

# A NEW CLASS OF GENERALIZED DERIVATIONS AND THE AUTOMATIC CONTINUITY OF ITS ELEMENTS

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ABSTRACT. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two algebras and let  $n$  be a positive integer. A linear mapping  $D : \mathcal{A} \rightarrow \mathcal{B}$  is called a *strongly generalized derivation of order  $n$* , if there exist the families of linear mappings  $\{E_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{F_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{G_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  and  $\{H_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  which satisfy  $D(ab) = \sum_{k=1}^n [E_k(a)F_k(b) + G_k(a)H_k(b)]$  for all  $a, b \in \mathcal{A}$ . The main purpose of this article is to study the automatic continuity of such derivations on Banach algebras and  $C^*$ -algebras.

## 1. INTRODUCTION AND PRELIMINARIES

Automatic continuity of derivations is an important topic in the theory of derivations and this topic has a fairly long history. The automatic continuity of a certain class of mappings, e.g. strongly generalized derivations, is the study of (algebraic) conditions on linear maps on a category of Banach algebras, e.g.  $C^*$ -algebras, which guarantee that every strongly generalized derivation is continuous. Results on automatic continuity of linear maps defined on Banach algebras comprise a fruitful area of research intensively developed during the last sixty years. The reader is referred to [4, 5, 24] for a deep and extensive study on this subject. Let us recall some basic definitions and set the notations which are used in what follows. Let  $\mathcal{A}$  be an algebra. A nonzero linear functional  $\varphi$  on  $\mathcal{A}$  is called a *character* if  $\varphi(ab) = \varphi(a)\varphi(b)$  for every  $a, b \in \mathcal{A}$ . The set of all characters on  $\mathcal{A}$  is denoted by  $\Phi_{\mathcal{A}}$  and is called the character space of  $\mathcal{A}$ . According to [4, Proposition 1.3.37], the kernel of  $\varphi$ ,  $\ker \varphi$ , is a maximal ideal of  $\mathcal{A}$  for every  $\varphi \in \Phi_{\mathcal{A}}$ . Recall that an algebra (or ring)  $\mathcal{A}$  is called prime if for  $a, b \in \mathcal{A}$ ,  $a\mathcal{A}b = \{0\}$  implies that  $a = 0$  or  $b = 0$ , and is semiprime if for  $a \in \mathcal{A}$ ,  $a\mathcal{A}a = \{0\}$  implies that  $a = 0$ .

Here we are going to introduce a new class of derivations called strongly generalized derivation of order  $n$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be two algebras and let  $n$  be a positive integer. A linear mapping  $D : \mathcal{A} \rightarrow \mathcal{B}$  is called a *strongly generalized derivation of order  $n$* , if there exist the families of linear mappings  $\{E_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{F_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{G_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  and  $\{H_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  which satisfy

$$D(ab) = \sum_{k=1}^n [E_k(a)F_k(b) + G_k(a)H_k(b)]$$

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for all  $a, b \in \mathcal{A}$ . Clearly, for  $n = 1$ , we have  $D(ab) = E(a)F(b) + G(a)H(b)$  for all  $a, b \in \mathcal{A}$ , where  $E, F, G, H : \mathcal{A} \rightarrow \mathcal{B}$  are linear mappings. If  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a strongly generalized derivation of order one associated with the mappings  $E, F, G, H : \mathcal{A} \rightarrow \mathcal{B}$ , then we say  $D$  is an  $(E, F, G, H)$ -derivation. Also, if  $D$  is a strongly generalized derivation of order  $n$  associated with the families of linear mappings  $\{E_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{F_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{G_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  and  $\{H_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ , we say  $D$  is an  $(\{E_k\}_{k=1}^n, \{F_k\}_{k=1}^n, \{G_k\}_{k=1}^n, \{H_k\}_{k=1}^n)$ -derivation. As can be seen, if  $D$  is a strongly generalized derivation of order one, then it covers the notion of derivation (if  $D = E = H$  and  $F = G = I$ ), the notion of generalized  $(\sigma, \tau)$ -derivation associated with a linear mapping  $d$  (if  $D = E$ ,  $F = \sigma$ ,  $G = \tau$  and  $H = d$ ), the notion of left  $\sigma$ -centralizer (if  $D = E$ ,  $F = \sigma$  and  $G$  or  $H$  is zero), the notion of right  $\tau$ -centralizer (if  $E$  or  $F$  is zero,  $G = \tau$  and  $H = D$ ), the notion of generalized derivation associated with a mapping  $d$  (if  $D = E$ ,  $F = G = I$  and  $H = d$ ), the notion of homomorphism (if  $D = E = F$  and  $G = 0$  or  $H = 0$ ), and the notion of ternary derivation (if  $F = G = I$ ). Let  $\mathcal{A}$  be an algebra. Recall that a triple of linear maps  $(D, E, H)$  of  $\mathcal{A}$  is a ternary derivation of  $\mathcal{A}$  if  $D(ab) = E(a)b + aH(b)$  for all  $a, b \in \mathcal{A}$ .

Derivations are used in quantum mechanics (see [2, 3]), and it is interesting to note that the applications of generalized types of derivations, such as generalized derivations and  $(\sigma, \tau)$ -derivations, to important physical topics have been recently studied. See, for example, [15] for the application of generalized derivations in general relativity, and [6, 13] for the application of  $(\sigma, \tau)$ -derivations in theoretical physics. Therefore, it is possible that *strongly generalized derivations of order  $n$*  might be considered by physicists in the future and used in the study of physical topics. Hence, it is interesting to investigate details of this general notion of derivations.

The main objective of this paper is to obtain some conclusions about the automatic continuity of strongly generalized derivations of order one on Banach algebras and  $C^*$ -algebras. Let us give a brief background in this regard. In 1958, Kaplansky [18] conjectured that every derivation on a  $C^*$ -algebra is continuous. Two years later, Sakai [23] answered this conjecture. Indeed, he proved that every derivation on a  $C^*$ -algebra is automatically continuous and later in 1972, Ringrose [22], by using the pioneering work of Bade and Curtis [1] concerning the automatic continuity of a module homomorphism between bimodules over  $C(K)$ -spaces, showed that every derivation from a  $C^*$ -algebra  $\mathcal{A}$  into a Banach  $\mathcal{A}$ -bimodule is automatically continuous. Also, Johnson and Sinclair [16] investigated the continuity of derivations on semisimple Banach algebras. In addition, in an interesting article, Peralta and Russo [21] investigated automatic continuity of derivations on  $C^*$ -algebras and  $JB^*$ -triples. Moreover, Hou and Ming [14] proved that if  $\mathcal{X}$  is simple and  $\sigma, \tau$  are surjective and continuous mappings on  $B(\mathcal{X})$ , then every  $(\sigma, \tau)$ -derivation on  $B(\mathcal{X})$  is continuous. It is worth to note that there are some other results on automatic continuity in the literature. For example, B. E. Johnson proved in [17] that every generalized  $*$ -homomorphism between  $C^*$ -algebras is continuous, while several extensions to generalized triple homomorphisms from a  $C^*$ -algebra or from a  $JB^*$ -triple are established in [8]. We further know from [7] that every generalized derivation on a von Neumann algebra and every linear mapping on a von Neumann algebra which is a derivation or a triplet derivation at zero is automatically

continuous. Furthermore, every generalized Jordan derivation from a  $C^*$ -algebra into a Jordan Banach module is continuous [9].

