathbb{S}^{n-1}} (P_\xi \eta)_j (P_\xi \eta)^\ell\, d\eta \right] dz d\xi.$$

The inner integral on (10) can be easily calculated. Indeed, since

$$(P_\xi \eta)_j (P_\xi \eta)^\ell = (\eta_j - \langle \xi, \eta\rangle \xi_j)(\eta^\ell - \langle \xi, \eta\rangle \xi^\ell),$$

we have

2.4 (11)
$$\begin{aligned} \int\limits_{\mathbb{S}^{n-1}} (P_\xi \eta)_j (P_\xi \eta)^\ell\, d\eta &= \int\limits_{\mathbb{S}^{n-1}} \eta_j \eta^\ell\, d\eta - \xi_j \int\limits_{\mathbb{S}^{n-1}} \langle \xi, \eta\rangle \eta^\ell\, d\eta \\ &\quad - \xi^\ell \int\limits_{\mathbb{S}^{n-1}} \langle \xi, \eta\rangle \eta_j\, d\eta + \xi_j \xi^\ell \int\limits_{\mathbb{S}^{n-1}} \langle \xi, \eta\rangle^2\, d\eta. \end{aligned}$$

Obviously,

$$\int\limits_{\mathbb{S}^{n-1}} \eta_j \eta^\ell\, d\eta = \delta_j^\ell \int\limits_{\mathbb{S}^{n-1}} \eta_1^2\, d\eta = \frac{\omega_n}{n} \delta_j^\ell,$$

where (δ_j^ℓ) is the Kronecker tensor and

2.5 (12)
$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the volume of the unit sphere $\mathbb{S}^{n-1}$. We have also

$$\int\limits_{\mathbb{S}^{n-1}} \langle \xi, \eta\rangle \eta_j\, d\eta = \xi_k \int\limits_{\mathbb{S}^{n-1}} \eta^k \eta_j\, d\eta = \frac{\omega_n}{n} \xi_j$$

and

$$\int\limits_{\mathbb{S}^{n-1}} \langle \xi, \eta\rangle^2\, d\eta = \frac{\omega_n}{n} \quad (|\xi| = 1).$$

Substitute last three values into (11) to obtain

$$\int\limits_{\mathbb{S}^{n-1}} (P_\xi\eta)_j(P_\xi\eta)^\ell\,d\eta=\frac{\omega_n}{n}(\delta_j^\ell-\xi_j\xi^\ell).$$

With the help of this, (10) takes the form

2.6 (13)

$$(\mathcal{N}^*\mathcal{N}f)_{ij}(x)=\frac{\omega_n}{n}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp} f_{kj}(x+z)\,\xi_i\xi^k\,dzd\xi-\frac{\omega_n}{n}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp} f_{k\ell}(x+z)\,\xi_i\xi_j\xi^k\xi^\ell\,dzd\xi.$$

Observe that the second integral on (13) depends on the symmetric part of the tensor f only.

For further transformations of (13), we use the following three formulas. For every function $g(z,\xi)$,

$$\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp} g(z,\xi)\,dzd\xi=\int\limits_{\mathbb{R}^n}\frac{1}{|z|}\int\limits_{\mathbb{S}^{n-1}\cap z^\perp} g(z,\xi)\,d^{n-2}\xi\,dz,$$

where $d^{n-2}\xi$ is the volume form of the sphere $\mathbb{S}^{n-1}\cap\xi^\perp$, see [8, Lemma 2.15.3]. Besides this,

$$\int\limits_{\mathbb{S}^{n-1}\cap z^\perp}\xi_i\xi_j\,d^{n-2}\xi=\frac{\pi^{(n-1)/2}}{\Gamma\big((n+1)/2\big)}\,\varepsilon_{ij}(z),$$

where the symmetric tensor field $\varepsilon\in C^\infty(\mathbb{R}^n\setminus\{0\};\mathbb{R}^n\otimes\mathbb{R}^n)$ is defined by

2.7 (14)

$$\varepsilon_{ij}(z)=\delta_{ij}-z_iz_j/|z|^2,$$

see [8, Lemma 2.15.4]. By the same Lemma,

$$\int\limits_{\mathbb{S}^{n-1}\cap z^\perp}\xi_i\xi_j\xi_k\xi_\ell\,d^{n-2}\xi=\frac{3\pi^{(n-1)/2}}{2\Gamma\big((n+3)/2\big)}\,\varepsilon^2_{ijk\ell}(z),$$

where ε^2 is the symmetrized square of ε. On using these formulas, we calculate the integrals participating in (13):

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp} f_{kj}(x+z)\,\xi_i\xi^k\,dzd\xi&=\int\limits_{\mathbb{R}^n}\frac{f_{kj}(x+z)}{|z|}\left[\int\limits_{\mathbb{S}^{n-1}\cap z^\perp}\xi_i\xi^k\,ds(\xi)\right]dz\\&=\frac{\pi^{(n-1)/2}}{\Gamma\big((n+1)/2\big)}\int\limits_{\mathbb{R}^n}\frac{f_{kj}(x+z)}{|z|}\varepsilon_i^k(z)\,dz;\end{aligned}$$

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp} f_{k\ell}(x+z)\,\xi_i\xi_j\xi^k\xi^\ell\,dzd\xi&=\int\limits_{\mathbb{R}^n}\frac{f_{k\ell}(x+z)}{|z|}\left[\int\limits_{\mathbb{S}^{n-1}\cap z^\perp}\xi_i\xi_j\xi^k\xi^\ell\,d^{n-2}\xi\right]dz\\&=\frac{3\pi^{(n-1)/2}}{2\Gamma\big((n+3)/2\big)}\int\limits_{\mathbb{R}^n}\frac{f_{k\ell}(x+z)}{|z|}(\varepsilon^2)^{k\ell}_{ij}(z)\,dz.\end{aligned}$$

Substituting these values into (13) and inserting the value (12) of ω_n, we obtain

$$(\mathcal{N}^*\mathcal{N}f)_{ij}(x) = \frac{2\pi^{n-1/2}}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} \int\limits_{\mathbb{R}^n} \frac{f_{kj}(x+z)}{|z|} \varepsilon_i^k(z)\,dz - \frac{3\pi^{n-1/2}}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+3}{2}\right)} \int\limits_{\mathbb{R}^n} \frac{f_{k\ell}(x+z)}{|z|} (\varepsilon^2)_{ij}^{k\ell}(z)\,dz.$$

This can be written as

2.8 (15)
$$(\mathcal{N}^*\mathcal{N}f)_{ij}(x) = \int\limits_{\mathbb{R}^n} N_{ijk\ell}(z)\, f^{k\ell}(x+z)\,dz \tag{15}$$

with the kernel

2.9 (16)
$$N_{ijl\ell}(z) = \frac{6\pi^{n-1/2}}{n(n+1)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} \frac{1}{|z|}\Big(\frac{n+1}{3}\delta_{j\ell}\varepsilon_{ik}(z) - (\varepsilon^2)_{ijk\ell}(z)\Big). \tag{16}$$

We remember that the Kronecker tensor belongs to the null-space of $\mathcal{N}$. Therefore the kernel must satisfy ${N_{ijp}}^p = 0$. Let us check this in order to control our calculations. By (16)

2.10 (17)
$$\frac{n(n+1)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{6\pi^{n-1/2}} |z|{N_{ijp}}^p = \frac{n+1}{3}\delta_j^p\varepsilon_{ip} - {(\varepsilon^2)_{ijp}}^p. \tag{17}$$

By the definition of the symmetrized square,

$$(\varepsilon^2)_{ijk\ell} = \frac{1}{3}\big(\varepsilon_{ij}\varepsilon_{k\ell} + \varepsilon_{ik}\varepsilon_{j\ell} + \varepsilon_{i\ell}\varepsilon_{jk}\big).$$

Therefore

$${(\varepsilon^2)_{ijp}}^p = \frac{1}{3}\big(\varepsilon_{ij}\varepsilon_p^p + 2\varepsilon_{ip}\varepsilon_j^p\big).$$

By (14),

$$\varepsilon_p^p = \delta_p^p - \frac{z_p z^p}{|z|^2} = n-1.$$

Substitute this into the previous formula

$${(\varepsilon^2)_{ijp}}^p = \frac{1}{3}\big((n-1)\varepsilon_{ij} + 2\varepsilon_{ip}\varepsilon_j^p\big).$$

Then we calculate

$$\varepsilon_{ip}\varepsilon_j^p = \Big(\delta_{ip} - \frac{z_i z_p}{|z|^2}\Big)\Big(\delta_j^p - \frac{z_j z^p}{|z|^2}\Big) = \delta_{ip}\delta_j^p - \delta_{ip}\frac{z_j z^p}{|z|^2} - \delta_j^p\frac{z_i z_p}{|z|^2} + \frac{z_i z_j z_p z^p}{|z|^4} = \delta_{ij} - \frac{z_i z_j}{|z|^2} = \varepsilon_{ij}.$$

Substitute this into the previous formula to obtain ${(\varepsilon^2)_{ijp}}^p = \frac{n+1}{3}\varepsilon_{ij}$. Together with (17), this gives ${N_{ijp}}^p = 0$.

On using the convolution, formula (15) is written as

2.11 (18)
$$(\mathcal{N}^*\mathcal{N}f)_{ij} = f^{k\ell} * N_{ijk\ell}. \tag{18}$$

We are going to find all tensor fields f satisfying $\mathcal{N}f = 0$. In order to get a system of algebraic equations, we are going to apply the Fourier transform to (18).

The Fourier transform of a tempered distribution $g \in \mathcal{S}'(\mathbb{R}^n)$ is denoted either by $F[g]$ or by $\hat{g}(y)$. The Fourier transform acts component-wise on tensor fields, i.e., $(\hat f)_{ij}=\widehat{f_{ij}}$. Recall that $\lambda\mapsto|x|^\lambda$ is the meromorphic $\mathcal{S}'(\mathbb{R}^n)$-valued function of $\lambda\in\mathbb{C}$ with simple poles at points $-n,-n-2,-n-4,\dots$. The Fourier transform of $|x|^\lambda$ is expressed by

2.12 (19)
$$F[|x|^\lambda]=\frac{2^{\lambda+n/2}\Gamma\Big(\frac{\lambda+n}{2}\Big)}{\Gamma(-\lambda/2)}|y|^{-\lambda-n}\quad(\lambda,-\lambda-n\notin 2\mathbb{Z}^+),$$
$$F[|x|^{2k}]=(2\pi)^{n/2}(-\Delta)^k\delta\quad(k\in\mathbb{Z}^+),$$

where δ is the Dirac function.

As is seen from (16), functions $N_{ijk\ell}(x)$ are locally integrable on $\mathbb{R}^n$ and decay at infinity (because $\varepsilon_{ij}(x)$ are bounded functions). Therefore $N_{ijk\ell}(x)$ can be considered as tempered distributions. Applying the Fourier transform to (18), we obtain

2.13 (20)
$$(\widehat{\mathcal{N}^*\mathcal{N}f})_{ij}=(2\pi)^{n/2}\hat f^{k\ell}\,F[N_{ijk\ell}].$$

The product on the right-hand side is now understood as a product of a function in $\mathcal{S}(\mathbb{R}^n)$ and of a tempered distribution. We shall soon see that the second factor is continuous on $\mathbb{R}^n\setminus\{0\}$ and hence the product can be understood in the conventional sense.

