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CAPACITIES IN FRACTIONAL SOBOLEV SPACES WITH VARIABLE EXPONENTS

AZEDDINE BAALAL AND MOHAMED BERGHOUT

ABSTRACT. In this paper, we develop a capacities theory connected with the fractional Sobolev spaces with variable exponents. Two kinds of capacities are studied: Sobolev capacity and relative capacity. Basic properties of capacities, including monotonicity, outer capacity, and several results, are studied. We prove that both capacities are a Choquet capacity and that all Borel sets are capacitable.

Keywords: Fractional Sobolev spaces with variable exponents, Sobolev capacity, relative capacity, Choquet capacity, outer capacity.

1. Introduction

Functions spaces with variable exponent have been intensely investigated in the recent years. One of such spaces is the Lebesgue and Sobolev spaces with variable exponent. They were introduced by W. Orlicz in 1931 [35]; their properties were further developed by H. Nakano as special cases of the theory of modular spaces [34]. In the ensuing decades they were primarily considered as important examples of modular spaces or the class of Musielak–Orlicz spaces [33]. In the beginning, these spaces had theoretical interest; later, at the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)-$Laplacian operator, which in its turn gave an essential impulse for the development of this theory. For more details on these spaces, see the monographs [12, 20, 28, 29].

The concept of capacity is indispensable to an understanding point-wise behavior of functions in a Sobolev space. In a sense, capacity is a measure of size for sets and it measures small sets more precisely than the usual Lebesgue measure.

Sobolev spaces and capacities theory is one of the significant aspects of the classical

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and nonlinear potential theory. In this setting, there are two natural kinds of capacities: Sobolev capacity and relative capacity. Both capacities have their advantages. The relative capacity is closely related to the Wiener criterion, thinness, fine topology and nonlinear fine potential theory; see [6, 7, 24, 31] and the monographs [8, 21, 32]. In contrast, Sobolev capacity plays a central role when studying quasicontinuous representative and fine properties for the equivalence class of Sobolev functions; see [9, 18, 19, 20, 25, 26, 31] and the monograph [12]. Fractional capacities for fixed exponent spaces have found a great number of uses, see for instance [37, 38, 39] and the references therein.

Throughout this paper we will use the following notations: $\mathbb{R}^d$ is the d-dimensional Euclidean space, and $d \in \mathbb{N}^*$ always stands for the dimension of the space. A domain $\Omega \subset \mathbb{R}^d$ is a connected open set equipped with the d-dimensional Lebesgue measure. For constants, we use the letter C whose value may change even within a string of estimates. The closure of a set A is denoted by $\overline{A}$. We use the usual convention of identifying two μ-measurable functions on A if they agree almost everywhere (a.e. in A, for short), i.e. if they agree up to a set of μ-measure zero. The Lebesgue integral of a measurable function $f : \Omega \longrightarrow \mathbb{R}$, is defined in the standard way and denoted by $\displaystyle\int_\Omega f(x)\, dx$. we use the symbol $:=$ to define the left-hand side by the right-hand side. For measurable functions $u, v : \Omega \longrightarrow \mathbb{R}$, we set $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. We denote by $L^0(\Omega)$ the space of all $\mathbb{R}-$ valued measurable functions on Ω. The set of smooth functions in Ω is denoted by $C^\infty(\Omega)$ - it consists of functions in Ω which are continuously differentiable arbitrarily many times. The set $C_0^\infty(\Omega)$ is the subset of $C^\infty(\Omega)$ of functions that have compact support. We denote by $C(\overline{\Omega})$ the space of uniformly continuous functions equipped with the supremum norm $\|f\|_\infty = \sup_{x\in\overline{\Omega}} |f(x)|$. By $C^k(\overline{\Omega})$, $k \in \mathbb{N}$, we denote the space of all function f, such that $\partial_\alpha f := \dfrac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n} \in C(\overline{\Omega})$ for all multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$; we denote also $C^\infty(\overline{\Omega}) = \bigcap_k C^k(\overline{\Omega})$ and this spaces is equipped with the norm $\sup_{|\alpha|\leq k} \|\partial_\alpha f\|_\infty$.

Let Ω be an open set of $\mathbb{R}^d$. We fix $s \in (0, 1)$ and we consider two variable exponents, that is, $q : \Omega \to [1, +\infty)$ and $p : \Omega \times \Omega \to [1, +\infty)$ be two measurable functions. The set of variable exponents $q : \Omega \to [1, +\infty)$ is denoted by $\mathcal{P}(\Omega)$ and the set of variable exponents $p : \Omega \times \Omega \to [1, +\infty)$ is denoted by $\mathcal{P}(\Omega \times \Omega)$. we set $p^- :=\text{essinf}_{(x,y)\in\Omega\times\Omega} p(x, y)$, $p^+ :=\text{esssup}_{(x,y)\in\Omega\times\Omega} p(x, y)$, $q^- :=\text{essinf}_{x\in\Omega} q(x)$ and $q^+ :=\text{esssup}_{x\in\Omega} q(x)$.