Now, we state some of the results in this paper. Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are two complex algebras such that  $\mathcal{A}$  is simple and unital with identity element  $\mathbf{1}$ , and  $\Phi_{\mathcal{B}} \neq \phi$ . Let  $D : \mathcal{A} \rightarrow \mathcal{B}$  be a  $(D, F, G, H)$ -derivation such that  $\varphi_0(D(\mathbf{1})) = \varphi_0(H(\mathbf{1})) \neq 0$  and  $\varphi_0(D(a_0)) \neq 0$  for some  $a_0 \in \mathcal{A} \setminus \{\mathbf{1}\}$  and some  $\varphi_0 \in \Phi_{\mathcal{B}}$ . Then  $\dim(\mathcal{A}) = 1$  and  $\dim(D(\mathcal{A})), \dim(F(\mathcal{A})), \dim(H(\mathcal{A})) \leq 1$ . In addition, if  $\mathcal{A}$  and  $\mathcal{B}$  are two normed algebras, then  $D, F$  and  $H$  are continuous (see Theorem 2.5).

In Theorem 2.7 we also obtain the following:

Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of  $B(\mathfrak{H})$ , the  $C^*$ -algebra of all bounded linear operators on a Hilbert space  $\mathfrak{H}$ , and let  $D : \mathcal{A} \rightarrow B(\mathfrak{H})$  be an  $*$ - $(E, F, G, H)$ -derivation, i.e.  $D, E, F, G$  and  $H$  are  $*$ -mappings. Then the following statements hold:

(i) If we suppose that  $E(I) = H(I) = I$ , where  $I$  is the identity mapping on  $\mathfrak{H}$ ,  $H(a)D(I)E(b) = -E(a)D(I)H(b)$  for all  $a, b \in \mathcal{A}$  and also  $(E + H)$  is a  $D$ -continuous mapping, then  $D$  is automatically continuous.

(ii) If we suppose that  $F(I) = G(I) = I$ ,  $G(a)D(I)F(b) = -F(a)D(I)G(b)$  for all  $a, b \in \mathcal{A}$  and also  $(F + G)$  is a  $D$ -continuous mapping, then  $D$  is automatically continuous. Using the just commented theorem, we also obtain a result on the automatic continuity of generalized  $(\sigma, \tau)$ -derivations on  $C^*$ -algebras.

Moreover, we prove that if  $\mathcal{A}$  is a Banach algebra,  $\mathcal{B}$  is a simple Banach algebra and  $D : \mathcal{A} \rightarrow \mathcal{B}$  is both a continuous  $(D, F, G, H)$ -derivation and continuous  $(H, F, G, D)$ -derivation such that  $H$  is a continuous linear mapping and  $F, G$  are surjective, then  $F$  and  $G$  are continuous or  $H$  and  $D$  are identically zero.

Some other results concerning the continuity and characterization of  $(E, F, G, H)$ -derivations are also discussed.

## 2. MAIN RESULTS

Throughout the paper,  $I$  denotes the identity mapping on an algebra and  $\mathbf{1}$  stands for the identity element of any unital algebra. We begin with the following definition.

**Definition 2.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two algebras and let  $n$  be a positive integer. A linear mapping  $D : \mathcal{A} \rightarrow \mathcal{B}$  is called a *strongly generalized derivation of order  $n$* , if there exist the families  $\{E_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{F_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{G_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  and  $\{H_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  of linear mappings which satisfy

$$D(ab) = \sum_{k=1}^n [E_k(a)F_k(b) + G_k(a)H_k(b)]$$

for all  $a, b \in \mathcal{A}$ . Clearly, for  $n = 1$ , we have  $D(ab) = E(a)F(b) + G(a)H(b)$  for all  $a, b \in \mathcal{A}$ , where  $E, F, G, H : \mathcal{A} \rightarrow \mathcal{B}$  are linear mappings.

If  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a strongly generalized derivation of order one associated with the mappings  $E, F, G, H : \mathcal{A} \rightarrow \mathcal{B}$ , then we say  $D$  is an  $(E, F, G, H)$ -derivation. Also, if  $D$  is a strongly generalized derivation of order  $n$  associated with the families  $\{E_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{F_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ ,  $\{G_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$  and  $\{H_k : \mathcal{A} \rightarrow \mathcal{B}\}_{k=1}^n$ , we say  $D$  is an  $(\{E_k\}_{k=1}^n, \{F_k\}_{k=1}^n, \{G_k\}_{k=1}^n, \{H_k\}_{k=1}^n)$ -derivation. As stated in Introduction,

concepts such as derivation, generalized derivation and generalized  $(\sigma, \tau)$ -derivation are examples of strongly generalized derivations of order one. Here, we give some examples of strongly generalized derivations of order  $n \geq 2$ . Mirzavaziri and Omidvar Tehrani [20] introduced the concept of a  $(\delta, \varepsilon)$ -double derivation as follows:

Let  $\mathcal{A}$  be an algebra and let  $\delta, \varepsilon : \mathcal{A} \rightarrow \mathcal{A}$  be linear mappings. A linear mapping  $d : \mathcal{A} \rightarrow \mathcal{A}$  is called a  $(\delta, \varepsilon)$ -double derivation if

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b),$$

for all  $a, b \in \mathcal{A}$ . By a  $\delta$ -double derivation we mean a  $(\delta, \delta)$ -double derivation. For more material about  $(\delta, \varepsilon)$ -double derivations, see, e.g. [11, 20].

**Example 2.2.** Every  $(\delta, \varepsilon)$ -double derivation is a strongly generalized derivation of order 2.

**Example 2.3.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two algebras. A sequence  $\{d_n\}$  of linear mappings from  $\mathcal{A}$  into  $\mathcal{B}$  is called a higher derivation if  $d_n(ab) = \sum_{k=0}^n d_{n-k}(a)d_k(b)$  for all  $a, b \in \mathcal{A}$  and all nonnegative integer  $n$ . Let  $n$  be a positive integer and let  $\{d_n\}$  be a higher derivation. Then every  $d_n$  is a strongly generalized derivation of order  $m$  in which

$$m = \begin{cases} \frac{n+2}{2} & n \text{ is even,} \\ \frac{n+1}{2} & n \text{ is odd.} \end{cases}$$

We now establish the following auxiliary result.

**Proposition 2.4.** *Let  $\mathcal{A}$  be a unital ring, let  $\mathcal{B}$  be a ring which is a domain and let  $D : \mathcal{A} \rightarrow \mathcal{B}$  be a  $(D, F, F, H)$ -derivation such that  $D(\mathbf{1}) = H(\mathbf{1})$ . If  $D(a_0) \neq 0$  for some  $a_0 \in \mathcal{A} \setminus \{\mathbf{1}\}$  and  $D(\mathbf{1}) \neq 0$ , then  $\ker(F) = \ker(D) \subseteq \ker(H)$  and further  $\ker(D)$  is an ideal of  $\mathcal{A}$ .*

*Proof.* First of all, we show that if  $D$  is a  $(D, F, F, H)$ -derivation such that  $D(\mathbf{1}) = H(\mathbf{1})$  and  $D(a_0) \neq 0$  for some  $a_0 \in \mathcal{A} \setminus \{\mathbf{1}\}$ , then  $D(\mathbf{1}) = 0$  if and only if  $F(\mathbf{1}) = \mathbf{1}$ . It is evident that if  $F(\mathbf{1}) = \mathbf{1}$ , then  $D(\mathbf{1}) = H(\mathbf{1}) = 0$ . Conversely, assume that  $D(\mathbf{1}) = 0$ . So, we have  $D(a_0) = D(a_0)F(\mathbf{1}) + F(a_0)H(\mathbf{1}) = D(a_0)F(\mathbf{1})$ . Since  $\mathcal{B}$  is a domain and  $D(a_0) \neq 0$ , we get that  $F(\mathbf{1}) = \mathbf{1}$ . Hence,  $D(\mathbf{1}) = 0$  if and only if  $F(\mathbf{1}) = \mathbf{1}$  and consequently, we deduce that  $D(\mathbf{1}) \neq 0$  if and only if  $F(\mathbf{1}) \neq \mathbf{1}$ .