We proceed to computing $F[N_{ijk\ell}]$. By (16),

2.14 (21)
$$F[N_{ijl\ell}]=\frac{6\pi^{n-1/2}}{n(n+1)\Gamma\Big(\frac{n}{2}\Big)\Gamma\Big(\frac{n+1}{2}\Big)}\Big(\frac{n+1}{3}\delta_{j\ell}F[|x|^{-1}\varepsilon_{ik}(x)]-F[|x|^{-1}(\varepsilon^2)_{ijk\ell}(x)]\Big).$$

Both Fourier transforms on the right-hand side of (21) can be easily found on the base of the equalities (see Lemma 2.11.1 of [8])

$$|x|^{-1}\varepsilon_{ik}(x)=\frac{\partial^2|x|}{\partial x_i\partial x_k},\quad |x|^{-1}(\varepsilon^2)_{ijk\ell}(x)=\frac{1}{9}\,\frac{\partial^4|x|^3}{\partial x_i\partial x_j\partial x_k\partial x_\ell}.$$

Applying the Fourier transform to these equalities and using the standard property of the Fourier transform, we get

$$F[|x|^{-1}\varepsilon_{ik}(x)]=-y_iy_kF[|x|],\quad F[|x|^{-1}(\varepsilon^2)_{ijk\ell}(x)]=\frac{1}{9}\,y_iy_jy_ky_\ell F[|x|^3].$$

By (19),

$$F[|x|]=-\frac{2^{n/2}\Gamma\Big(\frac{n+1}{2}\Big)}{\sqrt{\pi}}\,|y|^{-n-1},\quad F[|x|^3]=\frac{3\cdot 2^{n/2+1}\Gamma\Big(\frac{n+3}{2}\Big)}{\sqrt{\pi}}\,|y|^{-n-3}.$$

Substitute these values into previous formulas

$$F[|x|^{-1}\varepsilon_{ik}(x)]=\frac{2^{n/2}\Gamma\Big(\frac{n+1}{2}\Big)}{\sqrt{\pi}}\,\frac{y_iy_k}{|y|^{n+1}},$$
$$F[|x|^{-1}(\varepsilon^2)_{ijk\ell}(x)]=\frac{2^{n/2+1}\Gamma\Big(\frac{n+3}{2}\Big)}{3\sqrt{\pi}}\,\frac{y_iy_jy_ky_\ell}{|y|^{n+3}}.$$

Insert these values into (21)

$$F[N_{ijl\ell}]=\frac{2^{n/2+1}\pi^{n-1}}{n\Gamma(n/2)}\,\frac{1}{|y|^{n+3}}\big(|y|^2y_iy_k\delta_{j\ell}-y_iy_jy_ky_\ell\big).$$

Now, (20) takes the form

$$(\widehat{\mathcal{N}^*\mathcal{N}f})_{ij}(y)=\frac{2^{n+1}\pi^{(3n-2)/2}}{n\Gamma(n/2)}\,\frac{1}{|y|^{n+3}}\big(|y|^2y_iy_k\delta_{j\ell}-y_iy_jy_ky_\ell\big)\hat f^{k\ell}(y).$$

Equation $\mathcal{N}f=0$ is thus equivalent to the system

$$\big(|y|^2y_iy_k\delta_{j\ell}-y_iy_jy_ky_\ell\big)\hat f^{k\ell}(y)=0\quad(1\le i,j\le 3).$$

The system can be simplified. To this end we first rewrite it in the form

2.15 (22) $$y_\ell\,\varepsilon_{ij}(y)\big(y_k\hat f^{kj}(y)\big)=0\quad(0\ne y\in\mathbb{R}^n;1\le i,\ell\le n),$$

where $\varepsilon_{ij}(y)$ is defined by (14). On assuming $\hat f\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$, the system can be equivalently written as

2.16 (23) $$\varepsilon_{ij}(y)\big(y_k\hat f^{kj}(y)\big)=0\quad(0\ne y\in\mathbb{R}^n,1\le i\le n).$$

Indeed, (22) and (23) are equivalent for y satisfying $y_1y_2\dots y_n\ne 0$. This implies the validity of (23) by continuity.

Introduce the vector field $\hat g\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)$ by (i is the imaginary unit)

2.17 (24) $$\hat g^j(y)=i\,y_k\hat f^{kj}(y).$$

Then the system (23) is written as

2.18 (25) $$\varepsilon_{jk}(y)\,\hat g^k(y)=0\quad(0\ne y\in\mathbb{R}^n,1\le j\le n).$$

The geometric meaning of (25) is obvious: for $0\ne y\in\mathbb{R}^n$, the vector $\hat g(y)$ must be a scalar multiple of y. Indeed, let us remind the orthogonal projection $P_y:\mathbb{R}^n\to y^\perp$ which is defined by $P_y\hat g=\hat g-\frac{\langle\hat g,y\rangle}{|y|^2}y$. Equations (25) are equivalent to $P_y\hat g(y)=0$.

Thus, system (25) is equivalent to the existence of a function $\hat p\in C^\infty(\mathbb{R}^n\setminus\{0\})$ such that

2.19 (26) $$\hat g(y)+i\,\hat p(y)y=0\quad(0\ne y\in\mathbb{R}^n).$$

As follows from (24) and (26), $\hat p$ is expressed through $\hat f$ by

2.20 (27) $$\hat p(y)=-\frac{y^iy^j}{|y|^2}\,\hat f_{ij}(y)\quad(0\ne y\in\mathbb{R}^n).$$

This implies that the function $\hat p(y)$ is bounded, belongs to $C^\infty(\mathbb{R}^n\setminus\{0\})$ and fast decays together with all derivatives as $|y|\to\infty$. But a priori $\hat p(y)$ can have a singularity at $y=0$. Therefore the inverse Fourier transform $p(x)$ of the function $\hat p(y)$ belongs to $C^\infty(\mathbb{R}^n)$.

Let $f\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$. As is seen from (24), the inverse Fourier transform $g\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)$ of the vector field $\hat g$ satisfies

$$g_i=\sum_{j=1}^n\frac{\partial f_{ji}}{\partial x_j}.$$

Applying the inverse Fourier transform to the equation (26), we obtain

2.21 (28) $$\sum_{j=1}^n\frac{\partial f_{ji}}{\partial x_j}+\frac{\partial p}{\partial x_i}=0\quad(1\le i\le n).$$

We see from (28) that first order derivatives $\frac{\partial p}{\partial x_i}$ of the function p fast decay at infinity together with all their derivatives. This easily implies that the function p itself fast decays at infinity, i.e. $p\in\mathcal{S}(\mathbb{R}^n)$. We have thus proved

P2.1 **Proposition 1.** *A tensor field $f=(f_{ij})\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n\otimes\mathbb{C}^n)$ satisfies $\mathcal{N}f=0$ if and only if equations* (28) *hold with some function $p\in\mathcal{S}(\mathbb{R}^n)$.*

We can now prove the "only if" statement of Theorem 1. Given a divergence-free vector field $v\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)$, we set $f_{ij}=v_iv_j$ and see that (28) coincides with the Euler equations (8). This finishes the proof of Theorem 1.

The incompressibility equation (2) was used in our arguments for the passage from (1) to (8) only. We have thus proven a little bit more general statement.

Th2.1 **Theorem 2.** *A vector field $v\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^n)$ satisfies $\mathcal{N}(v\otimes v)=0$ if and only if there exists a function $p\in\mathcal{S}(\mathbb{R}^n)$ such that (v,p) is a solution to the Euler equations* (8).

We finally observe that Proposition 1 implies

C2.1 **Corollary 1.** *Under hypotheses of Proposition 1 Fourier transforms of f and p satisfy*

2.22
$$\hat f_{ij}(0)=-\hat p(0)\,\delta_{ij}, \tag{29}$$

where δ_{ij} is the Kronecker tensor.

Remark. In the case of $f=v\otimes v$, (29) is equivalent to one of so called *orthogonality relations*, see [9, formulas (1.10)–(1.11)].

3. The Nadirashvili – Vladuts potential

From now on we consider the three-dimensional case only. Some our statements can be generalized to the case of an arbitrary dimension but proofs become more complicated.

For $\xi\in\mathbb{R}^3$, by $j_\xi:\mathbb{C}^3\otimes\mathbb{C}^3\to\mathbb{C}^3$ we denote the operator of contraction with the vector ξ in the first index; it is expressed by $(j_\xi f)_j=f_{ij}\xi^i$ in coordinates.

For $0\neq\xi\in\mathbb{R}^3$, let $P_\xi:\mathbb{R}^3\to\xi^\perp=\{x\in\mathbb{R}^3\mid\langle\xi,x\rangle=0\}$ be the orthogonal projection. We consider $\xi^\perp$ as an oriented two-dimensional vector space. The orientation is defined by the rule: if (e_1,e_2) is a positive basis of $\xi^\perp$, then (e_1,e_2,ξ) should be positive basis of $\mathbb{R}^3$. Let $R_\xi:\xi^\perp\to\xi^\perp$ be the rotation through the right angle in the positive direction.

For $0\neq\xi\in\mathbb{R}^3$, we will also use the two-dimensional complex vector space $\mathbb{C}\otimes\xi^\perp=\{x\in\mathbb{C}^3\mid\langle\xi,x\rangle=x_i\xi^i=0\}$. The operators P_ξ and R_ξ are uniquely extended to linear operators between complex vector spaces $P_\xi:\mathbb{C}^3\to\mathbb{C}\otimes\xi^\perp$ and $R_\xi:\mathbb{C}\otimes\xi^\perp\to\mathbb{C}\otimes\xi^\perp$ respectively.

Introduce the 4-dimensional submanifold

$$T\mathbb{S}^2=\{(x,\xi)\mid|\xi|=1,\langle\xi,x\rangle=0\}$$

of $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\}$. It is the total space of the tangent bundle $T\mathbb{S}^2\to\mathbb{S}^2$ of the sphere $\mathbb{S}^2$. By the remark presented after (4), the Schwartz space $\mathcal{S}(T\mathbb{S}^2)$ of functions is well defined.

P3.1 **Proposition 2.** *Given a tensor field $f\in\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying*

3.1
$$\mathcal{N}f=0, \tag{30}$$

there exists a unique function $w\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ such that

(1) *the function satisfies*

3.2
$$w(x,t\xi)=|t|^{-1}w(x,\xi)\ \textit{for}\ 0\neq t\in\mathbb{R},\quad w(x+t\xi,\xi)=w(x,\xi)\ \textit{for}\ t\in\mathbb{R}; \tag{31}$$

(2) *the restriction of* w *to the manifold* $T\mathbb{S}^2$ *belongs to* $\mathcal{S}(T\mathbb{S}^2)$*;*
(3) *the equation*

3.3 (32)
$$\nabla_x w(x,\xi) = |\xi|^{-1}\int\limits_{-\infty}^{\infty} R_\xi P_\xi j_\xi f(x+t\xi)\,dt$$

holds on $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$*, where* $\nabla_x w$ *is the gradient of* w *with respect to the variable* x*.*

We call w the *Nadirashvili – Vladuts potential* determined by the tensors field $f\in\mathcal{S}(\mathbb{R}^3;S^2\mathbb{R}^3)$ satisfying (30). It was introduced in a different way in [5, Definition 3.2].

Proof. Fix a unit vector $\xi\in\mathbb{S}^2$ and define a vector field g on the plane $P=\xi^\perp$ by

$$g(x) = \int\limits_{-\infty}^{\infty} R_\xi P_\xi j_\xi f(x+t\xi)\,dt \quad (x\in P=\xi^\perp).$$

Obviously, $g\in\mathcal{S}(P;\xi^\perp)$. Let us demonstrate that

3.4 (33)
$$Ig=0,$$

where I is the ray transform on the plane P (see the definition of the operator I in [8, Section 2.1]). Indeed, for $x\in P,\eta\in\xi^\perp,|\eta|=1$,

3.5 (34)
$$(Ig)(x,\eta) = \int\limits_{-\infty}^{\infty}\langle g(x+s\eta),\eta\rangle\,ds = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\langle R_\xi P_\xi j_\xi f(x+t\xi+s\eta),\eta\rangle\,dsdt.$$

The operator P_ξ is self-adjoint while R_ξ satisfies $R_\xi^*=R_\xi^{-1}$. Therefore

$$\langle R_\xi P_\xi j_\xi f(x+t\xi+s\eta),\eta\rangle = \langle j_\xi f(x+t\xi+s\eta),P_\xi R_\xi^{-1}\eta\rangle = \langle j_\xi f(x+t\xi+s\eta),R_\xi^{-1}\eta\rangle.$$

The last equality holds because $R_\xi^{-1}\eta\in\xi^\perp$. Equation (34) takes now the form

3.6 (35)
$$(Ig)(x,\eta) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\langle j_\xi f(x+t\xi+s\eta),R_\xi^{-1}\eta\rangle\,dsdt.$$

Change integration variables in the latter integral by the formula $y=x+t\xi+s\eta$. The point y runs over the plane $\{y\in\mathbb{R}^3\mid\langle R_\xi^{-1}\eta,y\rangle=q\}$ with $q=\langle R_\xi^{-1}\eta,x\rangle$. Equation (35) takes now the form

$$(Ig)(x,\xi) = \int\limits_{\langle R_\xi^{-1}\eta,y\rangle=q} f_{ij}(y)\xi^i(R_\xi^{-1}\eta)^j\,dy.$$

By (30), the integral on the right-hand side is equal to zero. This proves (33).

By (33), the vector field $g\in\mathcal{S}(P;\xi^\perp)$ must be a potential vector field, i.e., there exists a function $w_{0,\xi}\in\mathcal{S}(P)$ on the plane $P=\xi^\perp$ such that $g=\nabla w_{0,\xi}$. Both g and $w_{0,\xi}$ depend smoothly on $\xi\in\mathbb{S}^2$. We can define the function $w_0\in\mathcal{S}(T\mathbb{S}^2)$ by $w_0(x,\xi)=w_{0,\xi}(x)$. The function satisfy

3.7 (36)
$$\nabla_x w_0(x,\xi) = g(x) = \int\limits_{-\infty}^{\infty} R_\xi P_\xi j_\xi f(x+t\xi)\,dt \quad\text{for}\quad (x,\xi)\in T\mathbb{S}^2.$$

The function $w_0(x,\xi)$ is even in ξ

3.8 (37) $$w_0(x,-\xi)=w_0(x,\xi).$$

Indeed, by (36)

$$\nabla_x w_0(x,-\xi)=\int_{-\infty}^{\infty} R_{-\xi}P_{-\xi}j_{-\xi}f(x-t\xi)\,dt.$$

Since $j_{-\xi}=-j_\xi, P_{-\xi}=P_\xi, R_{-\xi}=-R_\xi$, the previous formula takes the form

$$\nabla_x w_0(x,-\xi)=\int_{-\infty}^{\infty} R_\xi P_\xi j_\xi f(x-t\xi)\,dt.$$

After the change $t=-\tau$ of the integration variable, we obtain $\nabla_x w_0(x,-\xi)=\nabla_x w_0(x,\xi)$. This is equivalent to (37).