Throughout this paper, we assume that

$$1 < p^- \leq p(x, y) \leq p^+ < \infty, \tag{1.1}$$

$$1 < q^- \leq q(x) \leq q^+ < \infty. \tag{1.2}$$

Notice that by [12, Proposition 4.1.7], we can extend q and p to all of $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$ respectively. *The variable exponent Lebesgue space* $L^{q(.)}(\Omega)$ is the family of the equivalence classes of functions defined by

$$L^{q(.)}(\Omega) := \left\{ u \in L^0(\Omega) : \int_\Omega |u(x)|^{q(x)} dx < \infty \right\}.$$

We define a norm by

$$\|u\|_{L^{q(.)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{1}{\lambda}u(x)\right|^{q(x)} \, dx \leqslant 1\right\}.$$

Then $(L^{q(.)}(\Omega), \|.\|_{L^{q(.)}(\Omega)})$ is a Banach space. We define *the fractional Sobolev space with variable exponents* as follows:

$$W^{s,q(.),p(.,.)}(\Omega) := \left\{u \in L^{q(.)}(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{s+\frac{d}{p(.,.)}}} \in L^{p(.,.)}(\Omega \times \Omega)\right\}.$$

We define a *modular* on $W^{s,q(.),p(.,.)}(\Omega)$ by

$$\rho^{s,\Omega}_{q(.),p(.,.)}(u) := \int_\Omega |u(x)|^{q(x)} \, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} \, dx \, dy,$$

when $\Omega = \mathbb{R}^d$ we denote it by $\rho^s_{q(.),p(.,.)}$. The definition of the space $L^{q(.)}(\mathbb{R}^d)$ and $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$ is analogous to $L^{q(.)}(\Omega)$ and $W^{s,q(.),p(.,.)}(\Omega)$; one just changes every occurrence of Ω by $\mathbb{R}^d$. Let

$$[u]^{s,p(.,.)}(\Omega) := \inf\left\{\lambda > 0 : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{d+sp(x,y)}} dxdy \leq 1\right\},$$

be the corresponding *variable exponent Gagliardo semi-norm*. It is easy to see that $W^{s,q(.),p(.,.)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{s,q(.),p(.,.)}(\Omega)} := \|u\|_{L^{q(.)}(\Omega)} + [u]^{s,p(.,.)}(\Omega).$$

$W^{s,q(.),p(.,.)}(\Omega)$ can be seen as a natural extension of the classical fractional Sobolev space. The modular $\rho^{s,\Omega}_{q(.),p(.,.)}$ induces a norm by

$$\|u\|_{\rho^{s,\Omega}_{q(.),p(.,.)}} := \inf\left\{\lambda > 0 : \rho^{s,\Omega}_{q(.),p(.,.)}(\frac{1}{\lambda}u) \leqslant 1\right\},$$

which is equivalent to the norm $\|u\|_{W^{s,q(.),p(.,.)}(\Omega)}$. It is clear that $u \in W^{s,q(.),p(.,.)}(\Omega)$ if and only if $\rho^{s,\Omega}_{q(.),p(.,.)}(u) < \infty$. We refer the reader to [1, 2, 3, 13, 23].

Fractional Sobolev spaces with variable exponents have major applications in variational problems related to the well-known fractional version of the $p(x)$-Laplace operator; see for instance [4, 5, 12, 17].

In this paper, we develop a capacities theory connected with the fractional Sobolev spaces with variable exponents. Fundamental proprieties of capacity including Choquet capacity, capacitability and several results, are studied.

Let $s \in (0,1), q \in \mathcal{P}(\mathbb{R}^d)$ and $p \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. For a set $E \subset \mathbb{R}^d$ we denote

$$\mathcal{A}^s_{q(.),p(.,.)}(E) := \left\{u \in W^{s,q(.),p(.,.)}(\mathbb{R}^d) : u \geq 1 \text{ a.e. on a neighborhood of } E\right\}.$$

The fractional Sobolev $(s, q(.), p(.,.))$*-capacity* of the set E is the number defined by

$$C^s_{q(.),p(.,.)}(E) := \inf_{u \in \mathcal{A}^s_{q(.),p(.,.)}(E)} \rho^s_{q(.),p(.,.)}(u).$$

In case $\mathcal{A}^s_{q(.),p(.,.)}(E) = \varnothing$, we set $C^s_{q(.),p(.,.)}(E) = \infty$.

Let $\mathcal{T}$ be a topological space and let $\mathcal{P}(\mathcal{T})$ be the power set of $\mathcal{T}$. A mapping

$\mathcal{C} : \mathcal{P}(\mathcal{T}) \longrightarrow [0, \infty]$ is called a *Choquet capacity* on $\mathcal{T}$ if the following properties are satisfied:

($\mathcal{C}_0$): $\mathcal{C}(\varnothing) = 0$.
($\mathcal{C}_1$): $A \subset B \subset \mathcal{T}$ implies $\mathcal{C}(A) \leq \mathcal{C}(B)$.
($\mathcal{C}_2$): $(A_n)_n \subset \mathcal{T}$ an increasing sequence implies $\lim_{n\longrightarrow\infty} \mathcal{C}(A_n) = \mathcal{C}\left(\bigcup_{n=1}^{\infty} A_n\right)$.
($\mathcal{C}_3$): $(K_n)_n \subset \mathcal{T}$ a decreasing sequence, K_n compact, implies $\lim_{n\longrightarrow\infty} \mathcal{C}(K_n) = \mathcal{C}\left(\bigcap_{n=1}^{\infty} K_n\right)$.