For  $a \in \ker(F)$ , we have  $D(a) = D(a)F(\mathbf{1}) + F(a)H(\mathbf{1}) = D(a)F(\mathbf{1})$ , and since  $\mathcal{B}$  is a domain and  $F(\mathbf{1}) \neq \mathbf{1}$ , it is deduced that  $D(a) = 0$ . This means that  $a \in \ker(D)$ . So  $\ker(F) \subseteq \ker(D)$ . Now, suppose that  $a \in \ker(D)$ . Thus,  $0 = D(a) = D(a)F(\mathbf{1}) + F(a)H(\mathbf{1}) = F(a)H(\mathbf{1})$ . Since  $\mathcal{B}$  is a domain and  $H(\mathbf{1}) \neq 0$ , we get that  $F(a) = 0$ . It means that  $a \in \ker(F)$  and so  $\ker(D) \subseteq \ker(F)$ . Therefore, we obtain that  $\ker(F) = \ker(D)$ . For any  $a \in \ker(D)$ , we have  $0 = D(a) = D(\mathbf{1})F(a) + F(\mathbf{1})H(a) = F(\mathbf{1})H(a)$ , and since  $\mathcal{B}$  is a domain and  $D(\mathbf{1}) \neq 0$ , we arrive at  $H(a) = 0$ , which means that  $\ker(D) \subseteq \ker(H)$ . Our next task is to show that  $\ker(D)$  is an ideal of  $\mathcal{A}$ . Let  $a \in \ker(D)$  and  $b$  be an arbitrary element of  $\mathcal{A}$ . We know that  $\ker(F) = \ker(D) \subseteq \ker(H)$ . So we have

$$D(ab) = D(a)F(b) + F(a)H(b) = 0,$$

which means that  $\ker(D)$  is a right ideal of  $\mathcal{A}$ . Also, we have

$$D(ba) = D(b)F(a) + F(b)H(a) = 0,$$

which means that  $\ker(D)$  is a left ideal of  $\mathcal{A}$ . Consequently,  $\ker(D)$  is an ideal of  $\mathcal{A}$ , as desired.  $\square$

**Theorem 2.5.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are two complex algebras such that  $\mathcal{A}$  is unital and simple and also  $\Phi_{\mathcal{B}} \neq \phi$ . Let  $D : \mathcal{A} \rightarrow \mathcal{B}$  be a  $(D, F, F, H)$ -derivation such that  $\varphi_0(D(\mathbf{1})) = \varphi_0(H(\mathbf{1})) \neq 0$  and  $\varphi_0(D(a_0)) \neq 0$  for some  $a_0 \in \mathcal{A} \setminus \{\mathbf{1}\}$  and some  $\varphi_0 \in \Phi_{\mathcal{B}}$ . Then  $\dim(\mathcal{A}) = 1$  and  $\dim(D(\mathcal{A})), \dim(F(\mathcal{A})), \dim(H(\mathcal{A})) \leq 1$ . In addition, if  $\mathcal{A}$  and  $\mathcal{B}$  are in addition two normed algebras, then  $D, F$  and  $H$  are continuous.*

*Proof.* Let  $\varphi_0$  be a character on  $\mathcal{B}$  such that  $\varphi_0(D(\mathbf{1})) = \varphi_0(H(\mathbf{1})) \neq 0$  and  $\varphi_0(D(a_0)) \neq 0$  for some  $a_0 \in \mathcal{A} \setminus \{\mathbf{1}\}$ . Let  $\varphi_0 F = F_1, \varphi_0 H = H_1$  and  $\varphi_0 D = D_1$ . According to Proposition 2.4,  $\ker(D_1) = \ker(F_1)$  and  $\ker(D_1)$  is an ideal of  $\mathcal{A}$ . Since  $\mathcal{A}$  is simple,  $\ker(D_1) = \mathcal{A}$  or  $\ker(D_1) = \{0\}$  and since we are assuming that  $D_1(a_0) \neq 0$  for some  $a_0 \in \mathcal{A} \setminus \{\mathbf{1}\}$ ,  $\ker(D_1) \neq \mathcal{A}$ . So  $\ker(D_1) = \{0\}$  and this means that  $D_1$  is injective. It is a well-known fact that every non-zero linear functional is surjective. Hence,  $D_1 : \mathcal{A} \rightarrow \mathbb{C}$  is surjective, and therefore  $D_1$  is bijective, and clearly,  $\dim(\mathcal{A}) = 1$ . It follows from [19, Theorem 2.6-9] that  $\dim(D(\mathcal{A})), \dim(F(\mathcal{A})), \dim(H(\mathcal{A})) \leq 1$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are in addition two normed algebras, then [19, Theorem 2.7-8] implies the continuity of  $D, F$  and  $H$ .  $\square$

The concept of left (resp. right) continuity of a mapping was defined in [12] and we state the definition here.

**Definition 2.6.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two normed algebras and let  $T, S : \mathcal{A} \rightarrow \mathcal{B}$  be two mappings. The mapping  $T$  is left (resp. right)  $S$ -continuous if  $\lim_{x \rightarrow 0} T(x)S(b) = 0$  (resp.  $\lim_{x \rightarrow 0} S(b)T(x) = 0$ ) for all  $b \in \mathcal{B}$ . If  $T$  is both left and right  $S$ -continuous, then it is simply called  $S$ -continuous.

**Theorem 2.7.** *Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of the  $C^*$ -algebra  $B(\mathfrak{H})$ . Let  $D : \mathcal{A} \rightarrow B(\mathfrak{H})$  be a  $*$ - $(E, F, G, H)$ -derivation, i.e.  $D, E, F, G$  and  $H$  are  $*$ -mappings. The following statements hold:*

- (i) *Suppose  $E(I) = H(I) = I$ , where  $I$  is the identity mapping on  $\mathfrak{H}$ ,  $H(a)D(I)E(b) = -E(a)D(I)H(b)$  for all  $a, b \in \mathcal{A}$ , and  $(E + H)$  is a  $D$ -continuous mapping. Then  $D$  is automatically continuous.*
- (ii) *Suppose  $F(I) = G(I) = I$ ,  $G(a)D(I)F(b) = -F(a)D(I)G(b)$  for all  $a, b \in \mathcal{A}$ , and  $(F + G)$  is a  $D$ -continuous mapping. Then  $D$  is automatically continuous.*