There exists a unique extension of w_0 to a function $w\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ satisfying (31). Let us prove that the extension satisfies (32) on the whole of $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$. Indeed, the function w is expressed through w_0 by the explicit formula

$$w(x,\xi)=|\xi|^{-1}w_0\Big(x-\frac{\langle\xi,x\rangle}{|\xi|^2}\xi,\frac{\xi}{|\xi|}\Big).$$

Differentiate this equality to obtain

$$\frac{\partial w}{\partial x_i}(x,\xi)=|\xi|^{-1}\Big(\delta_i^j-\frac{\xi_i\xi^j}{|\xi|^2}\Big)\frac{\partial w_0}{\partial x_j}\Big(x-\frac{\langle\xi,x\rangle}{|\xi|^2}\xi,\frac{\xi}{|\xi|}\Big).$$

This can be written in the coordinate free form

$$\nabla_x w(x,\xi)=|\xi|^{-1}P_\xi\nabla_x w_0(P_\xi x,\xi/|\xi|).$$

By (36), $\nabla_x w_0(P_\xi x,\xi/|\xi|)\in\xi^\perp$, hence $P_\xi\nabla_x w_0(P_\xi x,\xi/|\xi|)=\nabla_x w_0(P_\xi x,\xi/|\xi|)$ and the previous formula is simplified to the following one:

3.9 (38) $$\nabla_x w(x,\xi)=|\xi|^{-1}\nabla_x w_0(P_\xi x,\xi/|\xi|).$$

On using (36) and (38), we derive

$$\nabla_x w(x,\xi)=|\xi|^{-1}\int_{-\infty}^{\infty} R_{\xi/|\xi|}P_{\xi/|\xi|}\,j_{\xi/|\xi|}f(P_\xi x+t\xi/|\xi|)\,dt.$$

Obviously,

$$R_{\xi/|\xi|}=R_\xi,\quad P_{\xi/|\xi|}=P_\xi,\quad j_{\xi/|\xi|}=|\xi|^{-1}j_\xi.$$

The previous formula takes the form

$$\nabla_x w(x,\xi)=|\xi|^{-2}\int_{-\infty}^{\infty} R_\xi P_\xi j_\xi f\Big(x+(\frac{1}{|\xi|}t-\frac{\langle\xi,x\rangle}{|\xi|^2})\xi\Big)\,dt.$$

After the change $\tau=\frac{1}{|\xi|}t-\frac{\langle\xi,x\rangle}{|\xi|^2}$ of the integration variable, this becomes

$$\nabla_x w(x,\xi)=|\xi|^{-1}\int_{-\infty}^{\infty} R_\xi P_\xi j_\xi f(x+\tau\xi)\,d\tau.$$

This proves (32).

We have thus proved the existence statement of the proposition. The uniqueness statement obviously follows from (31)–(32). ☐

C3.1 **Corollary 2.** *Under hypotheses of Proposition 2, the potential w can be explicitly expressed through the tensor field f as follows. Given $(x,\xi)\in\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$, choose a vector $\eta\in\mathbb{S}^2$ such that $\eta\neq\pm\xi/|\xi|$. Then*

3.10
$$w(x,\xi)=-|\xi|^{-1}\int\limits_0^\infty\int\limits_{-\infty}^\infty\langle R_\xi P_\xi j_\xi f(x+t\xi+s\eta),\eta\rangle\,dtds. \tag{39}$$

Proof. By the second statement of Proposition 2, $w(x+s\eta,\xi)\to0$ as $s\to\infty$. Therefore

$$w(x,\xi)=-\int\limits_0^\infty\frac{\partial w(x+s\eta,\xi)}{\partial s}\,ds=-\int\limits_0^\infty\langle(\nabla_x w)(x+s\eta,\xi),\eta\rangle\,ds.$$

Substituting the expression (32) for the gradient, we arrive to (39). ☐

The operator $R_\xi P_\xi$ is expressed by $R_\xi P_\xi v=|\xi|^{-1}\xi\times v$ for a vector $v\in\mathbb{R}^3$, where $\times$ stands for the vector product. The operator $v\mapsto\xi\times v$ is well defined for $v\in\mathbb{C}^3$ too. Therefore

$$\langle R_\xi P_\xi j_\xi f,\eta\rangle=\frac{1}{|\xi|}[\xi,j_\xi f,\eta],$$

where $[a,b,c]=\langle a\times b,c\rangle$. Formula (39) takes the form

3.11
$$w(x,\xi)=-|\xi|^{-2}\int\limits_0^\infty\int\limits_{-\infty}^\infty[\xi,j_\xi f(x+t\xi+s\eta),\eta]\,dtds, \tag{40}$$

Recall that η in (40) is an arbitrary unit vector subordinate the only condition $\eta\neq\pm\xi/|\xi|$. Assume for a moment that $\xi_3\neq0$. In such a case, we can choose either $\eta=(1,0,0)$ or $\eta=(0,1,0)$. In this way, we obtain two partial cases of (40):

$$w(x,\xi)=|\xi|^{-2}\int\limits_0^\infty\int\limits_{-\infty}^\infty\big(\xi_3(j_\xi f)_2-\xi_2(j_\xi f)_3\big)(x_1+t\xi_1+s,x_2+t\xi_2,x_3+t\xi_3)\,dtds,$$

$$w(x,\xi)=|\xi|^{-2}\int\limits_0^\infty\int\limits_{-\infty}^\infty\big(-\xi_3(j_\xi f)_1+\xi_1(j_\xi f)_3\big)(x_1+t\xi_1,x_2+t\xi_2+s,x_3+t\xi_3)\,dtds.$$

After obvious changes of integration variables, these formulas take the form

3.12
$$w(x,\xi)=|\xi|^{-2}\int\limits_{x_1+t\xi_1}^\infty\int\limits_{-\infty}^\infty\big(\xi_3(j_\xi f)_2-\xi_2(j_\xi f)_3\big)(s,x_2+t\xi_2,x_3+t\xi_3)\,dtds, \tag{41}$$

3.13
$$w(x,\xi)=|\xi|^{-2}\int\limits_{x_2+t\xi_2}^\infty\int\limits_{-\infty}^\infty\big(-\xi_3(j_\xi f)_1+\xi_1(j_\xi f)_3\big)(x_1+t\xi_1,s,x_3+t\xi_3)\,dtds. \tag{42}$$

Differentiating equations (41) and (42) with respect to x_1 and x_2 respectively, we obtain

3.14 $$\frac{\partial w}{\partial x_1}(x,\xi)=|\xi|^{-2}\int\limits_{-\infty}^{\infty}\big(\xi_3(j_\xi f)_2-\xi_2(j_\xi f)_3\big)(x+t\xi)\,dt, \tag{43}$$

3.15 $$\frac{\partial w}{\partial x_2}(x,\xi)=|\xi|^{-2}\int\limits_{-\infty}^{\infty}\big(-\xi_3(j_\xi f)_1+\xi_1(j_\xi f)_3\big)(x+t\xi)\,dt. \tag{44}$$

Formulas (43)–(44) imply a similar formula for $\partial w/\partial x_3$. Indeed, as is seen from (31),

3.16 $$\xi_1\frac{\partial w}{\partial x_1}+\xi_2\frac{\partial w}{\partial x_2}+\xi_3\frac{\partial w}{\partial x_3}=0, \tag{45}$$

i.e.,

$$\frac{\partial w}{\partial x_3}=-\frac{1}{\xi_3}\Big(\xi_1\frac{\partial w}{\partial x_1}+\xi_2\frac{\partial w}{\partial x_2}\Big).$$

Substituting values (43)–(44) into this equality, we obtain

3.17 $$\frac{\partial w}{\partial x_3}(x,\xi)=|\xi|^{-2}\int\limits_{-\infty}^{\infty}\big(\xi_2(j_\xi f)_1-\xi_1(j_\xi f)_2\big)(x+t\xi)\,dt. \tag{46}$$

Formulas (43)–(44), (46) have been proven for $\xi_3\neq 0$. Nevertheless, these formulas are valid on the whole of $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$ because they have no singularity at $\xi_3=0$. These formulas can be united as follows:

3.18 $$\nabla_x w(x,\xi)=-|\xi|^{-2}\,\xi\times\int\limits_{-\infty}^{\infty}j_\xi f(x+t\xi)\,dt. \tag{47}$$

The potential w can be eliminated from (47) by applying the operator curl_x to this equation. On using the identity

$$\operatorname{curl}(\xi\times v)=(\operatorname{div}v)\xi-\langle\xi,\partial_x\rangle v,\quad\text{where}\quad\langle\xi,\partial_x\rangle=\xi^i\frac{\partial}{\partial x_i},$$

which is valid for a constant vector ξ, we obtain

$$\Big(\int\limits_{-\infty}^{\infty}(\operatorname{div}j_\xi f)(x+t\xi)\,dt\Big)\xi-\int\limits_{-\infty}^{\infty}\Big(\xi_1\frac{\partial(j_\xi f)}{\partial x_1}+\xi_2\frac{\partial(j_\xi f)}{\partial x_2}+\xi_3\frac{\partial(j_\xi f)}{\partial x_3}\Big)(x+t\xi)\,dt=0.$$

The second integral is identically equal to zero. Indeed,

$$\int\limits_{-\infty}^{\infty}\Big(\xi_1\frac{\partial(j_\xi f)}{\partial x_1}+\xi_2\frac{\partial(j_\xi f)}{\partial x_2}+\xi_3\frac{\partial(j_\xi f)}{\partial x_3}\Big)(x+t\xi)\,dt=\int\limits_{-\infty}^{\infty}\frac{d}{dt}\big((j_\xi f)(x+t\xi)\big)\,dt=0.$$

We have thus obtained

3.19 $$\int\limits_{-\infty}^{\infty}(\operatorname{div}j_\xi f)(x+t\xi)\,dt=0. \tag{48}$$

Thus, (48) is a corollary of (30), i.e., the equation (48) holds for every tensor field $f \in \mathcal{S}(\mathbb{R}^3; S^2\mathbb{R}^3)$ satisfying (30). The corollary is not obvious because (30) involves a two-dimensional integral while (48) contains a one-dimensional integral.

Let us separately consider the case of $f = v \otimes v$, where a vector field $v \in \mathcal{S}(\mathbb{R}^3; \mathbb{R}^3)$ satisfies the Euler equations (1)–(2). In such a case, $j_\xi f = \langle \xi, v \rangle v$ and

3.20 (49)
$$\operatorname{div}(j_\xi f) = \langle \xi, v \rangle \operatorname{div} v + \sum_{i,j=1}^{3} \xi_i v_j \frac{\partial v_i}{\partial x_j} = -\langle \xi, \nabla p \rangle.$$

This implies the validity of (48). Thus, (48) follows from the Euler equations in the case of $f = v \otimes v$.

Now, following [5], we are going to compute values of some higher order differential operators on the potential w.

The *vertical (or fiber-wise) Laplacian*

$$\Delta^v : C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big) \to C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$$

is defined by

3.21 (50)
$$\begin{aligned}\Delta^v = \frac{1}{|\xi|^2}\Big(&(\xi_2^2 + \xi_3^2)\frac{\partial^2}{\partial x_1^2} + (\xi_1^2 + \xi_3^2)\frac{\partial^2}{\partial x_2^2} + (\xi_1^2 + \xi_2^2)\frac{\partial^2}{\partial x_3^2} \\ &- 2\xi_1\xi_2 \frac{\partial^2}{\partial x_1 \partial x_2} - 2\xi_1\xi_3 \frac{\partial^2}{\partial x_1 \partial x_3} - 2\xi_2\xi_3 \frac{\partial^2}{\partial x_2 \partial x_3}\Big).\end{aligned}$$

If a function $w \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$ satisfies (31), then $\Delta^v w$ satisfies (31) too.

Let us give a motivation of the definition (50). We first recall some standard facts of analysis on Riemannian manifolds.

