On reduction for eigenfunctions of graphs*

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Abstract

In this work, we prove a general version of the reduction lemmas for eigenfunctions of graphs admitting involutive automorphisms of a special type.

1 Introduction

Recently, for the eigenspaces of the Hamming and Johnson graphs, reduction lemmas were established (see [3, Lemma 1] and [6, Lemma 1]). In [1, 2, 3, 4, 5, 6, 7, 8], these lemmas were applied to study eigenfunctions and equitable 2-partitions of the Hamming and Johnson graphs. In this work, we generalize the reduction lemmas to graphs admitting involutive automorphisms of a special type. In particular, we prove that an analogue of the reduction lemmas holds for the halved n-cube.

The paper is organized as follows. In Section 2, we introduce basic definitions. In Section 3, we prove a general version of the reduction lemmas. Then, in Section 4, we apply this result to the Hamming graph, the Johnson graph, and the halved n-cube.

2 Basic definitions

Let G be a graph. The vertex set of G is denoted by $V(G)$. Given a vertex x of G, denote by $N_G(x)$ the set of all neighbors of x in G. For a set $W \subseteq V(G)$, denote by $G[W]$ the subgraph of G induced by W. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. An automorphism φ of G is called *involutive* if φ^2 is the identity automorphism.

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. Let G be a graph and let λ be an eigenvalue of G. A function $f : V(G) \longrightarrow \mathbb{R}$ is called a λ-*eigenfunction* of G if $f \not\equiv 0$ and the equality

$$\lambda \cdot f(x) = \sum_{y \in N_G(x)} f(y) \qquad (1)$$

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holds for any vertex $x \in V(G)$. The set of functions $f : V(G) \longrightarrow \mathbb{R}$ satisfying (1) for any vertex $x \in V(G)$ is called a λ-*eigenspace* of G. Denote by $U_\lambda(G)$ the λ-eigenspace of G.

Let G be a graph. Let φ be an automorphism of G and let $\{V_1, V_2, V_3\}$ be a partition of $V(G)$. The pair $(\varphi, \{V_1, V_2, V_3\})$ is called *special* if the following conditions hold:

1. $\varphi(V_1) = V_2$ and $\varphi(V_2) = V_1$, i.e., φ swaps V_1 and V_2.
2. For any vertex $x \in V_i$, where $i \in \{1, 2\}$, it holds $N_G(x) \cap V_{3-i} = \{\varphi(x)\}$.
3. $\varphi(x) = x$ for any vertex $x \in V_3$, i.e., φ stabilises V_3 pointwise.

Remark 1. *If $(\varphi, \{V_1, V_2, V_3\})$ is a special pair of a graph G, then the following properties hold:*

- *The graphs $G[V_1]$ and $G[V_2]$ are isomorphic.*
- *The graph $G[V_1 \cup V_2]$ is isomorphic to the Cartesian product of $G[V_1]$ and K_2.*
- *The automorphism φ is involutive.*

Let G be a graph with a special pair $P = (\varphi, \{V_1, V_2, V_3\})$. Let $G[V_1]$ and $G[V_2]$ be isomorphic to a graph G_0, and let $\varphi_1 : V_1 \longrightarrow V(G_0)$ and $\varphi_2 : V_2 \longrightarrow V(G_0)$ be the corresponding isomorphisms. Given a function $f : V(G) \longrightarrow \mathbb{R}$, we define a function $f_{P,\varphi_1,\varphi_2}$ on the vertices of G_0 as follows:

$$f_{P,\varphi_1,\varphi_2}(x) = f(\varphi_1^{-1}(x)) - f(\varphi_2^{-1}(x)).$$

Let $\{i_1, \dots, i_k\}$ be a subset of $\{1, 2, \dots, n\}$, where $1 \leq k < n$. For a vector $x \in \mathbb{Z}_q^n$, denote by $\Delta_{i_1,\dots,i_k}(x)$ the vector obtained from x by deleting coordinates with indices $i_1, \dots, i_k$.

Let $i, j \in \{1, 2, \dots, n\}$ and $i < j$. For a vector $x \in \mathbb{Z}_q^n$, denote by $\pi_{i,j}(x)$ the vector obtained from x by interchanging the ith and jth coordinates.