*Proof.* (i) First note that  $D(I) = F(I) + G(I)$ . For any  $a \in \mathcal{A}$ , we have

$$D(a) = E(a)F(I) + G(a)H(I) = E(a)F(I) + G(a).$$

So we have

$$G(a) = D(a) - E(a)F(I), \quad (a \in \mathcal{A}). \quad (2.1)$$

Similarly, we can get

$$F(a) = D(a) - G(I)H(a), \quad (a \in \mathcal{A}). \quad (2.2)$$

Using (2.1) and (2.2), we have the following expressions:

$$\begin{aligned}
D(ab) &= E(a)F(b) + G(a)H(b) \\
&= E(a)(D(b) - G(I)H(b)) + (D(a) - E(a)F(I))H(b) \\
&= E(a)D(b) - E(a)G(I)H(b) + D(a)H(b) - E(a)F(I)H(b) \\
&= D(a)H(b) + E(a)D(b) - E(a)D(I)H(b),
\end{aligned}$$

which means that

$$D(ab) = D(a)H(b) + E(a)D(b) - E(a)D(I)H(b), \quad (a, b \in \mathcal{A}). \quad (2.3)$$

Using (2.3), we have the following statements:

$$\begin{aligned}
D(ab) &= (D(ab)^*)^* = (D(b^*)H(a^*) + E(b^*)D(a^*) - E(b^*)D(I)H(a^*))^* \\
&= H(a)D(b) + D(a)E(b) - H(a)D(I)E(b),
\end{aligned}$$

which means that

$$D(ab) = H(a)D(b) + D(a)E(b) - H(a)D(I)E(b), \quad (a, b \in \mathcal{A}). \quad (2.4)$$

Adding (2.3) and (2.4) and using the assumption that  $H(a)D(I)E(b) = -E(a)D(I)H(b)$  for all  $a, b \in \mathcal{A}$ , we arrive at

$$2D(ab) = D(a)(H(b) + E(b)) + (H(a) + E(a))D(b), \quad (a, b \in \mathcal{A}). \quad (2.5)$$

Considering  $\Sigma = \frac{E+H}{2}$  in (2.5), we get that

$$D(ab) = D(a)\Sigma(b) + \Sigma(a)D(b), \quad (a, b \in \mathcal{A}).$$

This means that  $D$  is a  $\Sigma$ -derivation on  $\mathcal{A}$  and since we are assuming that  $\Sigma = \frac{E+H}{2}$  is  $D$ -continuous, it follows from [12, Theorem 4.4] that  $D$  is continuous, as desired.

(ii) This part is obtained in the same way as above.  $\square$

An immediate consequence of the above theorem about the continuity of generalized  $*$ - $(\sigma, \tau)$ -derivations on  $C^*$ -algebras reads as follows:

**Corollary 2.8.** *Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of the  $C^*$ -algebra  $B(\mathfrak{H})$ . Let  $D : \mathcal{A} \rightarrow B(\mathfrak{H})$  be a generalized  $*$ - $(\sigma, \tau)$ -derivation corresponding to a mapping  $H : \mathcal{A} \rightarrow B(\mathfrak{H})$ , i.e.  $D(ab) = D(a)\sigma(b) + \tau(a)H(b)$  for all  $a, b \in \mathcal{A}$ . Suppose that  $\sigma(I) = \tau(I) = I$ , where  $I$  is the identity mapping on  $\mathfrak{H}$ ,  $\tau(a)D(I)\sigma(b) = -\sigma(a)D(I)\tau(b)$  for all  $a, b \in \mathcal{A}$ , and  $(\sigma + \tau)$  is a  $D$ -continuous mapping. Then  $D$  is automatically continuous.*

*Proof.* See Theorem 2.7(ii).  $\square$

**Theorem 2.9.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and let  $D : \mathcal{A} \rightarrow \mathcal{A}$  be an  $(E, I, I, H)$ -derivation. Then  $D, E$  and  $H$  are generalized derivations which are automatically continuous under any of the following conditions:*

- (i)  $D$  is a  $*$ - $(E, I, I, H)$ -derivation, that is,  $D, E$  and  $H$  are  $*$ -maps.
- (ii) The  $C^*$ -algebra  $\mathcal{A}$  is commutative.

*Proof.* First note that the triplet  $(D, E, H)$  is actually a ternary derivation. We have  $D(ab) = E(a)b + aH(b)$  for all  $a, b \in \mathcal{A}$ . Since we are assuming  $D, E$  and  $H$  are  $*$ -maps, we have

$$D(ab) = D^*(ab) = (D(ab)^*)^* = (E(b^*)a^* + b^*H(a^*))^* = H(a)b + aE(b),$$

which means that

$$D(ab) = H(a)b + aE(b), \quad (a, b \in \mathcal{A}).$$

Considering these two equalities for  $D(ab)$ , we have

$$D(ab) = E(a)b + aH(b) = H(a)b + aE(b), \quad (a, b \in \mathcal{A}).$$

According to [10, Theorem 3.1], there exists a derivation  $\delta : \mathcal{A} \rightarrow \mathcal{A}$  such that  $D = \delta + L_{D(\mathbf{1})}$ ,  $E = \delta + L_{E(\mathbf{1})}$  and  $H = \delta + L_{H(\mathbf{1})}$ , where  $L_a(b) = ab$  for all  $a, b \in \mathcal{A}$ . Hence,  $D, E$  and  $H$  are generalized derivations. It follows from the main theorem of [23] that the derivation  $\delta$  is continuous and so  $D, E$  and  $H$  are continuous as well.

(ii) Since  $\mathcal{A}$  is commutative, one can easily get that

$$D(ab) = E(a)b + aH(b) = H(a)b + aE(b), \quad (a, b \in \mathcal{A}).$$

Arguing as the proof of (i), we obtain the required result.  $\square$

**Theorem 2.10.** *Let  $\mathcal{A}$  be a unital algebra and let  $D : \mathcal{A} \rightarrow \mathcal{A}$  be an  $(E, I, I, H)$ -derivation. Then  $D$  and  $H$  are generalized derivations. Furthermore, if we assume that every derivation on  $\mathcal{A}$  is continuous, then  $D, E$  and  $H$  are continuous linear mappings.*

*Proof.* It is easy to see that  $E(a) = D(a) - aH(\mathbf{1})$  and  $H(a) = D(a) - E(\mathbf{1})a$  for all  $a \in \mathcal{A}$ . Considering these equations, we get  $D(ab) = D(a)b + aD(b) - aD(\mathbf{1})b$  for all  $a, b \in \mathcal{A}$ . Define the mapping  $\Delta : \mathcal{A} \rightarrow \mathcal{A}$  by  $\Delta(a) = D(a) - D(\mathbf{1})a$ . We have

$$\begin{aligned} \Delta(ab) &= D(ab) - D(\mathbf{1})ab \\ &= D(a)b + aD(b) - aD(\mathbf{1})b - D(\mathbf{1})ab \\ &= (D(a) - D(\mathbf{1})a)b + a(D(b) - D(\mathbf{1})b) \\ &= \Delta(a)b + a\Delta(b), \end{aligned}$$

which means that  $\Delta$  is a derivation. Therefore,  $D$  is a generalized derivation. Moreover, we have  $H(a) = D(a) - E(\mathbf{1})a = \Delta(a) + D(\mathbf{1})a - E(\mathbf{1})a = \Delta(a) + H(\mathbf{1})a$ , which means that  $H$  is a generalized derivation. It is clear that the mappings  $D, E$  and  $H$  are continuous by hypothesis under which a derivation is continuous.  $\square$

Suppose that  $\mathcal{Y}$  and  $\mathcal{Z}$  are Banach spaces and  $T : \mathcal{Y} \rightarrow \mathcal{Z}$  is a linear mapping. Recall that the set

$$S(T) = \{z \in \mathcal{Z} : \exists \{y_n\} \subseteq \mathcal{Y} \text{ such that } y_n \rightarrow 0, T(y_n) \rightarrow z\}$$

is called the separating space of  $T$ . By the closed graph Theorem,  $T$  is continuous if and only if  $S(T) = \{0\}$ . For more material about the separating space and other results about this concept, see [4, 5].