For more details on the Choquet capacity, we refer the reader to [10]. The fractional Sobolev $(s, q(.), p(.,.))$-capacity turns out to be a Choquet capacity. More precisely, we have:

Theorem 3.4. *The fractional Sobolev $(s, q(.), p(.,.))$-capacity is a Choquet capacity.*

We define the *interior capacity* of E by

$$C_*(E) := \sup\left\{C^s_{q(.),p(.,.)}(K) : K \subset E,\ K \text{ compact}\right\}.$$

We define the *exterior capacity* of E by

$$C^*(E) := \inf\left\{C^s_{q(.),p(.,.)}(O) : E \subset O,\ O \text{ open}\right\}.$$

We say that the set E is *capacitable* if $C^*(E) = C_*(E)$. We obtain the following corollary of Theorem 3.4:

Corollary 3.5. *(Choquet's capacitability theorem). Let $s \in (0,1), q \in \mathcal{P}(\mathbb{R}^d)$ and $p \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Then all Borel sets $E \subset \mathbb{R}^d$ are capacitable.*

Now, we introduce an alternative to the fractional Sobolev $(s, q(.), p(.,.))$-capacity, in which the capacity of a set is taken relative to an open subset. Let $O \subset \overline{\Omega}$ be a relatively open set, that is, open with respect to the relative topology of $\overline{\Omega}$. By $C_c(\overline{\Omega})$ we design the space of continuous functions on $\overline{\Omega}$ with compact support in $\overline{\Omega}$. For $s \in (0,1), q \in \mathcal{P}(\Omega)$ and $p \in \mathcal{P}(\Omega \times \Omega)$, we let

$$\tilde{W}^{s,q(.),p(.,.)}(\Omega) := \overline{W^{s,q(.),p(.,.)}(\Omega) \cap C_c(\overline{\Omega})}^{W^{s,q(.),p(.,.)}(\Omega)}$$

and

$$\mathcal{R}^{s,\overline{\Omega}}_{q(.),p(.,.)}(O) := \left\{u \in \tilde{W}^{s,q(.),p(.,.)}(\Omega) : u \geq 1 \ \text{ a.e. on } O\right\}.$$

We define *the fractional relative $(s, q(.), p(.,.))$-capacity* of O, with respect to Ω, by

$$C^{s,\Omega}_{q(.),p(.,.)}(O) := \inf_{u \in \mathcal{R}^{s,\overline{\Omega}}_{q(.),p(.,.)}(O)} \rho^{s,\Omega}_{q(.),p(.,.)}(u).$$

For any set $E \subset \overline{\Omega}$,

$$C^{s,\Omega}_{q(.),p(.,.)}(E) = \inf\left\{C^{s,\Omega}_{q(.),p(.,.)}(O) : O \text{ relatively open in } \overline{\Omega} \text{ containing } E\right\}.$$

In analogy with Theorem 3.4 and Corollary 3.5 we have:

Theorem 3.9. *(Choquet's capacitability theorem). Let $s \in (0,1), q \in \mathcal{P}(\Omega)$ and $p \in \mathcal{P}(\Omega \times \Omega)$. Then the fractional relative $(s, q(.), p(.,.))$-capacity with respect to*

Ω, is a Choquet capacity on $\overline{\Omega}$. In particular, all Borel sets $E \subset \overline{\Omega}$ are capacitable, that is,

$$\begin{aligned} C^{s,\Omega}_{q(.),p(.,.)}(E) &= \inf\left\{C^{s,\Omega}_{q(.),p(.,.)}(O) : O \text{ relatively open in } \overline{\Omega} \text{ containing } E\right\} \\ &= \sup\left\{C^{s,\Omega}_{q(.),p(.,.)}(K) : K \subseteq E \subset \overline{\Omega},\ K \text{ compact}\right\}. \end{aligned}$$

This paper is organized as follows. In Section 2 we show some properties of fractional Sobolev spaces with variable exponents useful in this paper. In Section 3, we develop a capacities theory based on the definition of functions in fractional Sobolev spaces with variable exponents. Two kinds of capacities are studied: Sobolev capacity and relative capacity. Basic properties of capacities, including monotonicity, outer capacity and several results, are studied. We prove that both capacities are a Choquet capacity and that all Borel sets are capacitable.

2. Fractional Sobolev spaces with variable exponents

In this section, we introduce fractional Sobolev spaces with variable exponents as abstract modular spaces and we prove several useful properties for them.

2.1. **Modular spaces.** We refer to the monographs [12, 33, 34] for an exposition on modular spaces. We give here a brief review of this subject.

Let X be a (real or complex) vector space. A function $\rho : X \to [0, \infty]$ is called a *modular* on X if the following properties hold:

(1) $\rho(0_X) = 0$;
(2) $\rho(\lambda x) = \rho(x)$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(3) ρ is convex;
(4) ρ is left-continuous. That means $\lim_{\lambda \to 1^-} = \rho(x)$ for all $x \in X$;
(5) $\rho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0_X$;
(6) $\rho(x) = 0$ implies $x = 0_X$.

In general, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. If ρ is modular on X, then

$$X_\rho := \{x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

is called a *modular space.* we define a norm, called *the Luxemburg norm* by

$$\|x\|_\rho := \inf\left\{\lambda > 0 : \rho(\frac{1}{\lambda}x) \leqslant 1\right\}.$$

Then $(X_\rho, \|.\|_\rho)$ is a normed vector space.

Typical examples of modular spaces are Lebesgue-Sobolev spaces with variable exponent and Musielak–Orlicz spaces.

We say that a modular ρ on X satisfies the Δ_2-*condition* if there exists a constant $K \geq 2$ such that $\rho(2f) \leqslant K\rho(f)$ for all $f \in X_\rho$.