Given a Riemannian manifold (M, g), let $\pi : TM \to M$ be the tangent bundle. If $(U; x^1, \dots, x^n)$ is a local coordinate system on M, then the corresponding coordinate system $\big(\pi^{-1}(U); x^1, \dots, x^n, X^1, \dots, X^n\big)$ on TM is defined by $X = X^i \frac{\partial}{\partial x^i}$ for $X \in T_xM$, $x \in U$. Every tangent space T_xM is furnished by the dot product g_x, hence the Euclidean Laplacian $\Delta_x : C^\infty(T_xM) \to C^\infty(T_xM)$ is well defined. The Laplacian smoothly depends on x and defines the *vertical Laplacian* $\Delta^v : C^\infty(TM) \to C^\infty(TM)$. It is expressed in local coordinates by

$$\Delta^v = g^{ij}(x) \frac{\partial^2}{\partial X^i \partial X^j}.$$

Now, we apply this to the unit sphere $\mathbb{S}^2 = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 \xi_i^2 = 1\}$ which is considered as a two-dimensional Riemannian manifold with the metric induced from $\mathbb{R}^3$. Let (φ, ψ) be the geographic coordinates on $\mathbb{S}^2$ such that

$$\xi_1 = \cos\varphi \cos\psi, \quad \xi_2 = \cos\varphi \sin\psi, \quad \xi_3 = \sin\varphi.$$

The metric tensor is

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\varphi \end{pmatrix}, \quad \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^{-2}\varphi \end{pmatrix}.$$

Let $(\varphi, \psi, \Phi, \Psi)$ be corresponding coordinates on $T\mathbb{S}^2$. The vertical Laplacian on $T\mathbb{S}^2$ is expressed in geographic coordinates by

3.22 (51)
$$\Delta^v = \frac{\partial^2}{\partial \Phi^2} + \frac{1}{\cos^2\varphi} \frac{\partial^2}{\partial \Psi^2}.$$

The Cartesian coordinate (x_1, x_2, x_3) of a vector $X \in T_{(\varphi,\psi)}\mathbb{S}^2$ are related to the geographic coordinates by

$$x_1 = -\sin\varphi\cos\psi\,\Phi - \cos\varphi\sin\psi\,\Psi,\ x_2 = -\sin\varphi\sin\psi\,\Phi + \cos\varphi\cos\psi\,\Psi,\ x_3 = \cos\varphi\,\Phi.$$

This implies

$$\frac{\partial}{\partial\Phi} = -\sin\varphi\cos\psi\frac{\partial}{\partial x_1} - \sin\varphi\sin\psi\frac{\partial}{\partial x_2} + \cos\varphi\frac{\partial}{\partial x_3},$$
$$\frac{\partial}{\partial\Psi} = -\cos\varphi\sin\psi\frac{\partial}{\partial x_1} + \cos\varphi\cos\psi\frac{\partial}{\partial x_2}.$$

Substituting these values into (51), we obtain (50) for $(x, \xi) \in T\mathbb{S}^2$. Then we extend the Laplacian to the whole of $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ so that it preserves the homogeneity (31). The extension is given by (50) for an arbitrary point $(x, \xi) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$.

Let us compute $\Delta^v w$. To this end we observe that the definition (50) can be written in the form

3.23 (52)
$$\Delta^v = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{1}{|\xi|^2}\Big(\xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + \xi_3\frac{\partial}{\partial x_3}\Big)^2.$$

In view of (45), this gives

3.24 (53)
$$\Delta^v w = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)w.$$

Differentiating equations (43), (44) and (46) with respect to x_1, x_2 and x_3 respectively and substituting the results into (53), we obtain

$$\Delta^v w = |\xi|^{-2}\int\limits_{-\infty}^{\infty}\langle\xi, \operatorname{curl}_x(j_\xi f)(x + t\xi)\rangle\,dt.$$

In the case of $f = v \otimes v$, this becomes

3.25 (54)
$$\Delta^v w = |\xi|^{-2}\int\limits_{-\infty}^{\infty}\Big(\langle\xi, v\rangle\langle\xi, \operatorname{curl} v\rangle + [\xi, \nabla_x\langle\xi, v\rangle, v]\Big)(x + t\xi)\,dt.$$

Substituting the values

$$\langle\xi, \operatorname{curl} v\rangle = \xi_1\Big(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\Big) + \xi_2\Big(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\Big) + \xi_3\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big)$$

and

$$[\xi, \nabla_x\langle\xi, v\rangle, v] = (\xi_3 v_2 - \xi_2 v_3)\frac{\partial\langle\xi, v\rangle}{\partial x_1} + (\xi_1 v_3 - \xi_3 v_1)\frac{\partial\langle\xi, v\rangle}{\partial x_2} + (\xi_2 v_1 - \xi_1 v_2)\frac{\partial\langle\xi, v\rangle}{\partial x_3},$$

we write (54) in the coordinate form

3.26 (55)
$$\Delta^v w = \frac{1}{|\xi|^2}\int\limits_{-\infty}^{\infty}\Big[\langle\xi, v\rangle\Big(\xi_1\frac{\partial v_3}{\partial x_2} - \xi_1\frac{\partial v_2}{\partial x_3} + \xi_2\frac{\partial v_1}{\partial x_3} - \xi_2\frac{\partial v_3}{\partial x_1} + \xi_3\frac{\partial v_2}{\partial x_1} - \xi_3\frac{\partial v_1}{\partial x_2}\Big)$$
$$+ (\xi_3 v_2 - \xi_2 v_3)\frac{\partial\langle\xi, v\rangle}{\partial x_1} + (\xi_1 v_3 - \xi_3 v_1)\frac{\partial\langle\xi, v\rangle}{\partial x_2} + (\xi_2 v_1 - \xi_1 v_2)\frac{\partial\langle\xi, v\rangle}{\partial x_3}\Big](x + t\xi)\,dt.$$

Following [6, formula (2.1)], we introduce three second order differential operators on $C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$

$$J_1 = \frac{\partial^2}{\partial x_2 \partial \xi_3} - \frac{\partial^2}{\partial x_3 \partial \xi_2}, \quad J_2 = \frac{\partial^2}{\partial x_3 \partial \xi_1} - \frac{\partial^2}{\partial x_1 \partial \xi_3}, \quad J_3 = \frac{\partial^2}{\partial x_1 \partial \xi_2} - \frac{\partial^2}{\partial x_2 \partial \xi_1}.$$

They are called *John's operators*. We are going to compute the values of John's operators on the Nadirashvili – Vladuts potential w. We will present calculations for $J_3 w$ and then will write corresponding formulas for $J_1 w$ and for $J_2 w$ by analogy.

Differentiate (43) with respect to ξ_2

$$\begin{aligned}
\frac{\partial^2 w}{\partial x_1 \partial \xi_2}(x,\xi) = |\xi|^{-2} & \int\limits_{\infty}^{\infty} t\Big(\xi_3 \frac{\partial (j_\xi f)_2}{\partial x_2} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial x_2}\Big)(x+t\xi)\,dt \\
& + |\xi|^{-2} \int\limits_{-\infty}^{\infty} \Big(\xi_3 \frac{\partial (j_\xi f)_2}{\partial \xi_2} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial \xi_2}\Big)(x+t\xi)\,dt \\
& - |\xi|^{-2} \int\limits_{-\infty}^{\infty} (j_\xi f)_3(x+t\xi)\,dt \\
& - 2|\xi|^{-4}\xi_2 \int\limits_{-\infty}^{\infty} \big(\xi_3 (j_\xi f)_2 - \xi_2 (j_\xi f)_3\big)(x+t\xi)\,dt.
\end{aligned}$$

Then differentiate (44) with respect to ξ_1

$$\begin{aligned}
\frac{\partial^2 w}{\partial x_2 \partial \xi_1}(x,\xi) = |\xi|^{-2} & \int\limits_{-\infty}^{\infty} t\Big(-\xi_3 \frac{\partial (j_\xi f)_1}{\partial x_1} + \xi_1 \frac{\partial (j_\xi f)_3}{\partial x_1}\Big)(x+t\xi)\,dt \\
& + |\xi|^{-2} \int\limits_{-\infty}^{\infty} \Big(-\xi_3 \frac{\partial (j_\xi f)_1}{\partial \xi_1} + \xi_1 \frac{\partial (j_\xi f)_3}{\partial \xi_1}\Big)(x+t\xi)\,dt \\
& + |\xi|^{-2} \int\limits_{-\infty}^{\infty} (j_\xi f)_3(x+t\xi)\,dt \\
& - 2|\xi|^{-4}\xi_1 \int\limits_{-\infty}^{\infty} \big(-\xi_3 (j_\xi f)_1 + \xi_1 (j_\xi f)_3\big)(x+t\xi)\,dt.
\end{aligned}$$

Take the difference of two last equations (for brevity, we do not write arguments)

3.27 (56)

$$\begin{aligned} J_3 w = |\xi|^{-2} \int\limits_{-\infty}^{\infty} t\Big(\xi_3 \frac{\partial (j_\xi f)_1}{\partial x_1} + \xi_3 \frac{\partial (j_\xi f)_2}{\partial x_2} - \xi_1 \frac{\partial (j_\xi f)_3}{\partial x_1} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial x_2}\Big)\, dt \\ + |\xi|^{-2} \int\limits_{-\infty}^{\infty} \Big(\xi_3 \frac{\partial (j_\xi f)_1}{\partial \xi_1} + \xi_3 \frac{\partial (j_\xi f)_2}{\partial \xi_2} - \xi_1 \frac{\partial (j_\xi f)_3}{\partial \xi_1} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial \xi_2}\Big)\, dt \\ - 2|\xi|^{-2} \int\limits_{-\infty}^{\infty} (j_\xi f)_3\, dt \\ - 2|\xi|^{-4} \int\limits_{-\infty}^{\infty} \Big(\xi_1\xi_3 (j_\xi f)_1 + \xi_2\xi_3 (j_\xi f)_2 - \xi_1^2 (j_\xi f)_3 - \xi_2^2 (j_\xi f)_3\Big)\, dt. \end{aligned} \tag{56}$$

First of all we have to treat the first integral on the right-hand side of (56) containing the factor t in the integrand. We transform the integral as follows

$$\begin{aligned} &\int\limits_{-\infty}^{\infty} t\Big(\xi_3 \frac{\partial (j_\xi f)_1}{\partial x_1} + \xi_3 \frac{\partial (j_\xi f)_2}{\partial x_2} - \xi_1 \frac{\partial (j_\xi f)_3}{\partial x_1} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial x_2}\Big)(x + t\xi)\, dt \\ &= \int\limits_{-\infty}^{\infty} t\Big(\xi_3 \operatorname{div}(j_\xi f) - \xi_1 \frac{\partial (j_\xi f)_3}{\partial x_1} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial x_2} - \xi_3 \frac{\partial (j_\xi f)_3}{\partial x_3}\Big)(x + t\xi)\, dt \\ &= \xi_3 \int\limits_{-\infty}^{\infty} t \operatorname{div}(j_\xi f)(x + t\xi)\, dt - \int\limits_{-\infty}^{\infty} t \frac{d}{dt}\Big((j_\xi f)_3(x + t\xi)\Big)\, dt. \end{aligned}$$

After transforming the second integral on the right-hand side with the help of integration by parts, this gives

$$\begin{aligned} &\int\limits_{-\infty}^{\infty} t\Big(\xi_3 \frac{\partial (j_\xi f)_1}{\partial x_1} + \xi_3 \frac{\partial (j_\xi f)_2}{\partial x_2} - \xi_1 \frac{\partial (j_\xi f)_3}{\partial x_1} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial x_2}\Big)\, dt \\ &\qquad = \xi_3 \int\limits_{-\infty}^{\infty} t \operatorname{div}(j_\xi f)\, dt + \int\limits_{-\infty}^{\infty} (j_\xi f)_3\, dt. \end{aligned}$$

Substitute this value into (56)

3.28

$$
\begin{aligned}
J_3 w = {} & |\xi|^{-2}\xi_3 \int_{-\infty}^{\infty} t \operatorname{div}(j_\xi f)\, dt \\
& + |\xi|^{-2} \int_{-\infty}^{\infty} \Big(\xi_3 \frac{\partial (j_\xi f)_1}{\partial \xi_1} + \xi_3 \frac{\partial (j_\xi f)_2}{\partial \xi_2} - \xi_1 \frac{\partial (j_\xi f)_3}{\partial \xi_1} - \xi_2 \frac{\partial (j_\xi f)_3}{\partial \xi_2}\Big)\, dt \\
& - |\xi|^{-2} \int_{-\infty}^{\infty} (j_\xi f)_3\, dt \\
& - 2|\xi|^{-4} \int_{-\infty}^{\infty} \Big(\xi_1\xi_3 (j_\xi f)_1 + \xi_2\xi_3 (j_\xi f)_2 - \xi_1^2 (j_\xi f)_3 - \xi_2^2 (j_\xi f)_3\Big)\, dt.
\end{aligned} \tag{57}
$$

Thus, in the general case, we cannot eliminate the integral with the factor t in the integrand. Because of the difficulty, we continue the calculation for $f = v \otimes v$, where a vector field $v \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^3)$ satisfies the Euler equations (1)–(2). By (49),

$$
\int_{-\infty}^{\infty} t \operatorname{div}(j_\xi f)(x + t\xi)\, dt = -\int_{-\infty}^{\infty} t\, \langle \xi, (\nabla p)(x + t\xi)\rangle\, dt = -\int_{-\infty}^{\infty} t\, \frac{dp(x + t\xi)}{dt}\, dt.
$$

Transforming the last integral with the help of integration by parts, we obtain

$$
\int_{-\infty}^{\infty} t \operatorname{div}(j_\xi f)(x + t\xi)\, dt = (Ip)(x, \xi),
$$

where Ip is the ray transform of the pressure p. Substitute this and $j_\xi f = \langle \xi, v\rangle v$ into (57)

$$
\begin{aligned}
J_3 w = {} & |\xi|^{-2}\xi_3 Ip \\
& + |\xi|^{-2} \int_{-\infty}^{\infty} \Big(\xi_3 \frac{\partial (\langle \xi, v\rangle v_1)}{\partial \xi_1} + \xi_3 \frac{\partial (\langle \xi, v\rangle v_2)}{\partial \xi_2} - \xi_1 \frac{\partial (\langle \xi, v\rangle v_3)}{\partial \xi_1} - \xi_2 \frac{\partial (\langle \xi, v\rangle v)_3}{\partial \xi_2}\Big)\, dt \\
& - |\xi|^{-2} \int_{-\infty}^{\infty} \langle \xi, v\rangle v_3\, dt - 2|\xi|^{-4} \int_{-\infty}^{\infty} \langle \xi, v\rangle \big(\xi_1\xi_3 v_1 + \xi_2\xi_3 v_2 - \xi_1^2 v_3 - \xi_2^2 v_3\big)\, dt.
\end{aligned}
$$