**Theorem 2.11.** *Suppose that  $\mathcal{A}$  is a Banach algebra,  $\mathcal{B}$  is a simple Banach algebra and  $D : \mathcal{A} \rightarrow \mathcal{B}$  is both a continuous  $(D, F, G, H)$ -derivation and continuous  $(H, F, G, D)$ -derivation such that  $H$  is continuous and  $F, G$  are surjective. Then  $F$  and  $G$  are continuous or  $H$  and  $D$  are identically zero.*

*Proof.* First note that

$$D(ab) = D(a)F(b) + G(a)H(b) = H(a)F(b) + G(a)D(b), \quad (a, b \in \mathcal{A}).$$

Let  $c \in S(F)$ . Then there exists a sequence  $\{c_n\} \subseteq \mathcal{A}$  such that  $c_n \rightarrow 0$  and  $F(c_n) \rightarrow c$ . We have

$$0 = \lim_{n \rightarrow +\infty} D(ac_n) = \lim_{n \rightarrow +\infty} (D(a)F(c_n) + G(a)H(c_n)) = D(a)c, \quad (2.6)$$

for any  $a \in \mathcal{A}$ . Also, we have

$$0 = \lim_{n \rightarrow +\infty} D(ac_n) = \lim_{n \rightarrow +\infty} (H(a)F(c_n) + G(a)D(c_n)) = H(a)c, \quad (2.7)$$

for all  $a \in \mathcal{A}$ . Set

$$\mathfrak{X} = \{c \in \mathcal{B} : D(a)c = 0 = H(a)c \text{ for all } a \in \mathcal{A}\}.$$

According to the discussion above,  $S(F) \subseteq \mathfrak{X}$ . We shall show that  $\mathfrak{X}$  is an ideal of  $\mathcal{B}$ . Obviously, if  $c_1, c_2 \in \mathfrak{X}$  and  $\lambda \in \mathbb{C}$ , then  $c_1 + \lambda c_2 \in \mathfrak{X}$ . Now, suppose  $c$  and  $b$  are arbitrary elements of  $\mathfrak{X}$  and  $\mathcal{B}$ , respectively. We want to show that  $cb \in \mathfrak{X}$ . Let  $a$  be an arbitrary element of  $\mathcal{A}$ . We have  $D(a)(cb) = (D(a)c)b = 0$  and also  $H(a)(cb) = (H(a)c)b = 0$ , which means that  $cb \in \mathfrak{X}$  and consequently,  $\mathfrak{X}$  is a right ideal of  $\mathcal{B}$ . Since  $F$  is a surjective mapping, there exists an element  $a_1 \in \mathcal{A}$  such that  $F(a_1) = b$ . It follows from (2.6) and (2.7) that

$$0 = D(aa_1)c = D(a)F(a_1)c + G(a)H(a_1)c = D(a)bc,$$

and further

$$0 = D(aa_1)c = H(a)F(a_1)c + G(a)D(a_1)c = H(a)bc,$$

for all  $a \in \mathcal{A}$ . This means that  $\mathfrak{X}$  is a left ideal and so it is an ideal of  $\mathcal{B}$ . Since  $\mathcal{B}$  is a simple algebra, either  $\mathfrak{X} = \{0\}$  or  $\mathfrak{X} = \mathcal{B}$ . Suppose that  $\mathfrak{X} = \{0\}$ . Since  $S(F) \subseteq \mathfrak{X}$ , we infer that  $F$  is continuous. Now, suppose that  $\mathfrak{X} = \mathcal{B}$ , i.e.  $\mathfrak{X} = \{c \in \mathcal{B} : D(a)c = 0 = H(a)c \text{ for all } a \in \mathcal{A}\} = \mathcal{B}$ . So  $D(a)c = H(a)c = 0$  for all  $a \in \mathcal{A}$ ,  $c \in \mathcal{B}$ . We can thus see that  $D(a)\mathcal{B}D(a) = H(a)\mathcal{B}H(a) = \{0\}$  for all  $a \in \mathcal{A}$ . Since  $\mathcal{B}$  is a simple algebra and every simple algebra is semiprime, we deduce that  $D = H = 0$ .

We turn to the continuity of  $G$ . Suppose that  $g \in S(G)$ . Then there exists a sequence  $\{g_n\} \subseteq \mathcal{A}$  such that  $g_n \rightarrow 0$  and  $G(g_n) \rightarrow g$ . We have

$$0 = \lim_{n \rightarrow +\infty} D(g_n a) = \lim_{n \rightarrow +\infty} (H(g_n)F(a) + G(g_n)D(a)) = gD(a), \quad (2.8)$$

and also,

$$0 = \lim_{n \rightarrow +\infty} D(g_n a) = \lim_{n \rightarrow +\infty} (D(g_n)F(a) + G(g_n)H(a)) = gH(a), \quad (2.9)$$

for any  $a \in \mathcal{A}$ . Let

$$\mathfrak{J} = \{g \in \mathcal{B} : gD(a) = 0 = gH(a) \text{ for all } a \in \mathcal{A}\}.$$

The above discussion shows that  $S(G) \subseteq \mathfrak{J}$ . Our next task is to show that  $\mathfrak{J}$  is an ideal of  $\mathcal{B}$ . Clearly,  $\mathfrak{J}$  is a left ideal of  $\mathcal{B}$ . Let  $b$  and  $g$  be arbitrary elements of  $\mathcal{B}$  and  $\mathfrak{J}$ , respectively. Since  $G$  is surjective, there exists an element  $a_1 \in \mathcal{A}$  such that  $G(a_1) = b$ . Using (2.8) and (2.9), we have the following expressions:

$$0 = gD(a_1a) = gH(a_1)F(a) + gG(a_1)D(a) = gbD(a),$$

and further

$$0 = gD(a_1a) = gD(a_1)F(a) + gG(a_1)H(a) = gbH(a),$$

for all  $a \in \mathcal{A}$ . This implies that  $\mathfrak{J}$  is an ideal of  $\mathcal{B}$ . Since we are assuming that  $\mathcal{B}$  is a simple algebra, either  $\mathfrak{J} = \{0\}$ , which implies that  $G$  is continuous, or  $\mathfrak{J} = \mathcal{B}$  and in this case, we deduce that  $D = H = 0$ . From what has been said, we conclude that  $F$  and  $G$  are continuous mappings or  $D = H = 0$ .  $\square$

By similar arguments to those in the proof of the above theorem, we get the following result whose proof is left to the interested reader.

**Theorem 2.12.** *Suppose that  $\mathcal{A}$  is a Banach algebra,  $\mathcal{B}$  is a simple Banach algebra and  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a continuous  $(D, F, F, H)$ -derivation such that  $D(\mathcal{A}) \subseteq H(\mathcal{A})$  or  $H(\mathcal{A}) \subseteq D(\mathcal{A})$  and  $F$  is surjective. Then  $F$  is continuous or  $H$  and  $D$  are identically zero.*

**Corollary 2.13.** *Let  $\mathcal{A}$  be a Banach algebra and  $\mathcal{B}$  a simple Banach algebra.*

(i) *Suppose that  $D : \mathcal{A} \rightarrow \mathcal{B}$  is both a continuous  $(D, F, G, H)$ -derivation and a continuous  $(H, F, G, D)$ -derivation such that  $H$  is continuous and  $F, G$  are surjective. If  $F$  or  $G$  is a discontinuous mapping, then  $D$  and  $H$  are identically zero.*

(ii) *Suppose that  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a continuous  $(D, F, F, H)$ -derivation such that  $D(\mathcal{A}) \subseteq H(\mathcal{A})$  or  $H(\mathcal{A}) \subseteq D(\mathcal{A})$ . If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a discontinuous, surjective, linear mapping, then  $D$  and  $H$  are identically zero.*

In the following, we present some conditions under which the linear mappings  $F$  and  $G$  associated with a  $(D, F, G, H)$ -derivation  $D$  are automatically continuous.