A modular ρ on X is called *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho(\frac{f-g}{2}) \leq \varepsilon\ \frac{\rho(f)+\rho(g)}{2} \text{ or } \rho(\frac{f+g}{2}) \leq (1-\delta)\frac{\rho(f)+\rho(g)}{2}.$$

Theorem 2.1. *([12, Theorem 2.4.14]). Let ρ be a uniformly convex modular on X satisfies the Δ_2-condition. Then the norm $\|.\|_\rho$ on X_ρ is uniformly convex. Hence, X_ρ is uniformly convex.*

2.2. **Fractional Sobolev spaces on open sets.** For the investigation of classical Sobolev spaces, it is enough to stay in the framework of Banach spaces. In particular, the space and its topology are described in terms of a norm. However, in the context of Sobolev spaces with variable exponent, this is not the best way. Instead, it is better to start with the so-called modular which then induces a norm. In the case of classical fractional spaces for any $p \geq 1$ and $s \in (0,1)$ the modular is

$$\rho_p^s = \int |u(x)|^p \, dx + \int\int \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy$$

compared to the norm

$$\|u\|_{W^{s,p}} = \left(\int |u(x)|^p \, dx + \int\int \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$

In some cases, the modular has certain advantages compared to the norm since it inherits all the good properties of the integral. The modular spaces $L^{q(.)}(\Omega)$ and $W^{s,q(.),p(.,.)}(\Omega)$ capture this advantage.

From [14, Theorems 1.4], we obtain the following proposition:

Proposition 2.2. *Let $u_n, u \in L^{q(.)}(\Omega)$ such that $n \in \mathbb{N}$. Then the following statements are equivalent to each other:*

(1) $\lim_{n \longrightarrow +\infty} \|u_n - u\|_{L^{q(.)}(\Omega)} = 0$;
(2) $\lim_{n \longrightarrow +\infty} \int_\Omega |u_n(x) - u(x)|^{q(x)} \, dx = 0$;
(3) *u_n converge to u in Ω in measure and*

$$\lim_{n \longrightarrow +\infty} \int_\Omega |u_n(x)|^{q(x)} \, dx = \int_\Omega |u(x)|^{q(x)} \, dx.$$

Lemma 2.3. *Let $s \in (0,1)$, $q \in \mathcal{P}(\Omega)$ and $p \in \mathcal{P}(\Omega \times \Omega)$. Then the modular $\rho^{s,\Omega}_{q(.),p(.,.)}$ is uniformly convex and satisfies the Δ_2-condition.*

Proof. Let $u \in W^{s,q(.),p(.,.)}(\Omega)$. Since $q^+ < \infty$ and $p^+ < \infty$, we have that

$$\begin{aligned} \rho^{s,\Omega}_{q(.),p(.,.)}(2u) &\leqslant 2^{q^+} \int_\Omega |u(x)|^{q(x)} \, dx + 2^{p^+} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}} \, dx \, dy \\ &\leqslant \max\left\{2^{q^+}, 2^{p^+}\right\} \rho^{s,\Omega}_{q(.),p(.,.)}(u). \end{aligned}$$

Hence the modular $\rho^{s,\Omega}_{q(.),p(.,.)}$ satisfies the Δ_2-condition. In the same way in [12, Theorem 3.4.9], we can prove that the mappings $\varphi_{q(.)}(x,t) := t^{q(x)}$ and $\psi_{p(.,.)}((x,y),t) := t^{p(x,y)}$ are uniformly convex. Hence by [12, Theorem 3.4.11] the modular $\rho^{s,\Omega}_{q(.),p(.,.)}$ is uniformly convex. □

Next, we prove several properties of $W^{s,q(.),p(.,.)}(\Omega)$ that are needed to develop a capacities theory in $W^{s,q(.),p(.,.)}(\Omega)$.

Lemma 2.4. *Let $s \in (0,1)$, $q \in \mathcal{P}(\Omega)$ and $p \in \mathcal{P}(\Omega \times \Omega)$. Then the following assertions are held.*

(1) *If $u \in W^{s,q(.),p(.,.)}(\Omega)$, then $|u| \in W^{s,q(.),p(.,.)}(\Omega)$.*
(2) *If $u \in W^{s,q(.),p(.,.)}(\Omega)$, then $u^+ \in W^{s,q(.),p(.,.)}(\Omega)$.*
(3) *If $u \in W^{s,q(.),p(.,.)}(\Omega)$, then $u^- \in W^{s,q(.),p(.,.)}(\Omega)$.*

Proof. Let $s \in (0,1)$, $q \in \mathcal{P}(\Omega)$ and $p \in \mathcal{P}(\Omega \times \Omega)$.

(1) It is clear that $|u| \in L^{q(.)}(\Omega)$. Since

$$\big|\,|u(x)| - |u(y)|\,\big| \leqslant |u(x) - u(y)|\,,$$

we have that

$$\int_\Omega \int_\Omega \frac{\big|\,|u(x)| - |u(y)|\,\big|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy \leqslant \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy < \infty.$$

Hence, $|u| \in W^{s,q(.),p(.,.)}(\Omega)$.