We emphasize that derivatives on the integrand are understood in the following sense:

$$
\frac{\partial (\langle \xi, v\rangle v_i)}{\partial \xi_j} = \frac{\partial \big(\langle \xi, v\rangle v_i(y)\big)}{\partial \xi_j}\Big|_{y = x + t\xi} = (v_i v_j)(x + t\xi)
$$

because the derivatives $\frac{\partial v_i(x+t\xi)}{\partial\xi_j}$ have been already taken into account in the first integral on the right-hand side of (56). We thus obtain

$$J_3w=|\xi|^{-2}\xi_3Ip+|\xi|^{\lambda-1}\int\limits_{-\infty}^{\infty}\big(\xi_3v_1^2+\xi_3v_2^2-\xi_1v_1v_3-\xi_2v_2v_3\big)\,dt$$
$$-|\xi|^{-2}\int\limits_{-\infty}^{\infty}\langle\xi,v\rangle v_3\,dt-2|\xi|^{-4}\int\limits_{-\infty}^{\infty}\langle\xi,v\rangle\big(\xi_1\xi_3v_1+\xi_2\xi_3v_2-\xi_1^2v_3-\xi_2^2v_3\big)\,dt.$$

After grouping similar terms, we obtain

$$(J_3w)(x,\xi)=\frac{\xi_3}{|\xi|^2}(Ip)(x,\xi)+\frac{\xi_3}{|\xi|^2}\int\limits_{-\infty}^{\infty}|v(x+t\xi)|^2\,dt-2\frac{\xi_3}{|\xi|^4}\int\limits_{-\infty}^{\infty}\langle\xi,v(x+t\xi)\rangle^2\,dt.$$

This can be written in the form

$$J_3w=\frac{\xi_3}{|\xi|^2}I_0(p)+\frac{\xi_3}{|\xi|^2}I_0(|v|^2)-2\frac{\xi_3}{|\xi|^4}I_2(v\otimes v).$$

For definiteness, the notation I_m is used here for the ray transform of symmetric tensor fields of rank m. Let δ be the Kronecker tensor. Since $I_2(a\delta)=|\xi|^2I_0(a)$ for a scalar function a, the previous formula can be written as

$$J_3w=\frac{\xi_3}{|\xi|^4}I\big((p+|v|^2)\delta-2v\otimes v\big).$$

Formulas for J_1w and J_2w are obtained from this in an obvious way. We can write the final formula:

3.29 (58)
$$J_jw=\frac{\xi_j}{|\xi|^4}I\big((p+|v|^2)\delta-2v\otimes v\big),\quad (j=1,2,3).$$

Formula (58) generalizes Nadirashvili–Valaduts's formula [5, formula (4.2)] in the following sence: Nadirashvili–Valaduts's formula makes sense only for $(x,\xi)\in T\mathbb{S}^2$ while formula (58) holds on the whole of $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$. The factor $\frac{\xi_j}{|\xi|^4}$ on the right-hand side of (58) is very essential.

Next, we are going to compute J_i^2w $(1\le i\le 3)$. We start with computing J_3^2w. By (58),

$$J_3^2w=J_3\Big(\frac{\xi_3}{|\xi|^4}I\big((p+|v|^2)\delta-2v\otimes v\big)\Big).$$

Since the operator J_3 does not contain the derivative $\partial/\partial\xi_3$,

3.30 (59)
$$J_3^2w=\xi_3J_3\Big(\frac{1}{|\xi|^4}I\big((p+|v|^2)\delta-2v\otimes v\big)\Big).$$

Next, using $I\big((p+|v|^2)\delta\big)=|\xi|^2I(p+|v|^2)$, we calculate

$$J_3\Big(\frac{1}{|\xi|^4}I\big((p+|v|^2)\delta-2v\otimes v\big)\Big)=J_3\Big(\frac{1}{|\xi|^2}I(p+|v|^2)\Big)-2J_3\Big(\frac{1}{|\xi|^4}I(v\otimes v)\Big)$$

3.31 (60)
$$=\frac{1}{|\xi|^2}J_3I(p+|v|^2)-\frac{2}{|\xi|^4}J_3I(v\otimes v)$$
$$-\frac{2}{|\xi|^4}\Big(\xi_2\frac{\partial}{\partial x_1}-\xi_1\frac{\partial}{\partial x_2}\Big)I(p+|v|^2)+\frac{8}{|\xi|^6}\Big(\xi_2\frac{\partial}{\partial x_1}-\xi_1\frac{\partial}{\partial x_2}\Big)I(v\otimes v).$$

The first term on the right-hand side is equal to zero because $J_3Ia = 0$ for any scalar function a [8, Theorem 2.10.1]. Thus, (59) and (60) give

3.32 (61)
$$J_3^2 w = -\frac{2\xi_3}{|\xi|^4}\Big(\xi_2\frac{\partial}{\partial x_1} - \xi_1\frac{\partial}{\partial x_2}\Big) I(p+|v|^2) - \frac{2\xi_3}{|\xi|^4} J_3 I(v\otimes v) + \frac{8\xi_3}{|\xi|^6}\Big(\xi_2\frac{\partial}{\partial x_1} - \xi_1\frac{\partial}{\partial x_2}\Big) I(v\otimes v). \tag{61}$$

The first term on the right-hand side of (61) can be easily computed. Indeed, by the definition of the ray transform,

$$I(p+|v|^2) = \int\limits_{-\infty}^{\infty} (p+|v|^2)(x+t\xi)\,dt.$$

From this

3.33 (62)
$$\Big(\xi_2\frac{\partial}{\partial x_1} - \xi_1\frac{\partial}{\partial x_2}\Big) I(p+|v|^2) = \int\limits_{-\infty}^{\infty} \Big(\xi_2\frac{\partial(p+|v|^2)}{\partial x_1} - \xi_1\frac{\partial(p+|v|^2)}{\partial x_2}\Big)(x+t\xi)\,dt. \tag{62}$$

Now, we compute two last terms on the right-hand side of (61). By the definition of the ray transform,

$$I(v\otimes v) = \int\limits_{-\infty}^{\infty} \langle\xi, v\rangle^2(x+t\xi)\,dt.$$

From this,

3.34 (63)
$$\frac{\partial I(v\otimes v)}{\partial x_i} = \int\limits_{-\infty}^{\infty} \frac{\partial\langle\xi, v\rangle^2}{\partial x_i}(x+t\xi)\,dt. \tag{63}$$

In particular,

3.35 (64)
$$\Big(\xi_2\frac{\partial}{\partial x_1} - \xi_1\frac{\partial}{\partial x_2}\Big) I(v\otimes v) = 2\int\limits_{-\infty}^{\infty} \Big[\langle\xi, v\rangle\Big(\xi_2\frac{\partial\langle\xi, v\rangle}{\partial x_1} - \xi_1\frac{\partial\langle\xi, v\rangle}{\partial x_2}\Big)\Big](x+t\xi)\,dt. \tag{64}$$

Differentiate (63) with respect to ξ_k

$$\frac{\partial^2 I(v\otimes v)}{\partial x_i\partial\xi_k} = \int\limits_{-\infty}^{\infty} t\frac{\partial^2\langle\xi, v\rangle^2}{\partial x_i\partial x_k}(x+t\xi)\,dt + \int\limits_{-\infty}^{\infty} \frac{\partial^2\langle\xi, v\rangle^2}{\partial x_i\partial\xi_k}(x+t\xi)\,dt.$$

The first integral on the right-hand side is symmetric in (i,k). It will disappear after the alternation in these indices. We thus obtain

3.36 (65)
$$\begin{aligned} J_3 I(v\otimes v) &= \int\limits_{-\infty}^{\infty} \Big(\frac{\partial^2\langle\xi, v\rangle^2}{\partial x_1\partial\xi_2} - \frac{\partial^2\langle\xi, v\rangle^2}{\partial x_2\partial\xi_1}\Big)(x+t\xi)\,dt \\ &= 2\int\limits_{-\infty}^{\infty} \Big[v_2\frac{\partial\langle\xi, v\rangle}{\partial x_1} - v_1\frac{\partial\langle\xi, v\rangle}{\partial x_2} + \langle\xi, v\rangle\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big)\Big](x+t\xi)\,dt. \end{aligned} \tag{65}$$

Substituting values (62), (64)–(65) into (61), we obtain

3.37 (66)
$$
\begin{aligned}
J_3^2 w = &-\frac{2\xi_3}{|\xi|^4}\int\limits_{-\infty}^{\infty}\Big(\xi_2\frac{\partial(p+|v|^2)}{\partial x_1}-\xi_1\frac{\partial(p+|v|^2)}{\partial x_2}\Big)(x+t\xi)\,dt\\
&-\frac{4\xi_3}{|\xi|^4}\int\limits_{-\infty}^{\infty}\Big[v_2\frac{\partial\langle\xi,v\rangle}{\partial x_1}-v_1\frac{\partial\langle\xi,v\rangle}{\partial x_2}+\langle\xi,v\rangle\Big(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\Big)\Big](x+t\xi)\,dt\\
&+\frac{16\xi_3}{|\xi|^6}\int\limits_{-\infty}^{\infty}\Big[\langle\xi,v\rangle\Big(\xi_2\frac{\partial\langle\xi,v\rangle}{\partial x_1}-\xi_1\frac{\partial\langle\xi,v\rangle}{\partial x_2}\Big)\Big](x+t\xi)\,dt.
\end{aligned}
$$

Let us eliminate p from (66) with the help of the Euler equations. To this end we rewrite (66) in the form

$$
\begin{aligned}
J_3^2 w = &-\frac{2\xi_3}{|\xi|^4}\int\limits_{-\infty}^{\infty}\Big(\xi_2\frac{\partial p}{\partial x_1}-\xi_1\frac{\partial p}{\partial x_2}\Big)(x+t\xi)\,dt\\
&-\frac{2\xi_3}{|\xi|^4}\int\limits_{-\infty}^{\infty}\Big(\xi_2\frac{\partial|v|^2}{\partial x_1}-\xi_1\frac{\partial|v|^2}{\partial x_2}\Big)(x+t\xi)\,dt\\
&-\frac{4\xi_3}{|\xi|^4}\int\limits_{-\infty}^{\infty}\Big[v_2\frac{\partial\langle\xi,v\rangle}{\partial x_1}-v_1\frac{\partial\langle\xi,v\rangle}{\partial x_2}+\langle\xi,v\rangle\Big(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\Big)\Big](x+t\xi)\,dt\\
&+\frac{16\xi_3}{|\xi|^6}\int\limits_{-\infty}^{\infty}\Big[\langle\xi,v\rangle\Big(\xi_2\frac{\partial\langle\xi,v\rangle}{\partial x_1}-\xi_1\frac{\partial\langle\xi,v\rangle}{\partial x_2}\Big)\Big](x+t\xi)\,dt.
\end{aligned}
$$

By the Euler equations (1),

$$
\frac{\partial p}{\partial x_i}=-\langle v,\nabla v_i\rangle.
$$

Substitute these values into the previous equation

3.38 (67)
$$
\begin{aligned}
J_3^2 w = \frac{2\xi_3}{|\xi|^6}\int\limits_{-\infty}^{\infty}\Big[&|\xi|^2\Big(\xi_2\langle v,\nabla v_1\rangle-\xi_1\langle v,\nabla v_2\rangle-\xi_2\frac{\partial|v|^2}{\partial x_1}+\xi_1\frac{\partial|v|^2}{\partial x_2}\Big)\\
&-2|\xi|^2\Big(v_2\frac{\partial\langle\xi,v\rangle}{\partial x_1}-v_1\frac{\partial\langle\xi,v\rangle}{\partial x_2}+\langle\xi,v\rangle\frac{\partial v_2}{\partial x_1}-\langle\xi,v\rangle\frac{\partial v_1}{\partial x_2}\Big)\\
&+8\langle\xi,v\rangle\Big(\xi_2\frac{\partial\langle\xi,v\rangle}{\partial x_1}-\xi_1\frac{\partial\langle\xi,v\rangle}{\partial x_2}\Big)\Big](x+t\xi)\,dt.
\end{aligned}
$$

The corresponding formulas for $J_1^2 w$ and for $J_2^2 w$ are obtained from (67) by the cyclic permutation of indices