**Theorem 2.14.** *Suppose that  $\mathcal{A}$  is a Banach algebra,  $\mathcal{B}$  is a semiprime Banach algebra and  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a  $(D, F, G, H)$ -derivation such that  $D$  is surjective and  $H$  is continuous (or  $D$  is continuous and  $H$  is surjective). Then  $F$  and  $G$  are continuous.*

*Proof.* Suppose that  $D$  is surjective and  $H$  is continuous. As in (2.6) we have  $D(\mathcal{A})c = \{0\}$  for all  $c \in S(F)$ . Since  $D$  is surjective and  $\mathcal{B}$  is a semiprime algebra,  $c = 0$ . This means that  $F$  is continuous. The continuity of  $G$  is obtained via similar arguments to those employed in the proof of Theorem 2.11. If we assume that  $D$  is continuous and  $H$  is surjective, then our claim is easily obtained by (2.7) and (2.9).  $\square$

Let  $\mathcal{A}$  be an algebra and let  $\mathcal{A}_1$  denote the set of all pairs  $(a, \alpha)$ ,  $a \in \mathcal{A}$ ,  $\alpha \in \mathbb{C}$ , that is,  $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$ . Then  $\mathcal{A}_1$  becomes an algebra if the linear space operations and multiplication are defined by  $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$ ,  $\mu(a, \alpha) = (\mu a, \mu \alpha)$  and  $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha \beta)$  for  $a, b \in \mathcal{A}$  and  $\alpha, \beta \in \mathbb{C}$ . A simple calculation shows that the element  $\mathbf{1} = (0, 1) \in \mathcal{A}_1$  is the identity element of  $\mathcal{A}_1$ .

**Theorem 2.15.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two algebras and let  $\delta : \mathcal{A} \rightarrow \mathcal{B}$  be a linear mapping. Let  $D : \mathcal{A}_1 \rightarrow \mathcal{B}_1$  be defined as  $D(a, \alpha) = (\delta(a), \alpha)$  and let  $H : \mathcal{A}_1 \rightarrow \mathcal{B}_1$  be a linear mapping such that  $H(\mathbf{1}) = \mathbf{1}$ . Then  $D$  is a  $(D, F, G, H)$ -derivation for some  $F, G$  if and only if  $\delta$  is a homomorphism. Furthermore, if  $D$  is a  $(D, F, F, H)$ -derivation, then  $D = H$ .*

*Proof.* Let  $\delta : \mathcal{A} \rightarrow \mathcal{B}$  be a linear mapping and let  $D : \mathcal{A}_1 \rightarrow \mathcal{B}_1$  defined by  $D(a, \alpha) = (\delta(a), \alpha)$  be a  $(D, F, G, H)$ -derivation for some  $F, G$ . We know that  $\mathbf{1} = (0, 1)$  is the identity element of  $\mathcal{A}_1$  and clearly,  $D(\mathbf{1}) = \mathbf{1} = H(\mathbf{1})$ . So,

$$\mathbf{1} = D(\mathbf{1}) = D(\mathbf{1})F(\mathbf{1}) + G(\mathbf{1})H(\mathbf{1}) = F(\mathbf{1}) + G(\mathbf{1}). \quad (2.10)$$

Denoting every element  $(a, \alpha) \in \mathcal{A}_1$  by  $a_\alpha$ , we have the following expressions:

$$\begin{aligned} D(a_\alpha) &= D(a_\alpha)F(\mathbf{1}) + G(a_\alpha)H(\mathbf{1}) \\ &= D(a_\alpha)F(\mathbf{1}) + G(a_\alpha) \\ &= D(a_\alpha)(\mathbf{1} - G(\mathbf{1})) + G(a_\alpha) \\ &= D(a_\alpha) - D(a_\alpha)G(\mathbf{1}) + G(a_\alpha), \end{aligned}$$

which means that

$$G(a_\alpha) = D(a_\alpha)G(\mathbf{1}), \quad (a_\alpha \in \mathcal{A}_1). \quad (2.11)$$

Similarly, we can deduce that

$$F(a_\alpha) = D(a_\alpha) - G(\mathbf{1})H(a_\alpha), \quad (a_\alpha \in \mathcal{A}_1). \quad (2.12)$$

Using (2.11) and (2.12), we have the following expressions:

$$\begin{aligned} D(a_\alpha b_\beta) &= D(a_\alpha)F(b_\beta) + G(a_\alpha)H(b_\beta) \\ &= D(a_\alpha)(D(b_\beta) - G(\mathbf{1})H(b_\beta)) + D(a_\alpha)G(\mathbf{1})H(b_\beta) \\ &= D(a_\alpha)D(b_\beta), \end{aligned}$$

which means that  $D$  is a homomorphism on  $\mathcal{A}_1$  and then, so is  $\delta$  on  $\mathcal{A}$ . Conversely, suppose that  $\delta$  is a homomorphism on  $\mathcal{A}$ . A straightforward calculation shows that  $D$  is also a homomorphism on  $\mathcal{A}_1$ . Thus, we have

$$\begin{aligned} D(a_\alpha b_\beta) &= D(a_\alpha)D(b_\beta) \\ &= D(a_\alpha) \frac{D(b_\beta)}{2} + \frac{D(a_\alpha)}{2} D(b_\beta). \end{aligned}$$

Considering  $F = G = \frac{D}{2}$ , we see that  $D$  is a  $(D, F, F, D)$ -derivation on  $\mathcal{A}_1$ . Now, suppose  $D$  is a  $(D, F, F, H)$ -derivation. We get from (2.10) that  $F(\mathbf{1}) = \frac{1}{2}$  and it follows from (2.11) that

$$F(a_\alpha) = \frac{D(a_\alpha)}{2}, \quad (a_\alpha \in \mathcal{A}_1), \quad (2.13)$$

and it also follows from (2.12) and (2.13) that

$$\frac{D(a_\alpha)}{2} = D(a_\alpha) - \frac{H(a_\alpha)}{2}, \quad (a_\alpha \in \mathcal{A}_1).$$

The previous equality implies that  $D = H$ , as required.  $\square$

**Theorem 2.16.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two topological algebras and let  $D : \mathcal{A} \rightarrow \mathcal{B}$  be a  $(D, F, G, H)$ -derivation such that  $D(\mathbf{1}) = H(\mathbf{1}) = \mathbf{1}$ . Suppose that every homomorphism from  $\mathcal{A}$  into  $\mathcal{B}$  is continuous. Then  $D$  and  $G$  are continuous mappings and  $G$  is a right  $D$ -centralizer.*