(2) Since $0 \leqslant u^+ \leqslant |u|$, we have that $u^+ \in L^{q(.)}(\Omega)$.
Let $\Omega_1 := \{x \in \Omega : u(x) \geq 0\}$ and $\Omega_2 := \{x \in \Omega : u(x) < 0\}$. Then

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|u^+(x) - u^+(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy = &\int_{\Omega_1} \int_{\Omega_1} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy\, + \\ &\int_{\Omega_1} \int_{\Omega_2} \frac{|u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy\, + \\ &\int_{\Omega_2} \int_{\Omega_1} \frac{|u(x)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy.\end{aligned}$$

Since

$$|u(y)| \leqslant |u(x) - u(y)|$$

on

$$\Omega_2 \times \Omega_1 := \{(x,y) \in \Omega \times \Omega, u(x) < 0, u(y) > 0\}$$

and

$$|u(x)| \leqslant |u(x) - u(y)|$$

on

$$\Omega_1 \times \Omega_2 := \{(x,y) \in \Omega \times \Omega, u(x) \geq 0, u(y) < 0\}\,,$$

it follows that

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|u^+(x) - u^+(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy \leq &\int_{\Omega_1} \int_{\Omega_1} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy\, + \\ &\int_{\Omega_1} \int_{\Omega_2} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy\, + \\ &\int_{\Omega_2} \int_{\Omega_1} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy \\ \leq &\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{d+sp(x,y)}}\,dx\,dy.\end{aligned}$$

Hence $u^+ \in W^{s,q(.),p(.,.)}(\Omega)$.

(3) Since $u^- = |u| - u^+$, we have that $u^- \in W^{s,q(.),p(.,.)}(\Omega)$.

□

A (real-valued) function space is a *lattice* if the point-wise minimum and maximum of any two of its elements belong to the space. Next, we show that the fractional Sobolev space with a variable exponent is a lattice.

Proposition 2.5. *Let* $s \in (0,1)$, $q \in \mathcal{P}(\Omega)$ *and* $p \in \mathcal{P}(\Omega \times \Omega)$. *If* $u, v \in W^{s,q(.),p(.,.)}(\Omega)$, *then* $\min\{u,v\}$ *and* $\max\{u,v\}$ *are in* $W^{s,q(.),p(.,.)}(\Omega)$. *In particular*

$$\rho^{s,\Omega}_{q(.),p(.,.)}(\min\{1,u\}) \leqslant \rho^{s,\Omega}_{q(.),p(.,.)}(u).$$

Proof. Since $\max\{u,v\} = (u-v)^+ + v$ and $\min\{u,v\} = u-(u-v)^+$, then assertions follow from Lemma 2.4. □

Apart from our standard topology on $W^{s,q(.),p(.,.)}$, which was induced by the norm, it is possible to define another type of convergence using the modular $\rho^{s,\Omega}_{q(.),p(.,.)}$. Let $u_n, u \in W^{s,q(.),p(.,.)}(\Omega)$. We say that $u_n \longrightarrow u$ in modular if for some $\lambda > 0$, $\rho^{s,\Omega}_{q(.),p(.,.)}(\lambda(u_n - u)) \longrightarrow 0$ as $n \longrightarrow \infty$. From Proposition 2.2 we deduce the following proposition which characterizes this topology in terms of the modular.

Proposition 2.6. *Let* $s \in (0,1), q \in \mathcal{P}(\Omega)$ *and* $p \in \mathcal{P}(\Omega \times \Omega)$. *Let* $u_n, u \in W^{s,q(.),p(.,.)}(\Omega)$ *such that* $n \in \mathbb{N}$. *Then the following statements are equivalent to each other:*

(1) $\lim_{n\longrightarrow+\infty} \|u_n - u\|_{W^{s,q(.),p(.,.)}(\Omega)} = 0$;
(2) $\lim_{n\longrightarrow+\infty} \rho^{s,\Omega}_{q(.),p(.,.)}(u_n - u) = 0$;
(3) u_n *converge to* u *in* Ω *in measure and*

$$\lim_{n\longrightarrow+\infty} \rho^{s,\Omega}_{q(.),p(.,.)}(u_n) = \rho^{s,\Omega}_{q(.),p(.,.)}(u).$$

Proposition 2.7. *Let* $s \in (0,1), q \in \mathcal{P}(\Omega)$ *and* $p \in \mathcal{P}(\Omega \times \Omega)$. *The space* $W^{s,q(.),p(.,.)}(\Omega)$ *is separable, reflexive and uniformly convex.*

Proof. By [5, Lemma 3.1] $W^{s,q(.),p(.,.)}(\Omega)$ is a separable reflexive space. For the uniform convexity, we note that by Lemma 2.3 the modular $\rho^{s,\Omega}_{q(.),p(.,.)}$ is uniformly convex and satisfies the Δ_2-condition. Hence by Theorem 2.1 the space $W^{s,q(.),p(.,.)}(\Omega)$ is uniformly convex. □

A normed space X has the *Banach–Saks* property if $\dfrac{1}{m}\sum_{i=1}^m u_i \longrightarrow u$ whenever u_i is weakly convergent to u. By [22] every uniformly convex space has the Banach–Saks property. This together with Proposition 2.7 implies the following corollary.

Corollary 2.8. *Let* $s \in (0,1), q \in \mathcal{P}(\Omega)$ *and* $p \in \mathcal{P}(\Omega \times \Omega)$. *The space* $W^{s,q(.),p(.,.)}(\Omega)$ *has the Banach–Saks property.*

3. Capacities

In this section, we develop a capacities theory connected with the fractional Sobolev spaces with variable exponents. Two kinds of capacities are studied: Sobolev capacity and relative capacity. Basic properties of capacities, including monotonicity, outer capacity and several results, are studied. We prove that both capacities are a Choquet capacity and that all borel sets are capacitable.
Let $s \in (0,1), q \in \mathcal{P}(\mathbb{R}^d)$ and $p \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. For a set $E \subset \mathbb{R}^d$, we denote

$$\mathcal{A}^s_{q(.),p(.,.)}(E) := \Big\{ u \in W^{s,q(.),p(.,.)}(\mathbb{R}^d) : u \geq 1 \text{ a.e. on a neighborhood of } E \Big\}.$$

Functions $u \in \mathcal{A}^s_{q(.),p(.,.)}(E)$ are said to be $(s, q(.), p(., .))$-*admissible* for the capacity of the set E. Recall that the fractional Sobolev $(s, q(.), p(., .))$-capacity of the set E is the number defined by

$$C^s_{q(.),p(.,.)}(E) := \inf_{u \in \mathcal{A}^s_{q(.),p(.,.)}(E)} \rho^s_{q(.),p(.,.)}(u).$$

In case $\mathcal{A}^s_{q(.),p(.,.)}(E) = \varnothing$, we set $C^s_{q(.),p(.,.)}(E) = \infty$.