3.39 (68)
$$\begin{aligned}J_1^2 w = \frac{2\xi_1}{|\xi|^6}\int\limits_{-\infty}^{\infty} \Big[&|\xi|^2\Big(\xi_3\langle v,\nabla v_2\rangle - \xi_2\langle v,\nabla v_3\rangle - \xi_3\frac{\partial |v|^2}{\partial x_2} + \xi_2\frac{\partial |v|^2}{\partial x_3}\Big)\\ &- 2|\xi|^2\Big(v_3\frac{\partial\langle\xi,v\rangle}{\partial x_2} - v_2\frac{\partial\langle\xi,v\rangle}{\partial x_3} + \langle\xi,v\rangle\frac{\partial v_3}{\partial x_2} - \langle\xi,v\rangle\frac{\partial v_2}{\partial x_3}\Big)\\ &+ 8\langle\xi,v\rangle\Big(\xi_3\frac{\partial\langle\xi,v\rangle}{\partial x_2} - \xi_2\frac{\partial\langle\xi,v\rangle}{\partial x_3}\Big)\Big](x+t\xi)\,dt,\end{aligned}$$

3.40 (69)
$$\begin{aligned}J_2^2 w = \frac{2\xi_2}{|\xi|^6}\int\limits_{-\infty}^{\infty} \Big[&|\xi|^2\Big(\xi_1\langle v,\nabla v_3\rangle - \xi_3\langle v,\nabla v_1\rangle - \xi_1\frac{\partial |v|^2}{\partial x_3} + \xi_3\frac{\partial |v|^2}{\partial x_1}\Big)\\ &- 2|\xi|^2\Big(v_1\frac{\partial\langle\xi,v\rangle}{\partial x_3} - v_3\frac{\partial\langle\xi,v\rangle}{\partial x_1} + \langle\xi,v\rangle\frac{\partial v_1}{\partial x_3} - \langle\xi,v\rangle\frac{\partial v_3}{\partial x_1}\Big)\\ &+ 8\langle\xi,v\rangle\Big(\xi_1\frac{\partial\langle\xi,v\rangle}{\partial x_3} - \xi_3\frac{\partial\langle\xi,v\rangle}{\partial x_1}\Big)\Big](x+t\xi)\,dt.\end{aligned}$$

Take the sum of equations (67), (68) and (69). Many terms cancel each other in the sum and the result is as follows:

3.41 (70)
$$\begin{aligned}&(J_1^2+J_2^2+J_3^2)w =\\ &= -\frac{4}{|\xi|^4}\int\limits_{-\infty}^{\infty}\Big[\langle\xi,v\rangle\Big(\xi_1\frac{\partial v_3}{\partial x_2} - \xi_1\frac{\partial v_2}{\partial x_3} + \xi_2\frac{\partial v_1}{\partial x_3} - \xi_2\frac{\partial v_3}{\partial x_1} + \xi_3\frac{\partial v_2}{\partial x_1} - \xi_3\frac{\partial v_1}{\partial x_2}\Big)\\ &+ (\xi_3 v_2 - \xi_2 v_3)\frac{\partial\langle\xi,v\rangle}{\partial x_1} + (\xi_1 v_3 - \xi_3 v_1)\frac{\partial\langle\xi,v\rangle}{\partial x_2} + (\xi_2 v_1 - \xi_1 v_2)\frac{\partial\langle\xi,v\rangle}{\partial x_3}\Big](x+t\xi)\,dt.\end{aligned}$$

Comparing (55) and (70), we see that

3.42 (71)
$$\big(J_1^2+J_2^2+J_3^2+\frac{4}{|\xi|^2}\Delta^v\big)w = 0.$$

We have thus proved

P3.2 **Proposition 3.** *Given a solution $(v,p)\in\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)\times\mathcal{S}(\mathbb{R}^3)$ to the Euler equations* (1)–(2)*, let w be the Nadirashvili – Vladuts potential for $f = v\otimes v$. The function w solves the equation* (71)*.*

The equation (71) is the right extension of Nadirasvili–Vladuts's equation [5, formula (4.5)] to the whole of $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$.

The operator $H = J_1^2+J_2^2+J_3^2+\frac{4}{|\xi|^2}\Delta^v$ is a 4th order differential operator on $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$. Its principal part $J_1^2+J_2^2+J_3^2$ is a differential operator with constant coefficients. Observe that the operator H is "almost elliptic". Indeed, let y and η be Fourier dual variables for x and ξ respectively. The principle symbol of H is $-|y\times\eta|^2$. The symbol vanishes if and only if $y = t\eta$ $(t\in\mathbb{R})$. This property of the symbol is well agreed with the property (31) of the function w.

4. Solutions of the equation (71)

We are looking for solutions $w \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$ to the equation (71) satisfying (31) and such that $w|_{T\mathbb{S}^2}$ belongs to $\mathcal{S}(T\mathbb{S}^2)$. All solutions of such kind are described by the following

Th4.1 **Theorem 3.** *If $w \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$ is a solution to the equation (71) satisfying (31) and such that $w|_{T\mathbb{S}^2}$ belongs to $\mathcal{S}(T\mathbb{S}^2)$, then there exists a symmetric tensor field $\widehat a \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying the equation*

4.1
$$|y|^2 \mathrm{tr}\,\widehat a(y) - y^i y^j \widehat a_{ij}(y) = 0 \tag{72}$$

and such that

4.2
$$w(x,\xi) = \frac{\xi^i\xi^j}{2\pi|\xi|^3} \int\limits_{\xi^\perp} e^{\mathrm{i}\langle x,y\rangle} \widehat a_{ij}(y)\,dy. \tag{73}$$

Conversely, if a symmetric tensor field $\widehat a \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ solves the equation (72) then, being defined by (73), the function $w \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$ is a solution to the equation (71) satisfying (31) and such that $w|_{T\mathbb{S}^2}$ belongs to $\mathcal{S}(T\mathbb{S}^2)$.

A general solution $w \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$ to the equation (71) depends on two arbitrary functions belonging to $C^\infty(\mathbb{R}^3 \setminus \{0\})$.

The rest of the Section is devoted to the pretty long proof of Theorem 3. We start with repeating arguments from the proof of [8, Theorem 2.10.1].

Let $w \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$ be a solution to the equation (71) satisfying (31) and such that $w_0 = w|_{T\mathbb{S}^2} \in \mathcal{S}(T\mathbb{S}^2)$. Let $\widehat w_0 \in \mathcal{S}(T\mathbb{S}^2)$ be the Fourier transform of w_0 (see [8, Section 2.2] for the definition of the Fourier transform $F : \mathcal{S}(T\mathbb{S}^2) \to \mathcal{S}(T\mathbb{S}^2)$). Define the function $\widehat w \in C^\infty\big(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\big)$ by

$$\widehat w(y,\xi) = \widehat w_0\Big(y - \frac{\langle y,\xi\rangle}{|\xi|^2}\xi, \frac{\xi}{|\xi|}\Big).$$

Then

4.3
$$\widehat w|_{T\mathbb{S}^2} = \widehat w_0, \quad \widehat w(y,t\xi) = \widehat w(y,\xi)\ (0 \neq t \in \mathbb{R}), \quad \widehat w(y+t\xi,\xi) = \widehat w(y,\xi)\ (t \in \mathbb{R}) \tag{74}$$

and w is expressed through $\widehat w$ by

4.4
$$w(x,\xi) = (2\pi)^{-1}|\xi|^{-1} \int\limits_{\xi^\perp} e^{\mathrm{i}\langle x,y\rangle} \widehat w(y,\xi)\,dy. \tag{75}$$

Let us derive a differential equation for $\widehat w$ which follows from (71). To this end we rewrite (75) in the form

$$w(x,\xi) = (2\pi)^{-1} \int\limits_{\mathbb{R}^3} e^{\mathrm{i}\langle x,y\rangle} \delta(\langle \xi,y\rangle) \widehat w(y,\xi)\,dy,$$

where δ is the Dirac function. Differentiating this equality, we obtain

$$\begin{aligned}\frac{\partial^2 w(x,\xi)}{\partial x_j \partial \xi_k} = \mathrm{i}(2\pi)^{-1} &\int\limits_{\mathbb{R}^3} y_j e^{\mathrm{i}\langle x,y\rangle} \delta(\langle \xi,y\rangle) \frac{\partial \widehat w(y,\xi)}{\partial \xi_k}\,dy \\ &+ \mathrm{i}(2\pi)^{-1} \int\limits_{\mathbb{R}^3} y_j y_k e^{\mathrm{i}\langle x,y\rangle} \delta'(\langle \xi,y\rangle) \widehat w(y,\xi)\,dy.\end{aligned}$$

The second integral is symmetric in (j,k), it will disappear after the alternation in these indices. The result can be written as

$$\Big(\frac{\partial^2}{\partial x_j\partial\xi_k}-\frac{\partial^2}{\partial x_k\partial\xi_j}\Big)w(x,\xi)=i(2\pi)^{-1}|\xi|^{-1}\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}\Big(y_j\frac{\partial}{\partial\xi_k}-y_k\frac{\partial}{\partial\xi_j}\Big)\widehat{w}(y,\xi)\,dy.$$

Repeating this procedure, we obtain

$$\Big(\frac{\partial^2}{\partial x_j\partial\xi_k}-\frac{\partial^2}{\partial x_k\partial\xi_j}\Big)^2w(x,\xi)=-(2\pi)^{-1}|\xi|^{-1}\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}\Big(y_j\frac{\partial}{\partial\xi_k}-y_k\frac{\partial}{\partial\xi_j}\Big)^2\widehat{w}(y,\xi)\,dy.$$

Therefore

4.5 (76)
$$(J_1^2+J_2^2+J_3^2)w(x,\xi)=-(2\pi)^{-1}|\xi|^{-1}\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}L\widehat{w}(y,\xi)\,dy,$$

where L is the second order differential operator on $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})$ defined by

4.6 (77)
$$L=\Big(y_1\frac{\partial}{\partial\xi_2}-y_2\frac{\partial}{\partial\xi_1}\Big)^2+\Big(y_2\frac{\partial}{\partial\xi_3}-y_3\frac{\partial}{\partial\xi_2}\Big)^2+\Big(y_3\frac{\partial}{\partial\xi_1}-y_1\frac{\partial}{\partial\xi_3}\Big)^2.$$

One easily derives from (75) with the help of (53)

4.7 (78)
$$(\Delta^v w)(x,\xi)=\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\Big)w(x,\xi)=-(2\pi)^{-1}|\xi|^{-1}\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}|y|^2\widehat{w}(y,\xi)\,dy.$$

Substituting (76) and (78) into (71), we obtain

4.8 (79)
$$\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}\Big(L+4\frac{|y|^2}{|\xi|^2}\Big)\widehat{w}(y,\xi)\,dy=0.$$

Being valid for every $(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, equation (79) implies

4.9 (80)
$$\Big(L+4\frac{|y|^2}{|\xi|^2}\Big)\widehat{w}(y,\xi)=0\quad\text{for}\quad y\in\xi^\perp.$$

One easily derives from the definition (77) the following property of the operator L:

4.10 (81)
$$L(|\xi|^2\varphi)=|\xi|^2L\varphi+4|y|^2\langle\xi,\partial_\xi\rangle\varphi-4\langle y,\xi\rangle\langle y,\partial_\xi\rangle\varphi+4|y|^2\varphi$$

for any function $\varphi\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$, where

$$\langle\xi,\partial_\xi\rangle=\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}+\xi_3\frac{\partial}{\partial\xi_3},\quad\langle y,\partial_\xi\rangle=y_1\frac{\partial}{\partial\xi_1}+y_2\frac{\partial}{\partial\xi_2}+y_3\frac{\partial}{\partial\xi_3}.$$

By (74), the function $\widehat{w}(y,\xi$ is positively homogeneous of zero degree in ξ, hence $\langle\xi,\partial_\xi\rangle\widehat{w}=0$. The formula (81) is simplified for $\varphi=\widehat{w}=0$ as follows:

$$L(|\xi|^2\widehat{w})=|\xi|^2L\widehat{w}-4\langle y,\xi\rangle\langle y,\partial_\xi\rangle\widehat{w}+4|y|^2\widehat{w}.$$

Together with (80), this gives

$$L(|\xi|^2\widehat{w})=-4\langle y,\xi\rangle\langle y,\partial_\xi\rangle\widehat{w}.$$

In particular,

4.11 (82)
$$\big(L(|\xi|^2\widehat{w})\big)\big|_{T\mathbb{S}^2}=0.$$

L4.1 **Lemma 1.** *Let a function* $\varphi\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ *satisfy*

4.12 (83) $$\varphi(y,t\xi)=t^2\varphi(y,\xi)\ \ (0\neq t\in\mathbb{R}).$$

Assume that $\varphi|_{T\mathbb{S}^2}\in\mathcal{S}(T\mathbb{S}^2)$ *and* $(L\varphi)|_{T\mathbb{S}^2}=0$. *Then there exists a symmetric tensor field* $\widehat a\in\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ *satisfying the equation* (72) *and such that*

4.13 (84) $$\varphi(y,\xi)=\widehat a_{ij}(y)\xi^i\xi^j\quad \textit{for } (y,\xi)\in\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\ \textit{satisfying}\ \langle y,\xi\rangle=0.$$

We finish the proof of Theorem 3 with the help of the lemma. The proof of Lemma 1 is presented at the end of the section.