*Proof.* Using the proof of the previous theorem, we get

$$G(a) = D(a)G(\mathbf{1}), \quad a \in \mathcal{A}, \quad (2.14)$$

and also  $D$  is a homomorphism. Since we have all the conditions under which every homomorphism from  $\mathcal{A}$  into  $\mathcal{B}$  is continuous,  $D$  is continuous and it follows from (2.14) that so is  $G$ . Moreover, we have

$$G(ab) = D(ab)G(\mathbf{1}) = D(a)D(b)G(\mathbf{1}) = D(a)G(b), \quad a, b \in \mathcal{A},$$

which means that  $G$  is a right  $D$ -centralizer.  $\square$

**Theorem 2.17.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are two unital, normed algebras such that  $\mathcal{B}$  is a domain and assume that  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a  $(D, F, G, H)$ -derivation such that  $D(\mathbf{1}) = H(\mathbf{1}) \neq 0$ . If there exists a sequence  $\{a_n\} \subseteq \mathcal{A}$  such that the sequences  $\{D(a_n)\}$ ,  $\{F(a_n)\}$ ,  $\{G(a_n)\}$  and  $\{H(a_n)\}$  are convergent to a nonzero element  $a_0$ . Then  $G = D = \frac{F+H}{2}$ . In particular, if  $D = H$ , then  $D = F = G$ .*

*Proof.* We know that  $D(ab) = D(a)F(b) + G(a)H(b)$  for all  $a, b \in \mathcal{A}$ . So we have

$$\begin{aligned} a_0 &= \lim_{n \rightarrow \infty} D(a_n) \\ &= \lim_{n \rightarrow \infty} (D(a_n)F(\mathbf{1}) + G(a_n)H(\mathbf{1})) \\ &= a_0(F(\mathbf{1}) + H(\mathbf{1})) \\ &= a_0(F(\mathbf{1}) + D(\mathbf{1})) \end{aligned}$$

Since  $\mathcal{B}$  is a domain and  $a_0 \neq 0$ , we see that  $F(\mathbf{1}) + D(\mathbf{1}) = \mathbf{1}$ . We also have

$$\begin{aligned} D(\mathbf{1}) &= D(\mathbf{1})F(\mathbf{1}) + G(\mathbf{1})H(\mathbf{1}) \\ &= D(\mathbf{1})F(\mathbf{1}) + G(\mathbf{1})D(\mathbf{1}) \\ &= D(\mathbf{1})(\mathbf{1} - D(\mathbf{1})) + G(\mathbf{1})D(\mathbf{1}) \\ &= D(\mathbf{1}) - D(\mathbf{1})^2 + G(\mathbf{1})D(\mathbf{1}), \end{aligned}$$

and consequently,  $(G(\mathbf{1}) - D(\mathbf{1}))D(\mathbf{1}) = 0$ . Since  $\mathcal{B}$  is a domain and  $D(\mathbf{1}) \neq 0$ , we get that  $G(\mathbf{1}) = D(\mathbf{1})$ . Let  $a$  be an arbitrary element of  $\mathcal{A}$ . We have

$$\begin{aligned} D(a) &= D(a)F(\mathbf{1}) + G(a)H(\mathbf{1}) \\ &= D(a)(\mathbf{1} - D(\mathbf{1})) + G(a)D(\mathbf{1}) \\ &= D(a) - D(a)D(\mathbf{1}) + G(a)D(\mathbf{1}). \end{aligned}$$

Since  $\mathcal{B}$  is a domain and  $D(\mathbf{1}) \neq 0$ , we deduce that  $G = D$ . Using this fact, we see that

$$\begin{aligned} a_0 &= \lim_{n \rightarrow \infty} D(a_n) \\ &= \lim_{n \rightarrow \infty} (D(\mathbf{1})F(a_n) + G(\mathbf{1})H(a_n)) \\ &= 2D(\mathbf{1})a_0 \end{aligned}$$

Since  $\mathcal{B}$  is a domain,  $D(\mathbf{1}) = \frac{1}{2} = G(\mathbf{1})$ . Therefore, we have

$$\begin{aligned} D(a) &= D(\mathbf{1})F(a) + G(\mathbf{1})H(a) \\ &= \frac{F(a)}{2} + \frac{H(a)}{2} \end{aligned}$$

for all  $a \in \mathcal{A}$ . It implies that  $D = \frac{F+H}{2}$ . It is clear that if  $D = H$ , then we get that  $G = D = F$ , as desired.  $\square$

An immediate corollary reads as follows:

**Corollary 2.18.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are two unital, normed algebras such that  $\mathcal{B}$  is a domain and assume that  $D : \mathcal{A} \rightarrow \mathcal{B}$  is a  $(D, F, G, H)$ -derivation such that  $D(\mathbf{1}) = H(\mathbf{1}) \neq 0$ . If there exists an element  $a_0 \in \mathcal{A}$  such that  $D(a_0) = F(a_0) = G(a_0) = H(a_0) \neq 0$ , then  $G = D = \frac{F+H}{2}$ . In particular, if  $D = H$ , then  $D = F = G$ .*

For any  $a \in \mathcal{A}$ , we define the linear mappings  $L_a, R_a : \mathcal{A} \rightarrow \mathcal{A}$  by  $L_a(b) = ab$  and  $R_a(b) = ba$  for all  $b \in \mathcal{A}$ . A straightforward verification shows that  $R_b L_a = L_a R_b$ ,  $\lambda L_a = L_{\lambda a}$  and  $\lambda R_a = R_{\lambda a}$  for all  $a, b \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ . Clearly,  $L_a = R_a$  if and only if  $a \in Z(\mathcal{A})$ .

In the following theorem, we provide a characterization of the generalized  $(\sigma, \tau)$ -derivations using some functional equations. It is clear that a generalized  $(\sigma, \tau)$ -derivation can be considered as a strongly generalized derivation of order one.

**Theorem 2.19.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two unital rings and let  $\sigma, \tau : \mathcal{A} \rightarrow \mathcal{B}$  be two homomorphisms such that  $\sigma(\mathbf{1}) = \tau(\mathbf{1}) = \mathbf{1}$ . Let  $f, g, h : \mathcal{A} \rightarrow \mathcal{M}$  be additive mappings satisfying*

$$f(ab) = g(a)\sigma(b) + \tau(a)h(b) = h(a)\tau(b) + \sigma(a)g(b)$$

for all  $a, b \in \mathcal{A}$ . Then,  $f, g$  and  $h$  are generalized  $(\sigma, \tau)$ -derivations on  $\mathcal{A}$ . In particular, if  $\sigma = \tau$ , then all the mappings  $f, g$  and  $h$  are generalized  $\sigma$ -derivations associated with a  $\sigma$ -derivation.