Proposition 3.1. *Let $s \in (0, 1)$, $q \in \mathcal{P}(\mathbb{R}^d)$ and $p \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Then the fractional Sobolev $(s, q(.), p(., .))$-capacity has the following properties:*

1. $C^s_{q(.),p(.,.)}(\varnothing) = 0$.
2. *If $A \subset B \subset \mathbb{R}^d$, then* $C^s_{q(.),p(.,.)}(A) \leqslant C^s_{q(.),p(.,.)}(B)$.
3. *If $E \subset \mathbb{R}^d$, then*

$$C^s_{q(.),p(.,.)}(E) = \inf_{u \in \mathcal{B}^s_{q(.),p(.,.)}(E)} \rho^s_{q(.),p(.,.)}(u),$$

where $\mathcal{B}^s_{q(.),p(.,.)}(E) = \Big\{u \in \mathcal{A}^s_{q(.),p(.,.)}(E) : 0 \leqslant u \leqslant 1\Big\}$.

Proof. Since the zero function belongs to $\mathcal{A}^s_{q(.),p(.,.)}(\varnothing)$, property (1) follows.
To prove (2), let $A \subset B \subset \mathbb{R}^d$, then

$$\mathcal{A}^s_{q(.),p(.,.)}(B) \subset \mathcal{A}^s_{q(.),p(.,.)}(A).$$

Hence

$$\inf_{u \in \mathcal{A}^s_{q(.),p(.,.)}(A)} \rho^s_{q(.),p(.,.)}(u) \leq \inf_{u \in \mathcal{A}^s_{q(.),p(.,.)}(B)} \rho^s_{q(.),p(.,.)}(u),$$

and by definition, we get that

$$C^s_{q(.),p(.,.)}(A) \leqslant C^s_{q(.),p(.,.)}(B).$$

Now, we show the property (3). Let

$$\mathcal{B}^s_{q(.),p(.,.)}(E) = \Big\{u \in \mathcal{A}^s_{q(.),p(.,.)}(E) : 0 \leqslant u \leqslant 1\Big\},$$

then

$$\mathcal{B}^s_{q(.),p(.,.)}(E) \subset \mathcal{A}^s_{q(.),p(.,.)}(E).$$

Hence

$$C^s_{q(.),p(.,.)}(E) \leqslant \inf_{u \in \mathcal{B}^s_{q(.),p(.,.)}(E)} \rho^s_{q(.),p(.,.)}(u).$$

For the reverse inequality, let $\varepsilon > 0$. Then there exists $u \in \mathcal{A}^s_{q(.),p(.,.)}(E)$ such that

$$\rho^s_{q(.),p(.,.)}(u) \leq C^s_{q(.),p(.,.)}(E) + \varepsilon.$$

By Proposition 2.5 we have that $\min\{1, u\} \in \mathcal{A}^s_{q(.),p(.,.)}(E)$ and

$$\rho^s_{q(.),p(.,.)}(\min\{1, u\}) \leqslant \rho^s_{q(.),p(.,.)}(u).$$

Hence by Lemma 2.4, we have that $\max(0, \min\{1, u\})$ belongs to $\mathcal{B}^s_{q(.),p(.,.)}(E)$. Therefore

$$\inf_{u \in \mathcal{B}^s_{q(.),p(.,.)}(E)} \rho^s_{q(.),p(.,.)}(u) \leq \rho^s_{q(.),p(.,.)}(u) \leq C^s_{q(.),p(.,.)}(E) + \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$ we obtain

$$\inf_{u\in\mathcal{B}^s_{q(.),p(.,.)}(E)} \rho^s_{q(.),p(.,.)}(u) \leqslant C^s_{q(.),p(.,.)}(E).$$

This completes the proof. □

Notice that by Proposition 3.1 we can restrict ourselves in the definition of the fractional Sobolev $(s, q(.), p(.,.))$-capacity to those admissible functions u for which $0 \leqslant u \leqslant 1$.

Proposition 3.2. *Let $s \in (0,1), q \in \mathcal{P}(\mathbb{R}^d)$ and $p \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Then for all subset E of $\mathbb{R}^d$, the fractional Sobolev $(s, q(.), p(.,.))$-capacity is an outer capacity, that is,*

$$C^s_{q(.),p(.,.)}(E) = \inf\left\{C^s_{q(.),p(.,.)}(O) : E \subset O,\ O \text{ open}\right\}.$$

Proof. Let $E \subset \mathbb{R}^d$ and O be open such that $E \subset O$. By monotonicity

$$C^s_{q(.),p(.,.)}(E) \leq C^s_{q(.),p(.,.)}(O).$$

Hence

$$C^s_{q(.),p(.,.)}(E) \leq \inf\left\{C^s_{q(.),p(.,.)}(O) : E \subset O,\ O \text{ open}\right\}.$$