By (74) and (82), the function $\varphi=|\xi|^2\widehat w$ satisfies hypotheses of Lemma 1. Applying the lemma, we obtain

$$\widehat w(y,\xi)=|\xi|^{-2}\widehat a_{ij}(y)\xi^i\xi^j\quad\text{for } (y,\xi)\in\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\text{ satisfying } \langle y,\xi\rangle=0$$

with a symmetric tensor field $\widehat a\in\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying the equation (72). Substituting this expression into (75), we arrive to (73). This proves the first statement of Theorem 3.

We prove now the second statement of Theorem 3. Given a symmetric tensor field $\widehat a\in\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying (72), we define the function $w\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ by (73). The following properties of the function follow obviously from (73):

4.14 (85) $$w(y,t\xi)=|t|^{-1}w(y,\xi)\ \ (0\neq t\in\mathbb{R}),\quad w(y+t\xi,\xi)=w(y,\xi)\ \ (t\in\mathbb{R}).$$

We rewrite (73) in the form

4.15 (86) $$w(x,\xi)=\frac{1}{|\xi|^3}a_{ij}(x,\xi)\xi^i\xi^j,$$

where

4.16 (87) $$a_{ij}(x,\xi)=(2\pi)^{-1}\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}\widehat a_{ij}(y)\,dy.$$

It is clear now from (86)–(87) that $w_0=w|_{\mathbb{S}^2}\in\mathcal{S}(T\mathbb{S}^2)$. It remains to prove that the function w solves the equation (71).

First of all we compute $\Delta^v w$. By (52),

$$\Delta^v=\Delta_x-|\xi|^{-2}\langle\xi,\partial_x\rangle,$$

where

$$\Delta_x=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2},\quad \langle\xi,\partial_x\rangle=\xi_1\frac{\partial}{\partial x_1}+\xi_2\frac{\partial}{\partial x_2}+\xi_3\frac{\partial}{\partial x_3}.$$

As easily follows from (85), $\langle\xi,\partial_x\rangle w=0$. Therefore $\Delta^v w=\Delta_x w$. Together with (86), this gives

4.17 (88) $$(\Delta^v)w(x,\xi)=\frac{1}{|\xi|^3}(\Delta_x a_{ij})(x,\xi)\xi^i\xi^j.$$

From (87)

$$(\Delta_x a_{ij})(x,\xi)=-(2\pi)^{-1}\int\limits_{\xi^\perp}e^{i\langle x,y\rangle}|y|^2\widehat a_{ij}(y)\,dy.$$

Substitute this expression into (88) to obtain

4.18 (89)
$$(\Delta^v)w(x,\xi) = -\frac{1}{2\pi|\xi|^3}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}|y|^2\,\widehat{a}_{ij}(y)\,dy. \tag{89}$$

Next, we compute $(J_1^2+J_2^2+J_3^2)w$. The computation is similar to the arguments presented after (75). We will compute $J_3^2 w$ and then write down the corresponding formulas for $J_1^2 w$ and $J_2^2 w$ by analogy. First of all we rewrite (73) in the form

$$w(x,\xi) = \frac{\xi^p\xi^q}{2\pi|\xi|^2}\int\limits_{\mathbb{R}^3} e^{i\langle x,y\rangle}\delta(\langle\xi,y\rangle)\,\widehat{a}_{pq}(y)\,dy,$$

where δ is the Dirac function. Differentiate this equation to obtain

$$\begin{aligned}\frac{\partial^2 w(x,\xi)}{\partial x_1\partial\xi_2} &= i\,\frac{\xi^p\xi^q}{2\pi|\xi|^2}\int\limits_{\mathbb{R}^3} e^{i\langle x,y\rangle}y_1y_2\delta'(\langle\xi,y\rangle)\,\widehat{a}_{pq}(y)\,dy\\ &+2i\,\frac{\xi^p}{2\pi|\xi|^2}\int\limits_{\mathbb{R}^3} e^{i\langle x,y\rangle}y_1\delta(\langle\xi,y\rangle)\,\widehat{a}_{pk}(y)\,dy\\ &-2i\,\frac{\xi_k\xi^p\xi^q}{2\pi|\xi|^4}\int\limits_{\mathbb{R}^3} e^{i\langle x,y\rangle}y_1\delta(\langle\xi,y\rangle)\,\widehat{a}_{pq}(y)\,dy.\end{aligned}$$

The first integral on the right-hand side disappears after alternating with respect to the indices 1 and 2. Hence

$$\begin{aligned}J_3w(x,\xi) &= 2i\,\frac{\xi^p}{2\pi|\xi|^3}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\big(y_j\widehat{a}_{pk}(y) - y_k\widehat{a}_{pj}(y)\big)\,dy\\ &-2i\,\frac{\xi^p\xi^q}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_j\xi_k - y_k\xi_j)\widehat{a}_{pq}(y)\,dy.\end{aligned}$$

Repeating this procedure, we obtain

$$\begin{aligned}J_3^2w(x,\xi) = -2\frac{1}{2\pi|\xi|^3}&\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\big(y_2^2\widehat{a}_{11}(y) - 2y_1y_2\widehat{a}_{12}(y) + y_2^2\widehat{a}_{33}(y)(y)\big)\,dy\\ +8\frac{\xi^p}{2\pi|\xi|^5}&\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_1\xi_2 - y_2\xi_1)\big(y_1\widehat{a}_{2p}(y) - y_2\widehat{a}_{1p}(y)\big)\,dy\\ -8\frac{\xi^p\xi^q}{2\pi|\xi|^7}&\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_1\xi_2 - y_2\xi_1)^2\widehat{a}_{pq}(y)\,dy\\ +2\frac{\xi^p\xi^q}{2\pi|\xi|^5}&\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_1^2 + y_2^2)\widehat{a}_{pq}(y)\,dy.\end{aligned}$$

The corresponding formulas for $J_1^2 w$ and for $J_2^2 w$ are obtained by the cyclic permutation of indices

$$\begin{aligned}J_1^2 w(x,\xi) = -2\frac{1}{2\pi|\xi|^3}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\big(y_3^2\widehat a_{22}(y) - 2y_2y_3\widehat a_{23}(y) + y_2^2\widehat a_{33}(y)(y)\big)\,dy\\ + 8\frac{\xi^p}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_2\xi_3 - y_3\xi_2)\big(y_2\widehat a_{3p}(y) - y_3\widehat a_{2p}(y)\big)\,dy\\ - 8\frac{\xi^p\xi^q}{2\pi|\xi|^7}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_2\xi_3 - y_3\xi_2)^2\widehat a_{pq}(y)\,dy\\ + 2\frac{\xi^p\xi^q}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_2^2 + y_3^2)\widehat a_{pq}(y)\,dy,\end{aligned}$$

$$\begin{aligned}J_2^2 w(x,\xi) = -2\frac{1}{2\pi|\xi|^3}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\big(y_3^2\widehat a_{11}(y) - 2y_1y_3\widehat a_{13}(y) + y_1^2\widehat a_{33}(y)(y)\big)\,dy\\ + 8\frac{\xi^p}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_3\xi_1 - y_1\xi_3)\big(y_3\widehat a_{1p}(y) - y_1\widehat a_{3p}(y)\big)\,dy\\ - 8\frac{\xi^p\xi^q}{2\pi|\xi|^7}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_3\xi_1 - y_1\xi_3)^2\widehat a_{pq}(y)\,dy\\ + 2\frac{\xi^p\xi^q}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}(y_1^2 + y_3^2)\widehat a_{pq}(y)\,dy.\end{aligned}$$

Take the sum of three last equalities

4.19

$$\begin{aligned}&(J_1^2 + J_2^2 + J_3^2)w(x,\xi) =\\ &= -2\frac{1}{2\pi|\xi|^3}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\Big[(y_2^2+y_3^2)\widehat a_{11} + (y_1^2+y_3^2)\widehat a_{22} + (y_1^2+y_2^2)\widehat a_{33}\\ &\qquad - 2y_1y_2\widehat a_{12} - 2y_1y_3\widehat a_{13} - 2y_2y_3\widehat a_{23}\Big]\,dy\\ &+8\frac{\xi^p}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\Big[\big((y_2^2+y_3^2)\xi_1 - y_1y_2\xi_2 - y_1y_3\xi_3\big)\widehat a_{1p}\\ &\qquad + \big((y_1^2+y_3^2)\xi_2 - y_1y_2\xi_1 - y_2y_3\xi_3\big)\widehat a_{2p}\\ &\qquad + \big((y_1^2+y_2^2)\xi_3 - y_1y_3\xi_1 - y_2y_3\xi_2\big)\widehat a_{3p}\Big]\,dy\\ &-8\frac{\xi^p\xi^q}{2\pi|\xi|^7}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}|y\times\xi|^2\,\widehat a_{pq}(y)\,dy + 4\frac{\xi^p\xi^q}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}|y|^2\,\widehat a_{pq}(y)\,dy.\end{aligned} \tag{90}$$

Recall that the tensor field $\widehat a_{ij}$ is assumed to satisfy (72). The first integrand on the right-hand side of (90) is equal to zero in virtue of the latter equation. Indeed,

$$\begin{aligned}(y_2^2+y_3^2)\widehat a_{11} + (y_1^2+y_3^2)\widehat a_{22} + (y_1^2+y_2^2)\widehat a_{33} - 2y_1y_2\widehat a_{12} - 2y_1y_3\widehat a_{13} - 2y_2y_3\widehat a_{23}\\ = |y|^2\mathrm{tr}\,\widehat a - \widehat a_{ij}y^iy^j = 0.\end{aligned}$$

The fourth integrand on the right-hand side of (90) can be simplified since $|y\times\xi|^2 = |y|^2|\xi|^2$ for $y\in\xi^\perp$. We write the result in the preliminary form

4.20
$$(J_1^2+J_2^2+J_3^2)w(x,\xi) = -4\frac{\xi^p\xi^q}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}|y|^2\,\widehat{a}_{pq}(y)\,dy + (K\widehat{a})(x,\xi), \tag{91}$$

where

$$(K\widehat{a})(x,\xi) = 8\frac{1}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}\xi^p\Big[\big((y_2^2+y_3^2)\xi_1-y_1y_2\xi_2-y_1y_3\xi_3\big)\widehat{a}_{1p}$$
$$+\big((y_1^2+y_3^2)\xi_2-y_1y_2\xi_1-y_2y_3\xi_3\big)\widehat{a}_{2p}+\big((y_1^2+y_2^2)\xi_3-y_1y_3\xi_1-y_2y_3\xi_2\big)\widehat{a}_{3p}\Big]\,dy.$$

The last integrand can be also simplified with the help of the relation $\langle\xi,y\rangle=0$:

$$\begin{aligned}
&\xi^p\Big[\big((y_2^2+y_3^2)\xi_1-y_1y_2\xi_2-y_1y_3\xi_3\big)\widehat{a}_{1p}+\big((y_1^2+y_3^2)\xi_2-y_1y_2\xi_1-y_2y_3\xi_3\big)\widehat{a}_{2p}\\
&\qquad+\big((y_1^2+y_2^2)\xi_3-y_1y_3\xi_1-y_2y_3\xi_2\big)\widehat{a}_{3p}\Big]\\
&=\xi^p\Big[\big((y_2^2+y_3^2)\xi_1-y_1(y_2\xi_2+y_3\xi_3)\big)\widehat{a}_{1p}+\big((y_1^2+y_3^2)\xi_2-y_2(y_1\xi_1+y_3\xi_3)\big)\widehat{a}_{2p}\\
&\qquad+\big((y_1^2+y_2^2)\xi_3-y_3(y_1\xi_1+y_2\xi_2)\big)\widehat{a}_{3p}\Big]\\
&=\xi^p\Big[\big((y_2^2+y_3^2)\xi_1+y_1^2\xi_1\big)\widehat{a}_{1p}+\big((y_1^2+y_3^2)\xi_2+y_2^2\xi_2\big)\widehat{a}_{2p}+\big((y_1^2+y_2^2)\xi_3+y_3^2\xi_3\big)\widehat{a}_{3p}\Big]\\
&\qquad=|y|^2\xi^p\xi^q\widehat{a}_{pq}.
\end{aligned}$$

Substituting this value into (91), we obtain the final formula

4.21
$$(J_1^2+J_2^2+J_3^2)w(x,\xi) = 4\frac{\xi^p\xi^q}{2\pi|\xi|^5}\int\limits_{\xi^\perp} e^{i\langle x,y\rangle}|y|^2\,\widehat{a}_{pq}(y)\,dy. \tag{92}$$

By (89) and (92), $\Big(J_1^2+J_2^2+J_3^2+\frac{4}{|\xi|^2}\Delta^v\Big)w=0$. This proves the second statement of Theorem 3.

The last statement of Theorem 3 (dependence of a general solution on two arbitrary functions) will be explained after the proof of Lemma 1.