The *weight* of a vector $x \in \mathbb{Z}_q^n$, denoted by $|x|$, is the number of its non-zero coordinates.

3 Reduction for eigenfunctions of graphs

In this section, we prove the main theorem of this paper.

Theorem 1. *Suppose G is a graph with a special pair $P = (\varphi, \{V_1, V_2, V_3\})$ Let $G[V_1]$ and $G[V_2]$ be isomorphic to a graph G_0, and let $\varphi_1 : V_1 \longrightarrow V(G_0)$ and $\varphi_2 : V_2 \longrightarrow V(G_0)$ be the corresponding isomorphisms. If f is a λ-eigenfunction of G, then $f_{P,\varphi_1,\varphi_2} \in U_{\lambda+1}(G_0)$.*

Proof. For every $i \in \{1,2,3\}$, denote $G_i = G[V_i]$. Define a function h on the vertices of G as follows:

$$h(x) = f(x) - f(\varphi(x)).$$

Since f is a λ-eigenfunction of G and $\varphi \in \mathrm{Aut}(G)$, we have $h \in U_\lambda(G)$. The restriction of h to V_1 is denoted by h_1.

Let us prove that $h_1 \in U_{\lambda+1}(G_1)$. Consider a vertex $x \in V_1$. Since $h \in U_\lambda(G)$, we have

$$\lambda \cdot h(x) = \sum_{y \in N_G(x)} h(y).$$

Then

$$\begin{aligned}\lambda \cdot h(x) &= \sum_{y \in N_G(x) \cap V_1} h(y) + \sum_{y \in N_G(x) \cap V_2} h(y) + \sum_{y \in N_G(x) \cap V_3} h(y) = \\ &= \sum_{y \in N_{G_1}(x)} h(y) + h(\varphi(x)) + \sum_{y \in N_G(x) \cap V_3} h(y).\end{aligned}$$

Note that $h(\varphi(x)) = f(\varphi(x)) - f(x) = -h(x)$. Since φ stabilises V_3 pointwise, we have $h(y) = 0$ for any vertex $y \in V_3$. Hence we obtain that

$$(\lambda + 1) \cdot h(x) = \sum_{y \in N_{G_1}(x)} h(y).$$

Therefore, $h_1 \in U_{\lambda+1}(G_1)$. Finally, note that $f_{P,\varphi_1,\varphi_2} = h_1(\varphi_1^{-1})$. Since $h_1 \in U_{\lambda+1}(G_1)$ and φ_1 is an isomorphism between G_1 and G_0, we obtain that $f_{P,\varphi_1,\varphi_2} \in U_{\lambda+1}(G_0)$. □

4 Examples

In this section, we discuss how to apply Theorem 1 to the Hamming graph, the Johnson graph, and the halved n-cube. In particular, we show that these graphs admit special pairs.

4.1 Hamming graph

The *Hamming graph* $H(n,q)$ is defined as follows. The vertex set of $H(n,q)$ is $\mathbb{Z}_q^n$, and two vertices are adjacent if they differ in exactly one coordinate.

Let $k, m \in \mathbb{Z}_q$, $k \neq m$, and $r \in \{1,2,\ldots,n\}$. Denote

$$V_1 = \{x \in \mathbb{Z}_q^n : x_r = k\},$$

$$V_2 = \{x \in \mathbb{Z}_q^n : x_r = m\},$$

and $V_3 = \mathbb{Z}_q^n \setminus (V_1 \cup V_2)$. Denote $X = \{V_1, V_2, V_3\}$.

Define a map $\varphi : \mathbb{Z}_q^n \longrightarrow \mathbb{Z}_q^n$ as follows:

$$\varphi(x_1, \dots, x_n) = (x_1, \dots, x_{r-1}, (km)(x_r), x_{r+1}, \dots, x_n)$$

(here (km) is the transposition of k and m). Note that (φ, X) is a special pair of $H(n, q)$.