To prove the inequality in the other direction, let $\varepsilon > 0$. Then there exists $u \in \mathcal{A}^s_{q(.),p(.,.)}(E)$ such that

$$\rho^s_{q(.),p(.,.)}(u) \leq C^s_{q(.),p(.,.)}(E) + \varepsilon.$$

Since $u \in \mathcal{A}^s_{q(.),p(.,.)}(E)$ there is an open set O containing E such that $u \geq 1$ a.e. on O which implies

$$C^s_{q(.),p(.,.)}(O) \leq \rho^s_{q(.),p(.,.)}(u) \leq C^s_{q(.),p(.,.)}(E) + \varepsilon.$$

By letting $\varepsilon \longrightarrow 0$, we get that

$$C^s_{q(.),p(.,.)}(O) \leq C^s_{q(.),p(.,.)}(E).$$

Hence

$$\inf\left\{C^s_{q(.),p(.,.)}(O) : E \subset O,\ O \text{ open}\right\} \leq C^s_{q(.),p(.,.)}(E),$$

and this completes the proof. □

Corollary 3.3. *Let $(K_n)_n$ be a decreasing sequence of compact subsets of $\mathbb{R}^d$. Then*

$$C^s_{q(.),p(.,.)}\left(\bigcap_{n=1}^{\infty} K_n\right) = \lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(K_n).$$

Proof. Let $(K_n)_n$ be a decreasing sequence of compact subsets of $\mathbb{R}^d$ and let $K = \bigcap_{n=1}^{\infty} K_n$. First by the monotonicity

$$C^s_{q(.),p(.,.)}(K) \leqslant \lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(K_n).$$

Let O be an open subset of $\mathbb{R}^d$ such that $K \subset O$. Since every K_n is compact it follows that $K = \bigcap_{n=1}^{\infty} K_n$ is compact. Hence there exists $n_0 \in \mathbb{N}$ such that $K_n \subset O$ for all $n \geq n_0$. The monotonicity yields

$$C^s_{q(.),p(.,.)}(K) \leqslant \lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(K_n) \leqslant C^s_{q(.),p(.,.)}(O).$$

Since the fractional Sobolev $(s, q(.), p(.,.))$-capacity is an outer capacity, we obtain the claim by taking infimum over all open sets O containing K. $\square$

Next, we show that the fractional Sobolev $(s, q(.), p(.,.))$-capacity is a Choquet capacity.

Theorem 3.4. *The fractional Sobolev $(s, q(.), p(.,.))$-capacity is a Choquet capacity.*

Proof. Notice that by Proposition 3.1 the properties $\mathcal{C}_0$ and $\mathcal{C}_1$ hold and by Corollary 3.3 property $\mathcal{C}_3$ yields.
To prove $\mathcal{C}_2$, let $(A_n)_n$ be an increasing sequence subset of $\mathbb{R}^d$ and let $A := \bigcup_{n\geq 1} A_n$. The monotonicity implies that

$$\lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(A_n) \leqslant C^s_{q(.),p(.,.)}(A).$$

To get the converse inequality, let $n \in \mathbb{N}$. Then there exists $u_n \in \mathcal{A}^s_{q(.),p(.,.)}(A_n)$ such that

$$\rho^s_{q(.),p(.,.)}(u_n) \leq C^s_{q(.),p(.,.)}(A_n) + 2^{-n}.$$

Since (u_n) is a bounded sequence in the reflexive Banach space $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$, hence there is a subsequence of (u_n) which converges weakly to a function $u \in W^{s,q(.),p(.,.)}(\mathbb{R}^d)$. By Banach-Saks property, we obtain that $\dfrac{1}{m}\sum_{n=1}^m u_n \longrightarrow u$ in $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$ as $m \longrightarrow \infty$. We set

$$v_j = \frac{1}{j}\sum_{n=1}^{j} u_n.$$

Since $u_n \geq 1$ a.e. on every open set O_n containing A_n, there exists an open set $O'_n = \bigcap_{j\geq n} O_j$ containing A_n such that $v_n \geq 1$ a.e. on O'_n. Since $(v_j)_j$ converges to u in $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$, we may assume that

$$\|v_{j+1} - v_j\|_{W^{s,q(.),p(.,.)}(\mathbb{R}^d)} \leqslant \frac{1}{2^j}.$$

Let

$$w_j := v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i|.$$

Observe that

$$w_j \geq v_j + \sum_{i=j}^{k-1} |v_{i+1} - v_i| = v_k, \text{ for } k \geq j.$$

Then $w_j \in W^{s,q(.),p(.,.)}(\mathbb{R}^d)$ and $w_j \geq 1$ a.e on $A \subset O = \bigcup_{i=1}^{\infty} O_i$. Hence

$$w_j \in \mathcal{A}^s_{q(.),p(.,.)}(A) \text{ and } C^s_{q(.),p(.,.)}(A) \leq \rho^s_{q(.),p(.,.)}(w_j) \text{ for } j = 1, 2, ...$$

We also find that

$$\begin{aligned} \|w_j - v_j\|_{W^{s,q(.),p(.,.)}(\mathbb{R}^d)} &\leq \sum_{i=j}^{\infty} \|v_{i+1} - v_i\|_{W^{s,q(.),p(.,.)}(\mathbb{R}^d)} \\ &\leq \sum_{i=j}^{\infty} \frac{1}{2^i} = \frac{1}{2^{i-1}}, \end{aligned}$$

hence

$$\|w_j - v_j\|_{W^{s,q(.),p(.,.)}(\mathbb{R}^d)} \longrightarrow 0 \text{ as } j \longrightarrow \infty.$$