Proof of Lemma 1. For $0\neq y\in\mathbb{R}^3$, set $y^\perp=\{\xi\in\mathbb{R}^3\mid\langle\xi,y\rangle=0\}$. Let $\Delta_{y^\perp}: C^\infty(y^\perp)\to C^\infty(y^\perp)$ be the Euclidean Laplacian on the plane $y^\perp$. Given a function $\varphi\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ and vector $0\neq y\in\mathbb{R}^3$, define the function $\varphi_y\in C^\infty(y^\perp\setminus\{0\})$ by $\varphi_y(\xi)=\varphi(y,\xi)$. Then the equality

4.22
$$(L\varphi)(y)=|y|^2\Delta_{y^\perp}\varphi_y \tag{93}$$

holds for every function $\varphi\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ and every vector $0\neq y\in\mathbb{R}^3$. Indeed, for $0\neq y=(0,0,|y|)$, the equality obviously follows from the definition (77) of the operator L. It remains to observe that both sides of (93) are invariant under the action of the orthogonal group $O(3)$.

Let a function $\varphi\in C^\infty\big(\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\big)$ satisfy hypotheses of Lemma 1. By (83), $\varphi(x,\xi)$ is positively homogeneous of second degree in ξ. Therefore φ can be extended to a continuous function on $\mathbb{R}^3\times\mathbb{R}^3$ by setting $\varphi(y,0)=0$. We denote the extension by φ again.

Let us introduce the 5-dimensional submanifold $M \subset \mathbb{R}^3 \times \mathbb{R}^3$ by

$$M = \{(y,\xi) \mid y \neq 0, \langle y,\xi\rangle = 0\}.$$

Then

4.23 (94)
$$M \to \mathbb{R}^3 \setminus \{0\}, \quad (y,\xi) \mapsto y$$

is the two-dimensional vector bundle with the fiber $y^\perp$ over a point $0 \neq y \in \mathbb{R}^3$. Let $\psi \in C(M)$ be the restriction of φ to M. We will see soon that actually $\psi \in C^\infty(M)$.

In virtue of (93), the hypothesis $(L\varphi)|_{T\mathbb{S}^2} = 0$ of Lemma 1 means that, for every $0 \neq y \in \mathbb{R}^3$, the restriction ψ_y of the function ψ to the fiber $y^\perp$ of the bundle (94) satisfies

$$(\Delta_{y^\perp}\psi_y)(\xi) = 0 \quad \text{for} \quad \xi \neq 0.$$

This equality must hold at $\xi = 0$ as well since ψ_y is a continuous function on $y^\perp$. Thus,

4.24 (95)
$$\Delta_{y^\perp}\psi_y = 0,$$

i.e. ψ_y is a harmonic function on $y^\perp$. Besides this, $\psi_y(\xi)$ is a positively homogeneous function of second degree by (83). As well known (and can be easily proved) a harmonic second degree positively homogeneous function is a homogeneous polynomial of second degree. Thus, given an orthonormal basis (f^1, f^2) of $y^\perp$, the function ψ_y can be represented in the form

4.25 (96)
$$\psi_y(\xi_1 f^1 + \xi_2 f^2) = \widehat{c'}_{11}\xi_1^2 + 2\widehat{c'}_{12}\xi_1\xi_2 + \widehat{c'}_{22}\xi_2^2$$

with uniquely determined coefficients $\widehat{c'}_{ij} = \widehat{c'}_{ji} \in \mathbb{C}$ $(1 \leq i,j \leq 2)$. The equation (95) is equivalent to the equality

4.26 (97)
$$\widehat{c'}_{11} + \widehat{c'}_{22} = 0.$$

We can now prove smoothness of the function ψ. For a fixed $0 \neq y_0 \in \mathbb{R}^3$, we can choose an orthonormal basis (f_y^1, f_y^2) of the space $y^\perp$ smoothly depending on a point y belonging to some neighborhood $U \subset \mathbb{R}^3 \setminus \{0\}$ of the point y_0. By (96),

4.27 (98)
$$\psi(y, \xi_1 f_y^1 + \xi_2 f_y^2) = \varphi(y, \xi_1 f_y^1 + \xi_2 f_y^2) = \widehat{c'}_{11}(y)\xi_1^2 + 2\widehat{c'}_{12}(y)\xi_1\xi_2 + \widehat{c'}_{22}(y)\xi_2^2 \quad (y \in U)$$

with uniquely determined coefficients $\widehat{c'}_{ij} \in C(U)$. The right hand side of (98) smoothly depends on $(y;\xi_1,\xi_2) \in U \times \mathbb{R}^2$, at least for $\xi_1^2 + \xi_2^2 \neq 0$, since $\varphi \in C^\infty\big(\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})\big)$. This implies that $\widehat{c'}_{ij} \in C^\infty(U)$ and hence $\psi \in C^\infty(M)$.

We cannot write (98) for all points $y \in \mathbb{R}^3 \setminus \{0\}$ simultaneously. Instead of that, we will write some coordinate-free formula equivalent to (98). To this end we have to use so called tangential tensor fields introduced in [4, Section 4].

We think on $\mathbb{R}^3 \setminus \{0\}$ as the disjoint union (= foliation) of spheres centered at the origin

$$\mathbb{R}^3 \setminus \{0\} = \bigcup_{\rho>0} \mathbb{S}^2_\rho, \qquad \mathbb{S}^2_\rho = \{y \in \mathbb{R}^3 \mid |y| = \rho\}.$$

The manifold M is also presented as the disjoint union

$$M = \bigcup_{\rho>0} T\mathbb{S}^2_\rho, \qquad T\mathbb{S}^2_\rho = \{(y,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |y| = \rho, \xi \in y^\perp\}.$$

Introduce the operator

$$i_y : C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3)\to C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)$$

of symmetric multiplication by y and operator

$$j_y : C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)\to C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3)$$

of contraction with the vector y by the formulas

$$(i_y v)_{k\ell}=\frac{1}{2}(y_k v_\ell+y_\ell v_k),\quad (j_y f)_k=y^\ell f_{k\ell}.$$

We emphasize that i_y and j_y are invariant operators (independent of coordinate choice) although last formulas are written in Cartesian coordinates. The same is true for the operator $\operatorname{tr}: C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)\to C^\infty(\mathbb{R}^3)$, $\operatorname{tr} f=f_{11}+f_{22}+f_{33}$.

We say that $v\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3)$ is *a tangential vector field* if $\langle y,v(y)\rangle=0$ for all $0\neq y\in\mathbb{R}^3$. In other words, the vector $v(y)$ is tangent to the sphere $\mathbb{S}^2_{|y|}$. Quite similarly, a symmetric $f\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)$ is called *a tangential tensor field* if $j_y f(y)\rangle=0$ for all $0\neq y\in\mathbb{R}^3$.

We can now present an invariant version of (97)–(98). Under hypotheses of Lemma 1, there exists a symmetric tangential tensor field $\widehat{c}\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying

4.28 (99) $$\operatorname{tr}\widehat{c}=0$$

and such that

4.29 (100) $$\varphi(y,\xi)=\widehat{c}_{ij}(y)\xi^i\xi^j\quad(0\neq y\in\mathbb{R}^3,\xi\in y^\perp).$$

Indeed, choosing an orthonormal basis (f^1_y,f^2_y) of $y^\perp$ smoothly depending on a point y belonging to some neighborhood U of a fixed point $0\neq y_0\in\mathbb{R}^3$, one easily checks that formulas (99)–(100) are equivalent to (97)–(98) for $y\in U$.

Recall [8, Lemma 2.6.1] that every symmetric tensor field $a\in C^\infty(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ can be uniquely represented in the form

$$a_{ij}(y)={}^t a_{ij}(y)+\frac{1}{2}\big(y_i b_j(y)+y_j b_i(y)\big)\quad(y\neq 0)$$

with a vector field $b\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3)$ and symmetric tensor field ${}^t a\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying

4.30 (101) $$y^j\,{}^t a_{ij}(y)=0\quad(i=1,2,3).$$

In terminology of [8], ${}^t a$ is the *tangential component* and $\frac{1}{2}(y_i b_j+y_j b_i)$ is the *radial component* of the tensor field a. In our terminology, (101) means that ${}^t a$ is a tangential tensor field.

Recall also the important *theorem on the tangential component* [8, Theorem 2.7.1]. It states that a symmetric tangential tensor field $c\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)$ serves as the tangential component of some symmetric tensor field $a\in C^\infty(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ if and only if the restriction of the function $c_{ij}(y)\xi^i\xi^j$ to $T\mathbb{S}^2$ belongs to $C^\infty(T\mathbb{S}^2)$. The most important (and difficult to prove) part of this statement is the smoothness of $a(y)$ at $y=0$.

Returning to the proof of Lemma 1, we apply the theorem on the tangential component to the symmetric tangential tensor field $\widehat{c}\in C^\infty(\mathbb{R}^3\setminus\{0\};\mathbb{C}^3\otimes\mathbb{C}^3)$ satisfying (99)–(100). The hypothesis $\varphi|_{T\mathbb{S}^2}\in\mathcal{S}(T\mathbb{S}^2)$ of Lemma 1 means that the restriction of the function $\widehat{c}_{ij}(y)\xi^i\xi^j$ to $T\mathbb{S}^2$ belongs to $\mathcal{S}(T\mathbb{S}^2)$. The theorem on the tangential component gives us a smooth symmetric tensor field $\widehat{a} \in C^\infty(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ such that

4.31 (102) $$\widehat{a}_{ij}(y) = \widehat{c}_{ij}(y) + \frac{1}{2}\big(y_i b_j(y) + y_j b_i(y)\big) \quad (y \neq 0)$$

with some vector field b. Moreover, we can state that $\widehat{a} \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ since the restriction of the function $\widehat{a}_{ij}(y)\xi^i\xi^j$ to $T\mathbb{S}^2$ belongs to $\mathcal{S}(T\mathbb{S}^2)$.

Comparing (100) and (102), we obtain the statement (84) of Lemma 1.

Since $\widehat{c}$ is a symmetric tangential tensor field, it satisfies

4.32 (103) $$y^j\widehat{c}_{ij}(y) = 0 \quad (i = 1, 2, 3).$$

Together with (102), this gives

$$y^iy^j\widehat{a}_{ij}(y) = |y|^2\langle y, b(y)\rangle.$$

On the other hand, (99) and (101) give

$$\mathrm{tr}\,\widehat{a}(y) = \langle y, b(y)\rangle.$$

From two last formulas

$$|y|^2\mathrm{tr}\,\widehat{a}(y) - y^iy^j\widehat{a}_{ij}(y) = 0,$$

i.e., $\widehat{a}$ solves the equation (72). □

We still have to justify the last statement of Theorem 3. We emphasize that the tensor field $\widehat{a}$ in Theorem 3 and Lemma 1 is not unique. Only its tangential component $\widehat{c} = {}^t\widehat{a}$ is unique. In principle, Theorem 3 can be formulated in terms of a tensor field $\widehat{c}$ with the equation (72) replaced by (99). Nevertheless, we prefer to formulate Theorem 3 in terms of the field $\widehat{a}$ because the tangential component of a smooth symmetric tensor field has a specific singularity at the origin. Six components of the tensor field $\widehat{c}$ are subordinated to four linear equations (99) and (103). Therefore $\widehat{c}(y)$ is determined by two arbitrary functions belonging to $C^\infty(\mathbb{R}^3 \setminus \{0\})$.

5. Some open question

For a tensor field $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ belonging to the kernel of the operator $\mathcal{N}$, we defined the Nadirasvili – Valaduts potential w, see Proposition 2. Let us write $w[f]$ instead of w to emphasize the dependence on f. As is seen from the proof of Proposition 2, $w[f]$ depends linearly on f. The following question is still open:

Pr5.1 **Problem 1.** *Does the equality $w[f] = 0$ imply $f = 0$ for a tensor field $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ belonging to the kernel of $\mathcal{N}$? More generally, is it possible to describe explicitly the subspace of $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3\otimes\mathbb{C}^3)$ consisting of tensor fields f satisfying $\mathcal{N}f = 0$ and $w[f] = 0$?*

Then after (57), we discussed the Nadirasvili – Valaduts potential $w[v\otimes v]$ for a vector field $v \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)$ satisfying the Euler equations (1)–(2). Since $w[v\otimes v]$ depends quadratically on v, the corresponding question takes the form:

Pr5.2 **Problem 2.** *Does the equality $w[v\otimes v] = w[\tilde v\otimes\tilde v]$ imply $v = \tilde v$ for two vector fields $v, \tilde v \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)$ satisfying the Euler equations? More generally, given a solution $(v, p) \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)\times\mathcal{S}(\mathbb{R}^3)$ to the Euler equations, is it possible to describe all solution $(\tilde v, \tilde p) \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)\times\mathcal{S}(\mathbb{R}^3)$ to the Euler equations satisfying $w[\tilde v\otimes\tilde v] = w[v\otimes v]$?*

Nadirasvili and Valaduts [5] proved that $w[v \otimes v] = 0$ implies $v = 0$ in the case of a real v. But this does not answer Problem 2. If the first question of Problem 2 was answered "yes", Theorem 3 would imply that a general solution $(v, p) \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^3) \times \mathcal{S}(\mathbb{R}^3)$ to the Euler equations is determined by two arbitrary functions of $y \in \mathbb{R}^3 \setminus \{0\}$. In this way we hope to answer the following question:

Pr5.3 **Problem 3.** *Is it possible to classify all solutions (v, p) to the Euler equations such that $v_1, v_2, v_3, p \in \mathcal{S}(\mathbb{R}^3)$?*

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