*Proof.* Putting  $a = \mathbf{1}$  and  $b = \mathbf{1}$ , we have the following expressions:

$$\begin{aligned} f(a) &= g(a) + \tau(a)h(\mathbf{1}), \\ f(b) &= h(\mathbf{1})\tau(b) + g(b), \end{aligned}$$

for all  $a, b \in \mathcal{A}$ . It means that

$$\tau(a)h(\mathbf{1}) = h(\mathbf{1})\tau(a), \quad (a \in \mathcal{A}). \quad (2.15)$$

Similarly, we obtain that

$$g(\mathbf{1})\sigma(a) = \sigma(a)g(\mathbf{1}), \quad (a \in \mathcal{A}) \quad (2.16)$$

Since  $f(a) = g(a) + \tau(a)h(\mathbf{1}) = h(a) + \sigma(a)g(\mathbf{1})$  for all  $a \in \mathcal{A}$ , we get that

$$h(a) = g(a) + \tau(a)h(\mathbf{1}) - \sigma(a)g(\mathbf{1}), \quad (a \in \mathcal{A}), \quad (2.17)$$

and further, we have

$$g(a)\sigma(b) + \tau(a)h(b) = f(ab) = g(ab) + \tau(ab)h(\mathbf{1}).$$

Hence, for all  $a, b \in \mathcal{A}$ , we have the following identities:

$$\begin{aligned} g(ab) &= g(a)\sigma(b) + \tau(a)h(b) - \tau(a)\tau(b)h(\mathbf{1}) \\ &= g(a)\sigma(b) + \tau(a)(g(b) + \tau(b)h(\mathbf{1}) - \sigma(b)g(\mathbf{1})) - \tau(a)\tau(b)h(\mathbf{1}) \quad (\text{see 2.17}) \\ &= g(a)\sigma(b) + \tau(a)(g(b) - \sigma(b)g(\mathbf{1})), \end{aligned}$$

which means that

$$g(ab) = g(a)\sigma(b) + \tau(a)(g(b) - \sigma(b)g(\mathbf{1})), \quad (2.18)$$

Define a mapping  $D : \mathcal{A} \rightarrow \mathcal{M}$  by  $D(a) = g(a) - \sigma(a)g(\mathbf{1})$ . We have the following statements:

$$\begin{aligned} D(ab) &= g(ab) - \sigma(ab)g(\mathbf{1}) \\ &= g(a)\sigma(b) + \tau(a)g(b) - \tau(a)\sigma(b)g(\mathbf{1}) - \sigma(a)\sigma(b)g(\mathbf{1}) \\ &= (g(a) - \sigma(a)g(\mathbf{1}))\sigma(b) + \tau(a)(g(b) - \sigma(b)g(\mathbf{1})) \\ &= D(a)\sigma(b) + \tau(a)D(b), \end{aligned}$$

which means that  $D$  is a  $(\sigma, \tau)$ -derivation. This fact with (2.18) imply that  $g$  is a generalized  $(\sigma, \tau)$ -derivation associated with the  $(\sigma, \tau)$ -derivation  $D$ . Indeed, we have

$$g(ab) = g(a)\sigma(b) + \tau(a)D(b).$$

Now, we define a mapping  $\delta : \mathcal{A} \rightarrow \mathcal{M}$  by  $\delta(a) = g(a) - \tau(a)g(\mathbf{1})$ . Using (2.16) and (2.18), we have

$$\begin{aligned} g(ab) &= g(a)\sigma(b) + \tau(a)g(b) - \tau(a)\sigma(b)g(\mathbf{1}) \\ &= (g(a) - \tau(a)g(\mathbf{1}))\sigma(b) + \tau(a)g(b) \\ &= \delta(a)\sigma(b) + \tau(a)g(b). \end{aligned}$$

A straightforward verification shows that

$$\delta(ab) = \delta(a)\sigma(b) + \tau(a)\delta(b)$$

for all  $a, b \in \mathcal{A}$ . We know that  $g(ab) = g(a)\sigma(b) + \tau(a)(h(b) - \tau(b)h(\mathbf{1}))$  and defining a mapping  $d : \mathcal{A} \rightarrow \mathcal{M}$  by  $d(a) = h(a) - \tau(a)h(\mathbf{1})$ , we have  $g(ab) = g(a)\sigma(b) + \tau(a)d(b)$  for all  $a, b \in \mathcal{A}$ . On the other hand, we know that  $g$  is a generalized  $(\sigma, \tau)$ -derivation associated with the  $(\sigma, \tau)$ -derivation  $D$ , i.e.  $g(ab) = g(a)\sigma(b) + \tau(a)D(b)$  for all  $a, b \in \mathcal{A}$ . So, we infer that  $d = D$  and this means that  $d$  is also a  $(\sigma, \tau)$ -derivation. Hence,  $h(a) = d(a) + \tau(a)h(\mathbf{1})$  is a generalized  $(\sigma, \tau)$ -derivation associated with the  $(\sigma, \tau)$ -derivation  $d = D$  and we see that

$$h(ab) = D(a)\sigma(b) + \tau(a)h(b),$$

for all  $a, b \in \mathcal{A}$ . Furthermore, we know that  $f(ab) = h(ab) + \sigma(ab)g(\mathbf{1})$  for all  $a, b \in \mathcal{A}$ . Thus, we have

$$h(a)\tau(b) + \sigma(a)g(b) = f(ab) = h(ab) + \sigma(ab)g(\mathbf{1})$$

for all  $a, b \in \mathcal{A}$ . So, we can write that

$$\begin{aligned} h(ab) &= h(a)\tau(b) + \sigma(a)g(b) - \sigma(ab)g(\mathbf{1}) \\ &= h(a)\tau(b) + \sigma(a)(g(b) - \sigma(b)g(\mathbf{1})) \\ &= h(a)\tau(b) + \sigma(a)D(b). \end{aligned}$$

We know that the linear mapping  $\delta = g - R_{g(\mathbf{1})}\tau$  is a  $(\sigma, \tau)$ -derivation and further,  $f = g + R_{h(\mathbf{1})}\tau$ . Using these two previous equations, we can thus deduce that

$$f = \delta + R_{g(\mathbf{1})}\tau + R_{h(\mathbf{1})}\tau = \delta + R_{f(\mathbf{1})}\tau, \quad (2.19)$$

which means that  $f$  is a generalized  $(\sigma, \tau)$ -derivation associated with the  $(\sigma, \tau)$ -derivation  $\delta$ . Moreover, since  $g = D + R_{g(\mathbf{1})}\sigma$ , we see that

$$f = g + R_{h(\mathbf{1})}\tau = D + R_{g(\mathbf{1})}\sigma + R_{h(\mathbf{1})}\tau. \quad (2.20)$$

Now, suppose that  $\sigma = \tau$ . Then, we have  $\delta = g - R_{g(\mathbf{1})}\tau = g - R_{g(\mathbf{1})}\sigma = D$ . It follows from (2.19) or (2.20) that

$$f = D + R_{f(\mathbf{1})}\sigma. \quad (2.21)$$

So, we have

$$\begin{aligned} f(ab) &= D(ab) + R_{f(\mathbf{1})}\sigma(ab) \\ &= D(a)\sigma(b) + \sigma(a)D(b) + f(\mathbf{1})\sigma(a)\sigma(b) \\ &= (D(a) + f(\mathbf{1})\sigma(a))\sigma(b) + \sigma(a)D(b) \\ &= f(a)\sigma(b) + \sigma(a)D(b) \end{aligned}$$

for all  $a, b \in \mathcal{A}$ . Also, note that  $f(\mathbf{1})\sigma(a) = \sigma(a)f(\mathbf{1})$  for all  $a \in \mathcal{A}$ , since  $\sigma = \tau$ . One can easily deduce that

$$f(ab) = D(a)\sigma(b) + \sigma(a)f(b)$$

for all  $a, b \in \mathcal{A}$ . Therefore, all the mappings  $f$ ,  $g$  and  $h$  are generalized  $\sigma$ -derivations associated with the  $\sigma$ -derivation  $D$ .  $\square$

We conclude this article with the following conclusion.

**Corollary 2.20.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two unital topological algebras and let the mappings  $f$ ,  $g$  and  $h$  satisfy the equations of Theorem 2.19 and  $\sigma(\mathbf{1}) = \mathbf{1} = \tau(\mathbf{1})$ . If we have all the conditions under which every  $(\sigma, \tau)$ -derivation is continuous, then the mappings  $f$ ,  $g$  and  $h$  are continuous under the same conditions.*

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