Let $G_0 = H(n-1, q)$. Define maps $\varphi_1 : V_1 \longrightarrow V(G_0)$ and $\varphi_2 : V_2 \longrightarrow V(G_0)$ as follows:

$$\varphi_1(x) = \Delta_r(x)$$

and

$$\varphi_2(y) = \Delta_r(y).$$

One can check that $G[V_1]$ and $G[V_2]$ are isomorphic to G_0, and φ_1 and φ_2 are the corresponding isomorphisms. Thus, (φ, X), G_0, φ_1 and φ_2 satisfy the conditions of Theorem 1.

4.2 Johnson graph

The *Johnson graph* $J(n, k)$ is defined as follows. The vertex set of $J(n, k)$ is $\{x \in \mathbb{Z}_2^n : |x| = k\}$, and two vertices are adjacent if they differ in exactly two coordinates.

Let $i, j \in \{1, 2, \dots, n\}$ and $i < j$. Denote

$$V_1 = \{x \in \mathbb{Z}_2^n : |x| = k, x_i = 1, x_j = 0\},$$

$$V_2 = \{x \in \mathbb{Z}_2^n : |x| = k, x_i = 0, x_j = 1\},$$

and $V_3 = V(J(n, k)) \setminus (V_1 \cup V_2)$. Denote $X = \{V_1, V_2, V_3\}$.

Define a map $\varphi : V(J(n, k)) \longrightarrow V(J(n, k))$ as follows:

$$\varphi(x) = \pi_{i,j}(x).$$

Note that (φ, X) is a special pair of $J(n, k)$.

Let $G_0 = J(n-2, k-1)$. Define maps $\varphi_1 : V_1 \longrightarrow V(G_0)$ and $\varphi_2 : V_2 \longrightarrow V(G_0)$ as follows:

$$\varphi_1(x) = \Delta_{i,j}(x)$$

and

$$\varphi_2(y) = \Delta_{i,j}(y).$$

One can check that $G[V_1]$ and $G[V_2]$ are isomorphic to G_0, and φ_1 and φ_2 are the corresponding isomorphisms. Thus, (φ, X), G_0, φ_1 and φ_2 satisfy the conditions of Theorem 1.

4.3 Halved n-cube

The *halved n-cube* $\frac{1}{2}H(n)$ is defined as follows. The vertex set of $\frac{1}{2}H(n)$ is $\{x \in \mathbb{Z}_2^n : |x| \text{ is even}\}$, and two vertices are adjacent if they differ in exactly two coordinates.

Let $i, j \in \{1, 2, \ldots, n\}$ and $i < j$. Denote

$$V_1 = \{x \in \mathbb{Z}_2^n : |x| \text{ is even}, x_i = 1, x_j = 0\},$$

$$V_2 = \{x \in \mathbb{Z}_2^n : |x| \text{ is even}, x_i = 0, x_j = 1\},$$

and $V_3 = V(\frac{1}{2}H(n)) \setminus (V_1 \cup V_2)$. Denote $X = \{V_1, V_2, V_3\}$.

Define a map $\varphi : V(\frac{1}{2}H(n)) \longrightarrow V(\frac{1}{2}H(n))$ as follows:

$$\varphi(x) = \pi_{i,j}(x).$$

Note that (φ, X) is a special pair of $\frac{1}{2}H(n)$.

We define a graph G_0 as follows. The vertex set of G_0 is $\{x \in \mathbb{Z}_2^{n-2} : |x| \text{ is odd}\}$, and two vertices are adjacent if they differ in exactly two coordinates. Note that G_0 is isomorphic to $\frac{1}{2}H(n-2)$. Define maps $\varphi_1 : V_1 \longrightarrow V(G_0)$ and $\varphi_2 : V_2 \longrightarrow V(G_0)$ as follows:

$$\varphi_1(x) = \Delta_{i,j}(x)$$

and

$$\varphi_2(y) = \Delta_{i,j}(y).$$

One can check that $G[V_1]$ and $G[V_2]$ are isomorphic to G_0, and φ_1 and φ_2 are the corresponding isomorphisms. Thus, (φ, X), G_0, φ_1 and φ_2 satisfy the conditions of Theorem 1.

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