Consequently, by Lemma 2.6 we get that

$$\rho^s_{q(.),p(.,.)}(w_j - v_j) \longrightarrow 0 \text{ as } j \longrightarrow \infty.$$

Therefore

$$C^s_{q(.),p(.,.)}(A) \leqslant \lim_{j\longrightarrow\infty} \rho^s_{q(.),p(.,.)}(w_j) = \lim_{j\longrightarrow\infty} \rho^s_{q(.),p(.,.)}(v_j).$$

By the convexity of the modular and $(\mathcal{C}_1)$ property, we obtain

$$\begin{aligned} \rho^s_{q(.),p(.,.)}(v_j) &\leqslant \frac{1}{j}\sum_{n=1}^{j} \rho^s_{q(.),p(.,.)}(u_n) \\ &\leqslant \sup_{n\geq j} \rho^s_{q(.),p(.,.)}(u_n) \\ &\leqslant \sup_{n\geq j}(C^s_{q(.),p(.,.)}(A_n) + 2^{-n}) \\ &\leqslant \lim_{n\longrightarrow\infty}(C^s_{q(.),p(.,.)}(A_n) + 2^{-j}). \end{aligned}$$

Hence

$$C^s_{q(.),p(.,.)}(A) \leqslant \lim_{j\longrightarrow\infty}\lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(A_n) + 2^{-j} = \lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(A_n).$$

Therefore

$$C^s_{q(.),p(.,.)}(A) \leqslant \lim_{n\longrightarrow\infty} C^s_{q(.),p(.,.)}(A_n),$$

and the proof is finished. □

Corollary 3.5. *(Choquet's capacitability theorem). Let $s \in (0,1), q \in \mathcal{P}(\mathbb{R}^d)$ and $p \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Then all Borel sets $E \subset \mathbb{R}^d$ are capacitable, that is,*

$$\begin{aligned} C^s_{q(.),p(.,.)}(E) &= \inf\left\{C^s_{q(.),p(.,.)}(O) : E \subset O,\ O \text{ open}\right\} \\ &= \sup\left\{C^s_{q(.),p(.,.)}(K) : K \subset E,\ K \text{ compact}\right\}. \end{aligned}$$

Proof. From [10] every Suslin set is capacitable for a Choquet capacity and from [15] every Borel set is a Suslin set. □

We can derive a useful form of the fractional Sobolev $(s, q(.), p(.,.))-$capacity even if the Sobolev functions $u \in W^{s,q(.),p(.,.)}(\mathbb{R}^d)$ can be approximated by more regular functions, such as smooth functions. When they are the case, we need to consider only a limited class of $(s, q(.), p(.,.))$-admissible functions when calculating the fractional Sobolev $(s, q(.), p(.,.))$-capacity. In connection with the density problem for smooth function, we introduce the most important condition on the exponent in the study of variable exponent spaces, the well-known log-Hölder continuity condition. We say that a function $q : \Omega \to \mathbb{R}$ is *log-Hölder continuous* on Ω if there exists $C > 0$ such that

$$|q(x) - q(y)| \leqslant \frac{C}{-\log|x-y|} \tag{3.1}$$

for all $x, y \in \Omega$ such that $|x - y| \leqslant \frac{1}{2}$.
In [2] Baalal and Berghout generalize conditions (3.1) for an exponent $p \in \mathcal{P}(\Omega\times\Omega)$.

We say that a function $p : \Omega \times \Omega \to \mathbb{R}$ satisfies *condition (B-B)* on $\Omega \times \Omega$ if there exists $C > 0$ such that

$$\left|p(x,y) - p(x',y')\right| \leqslant \frac{C}{-\log(|x-x'|+|y-y'|)} \tag{3.2}$$

for all $(x,y),(x',y') \in \Omega \times \Omega$ such that $\left|x-x'\right| + \left|y-y'\right| \leqslant \frac{1}{2}$.
Notice that conditions (3.1) and (3.2) ensure that the typical mollifier function approximates the function in Lebesgue Spaces with variable exponents (we refer the reader to [2, 11, 36]).
We define the following class of variable exponents

$$\mathcal{P}^{log}(\Omega) := \{q : \Omega \to \mathbb{R} : q \text{ is measurable and log-Hölder continuous }\}$$

and

$$\mathcal{P}^{log}(\Omega \times \Omega) := \{p : \Omega \times \Omega \to \mathbb{R} : p \text{ is measurable and satisfies condition (B-B)}\}.$$

We assume also that the variable exponent $p \in \mathcal{P}(\Omega \times \Omega)$ satisfies the condition

$$p((x,y) - (z,z)) = p(x,y), \forall (x,y),(z,z) \in \Omega \times \Omega. \tag{3.3}$$

We say that $\Omega \subset \mathbb{R}^d$ is a $W^{s,q(.),p(.,.)}-$*extension domain* if there exists a continuous linear extension operator

$$\mathcal{E} : W^{s,q(.),p(.,.)}(\Omega) \longrightarrow W^{s,q(.),p(.,.)}(\mathbb{R}^d)$$

such that $\mathcal{E}u|_\Omega = u$ for each $u \in W^{s,q(.),p(.,.)}(\Omega)$. A typical example for $W^{s,q(.),p(.,.)}$-extension domain is a domain with a Lipschitz boundary (we refer the reader to [1]). From [2, Theorem 3.2 and Theorem 3.3], we have the following denseness theorems:

Theorem 3.6. *Let $q \in \mathcal{P}^{log}(\mathbb{R}^d)$ and $p \in \mathcal{P}^{log}(\mathbb{R}^d \times \mathbb{R}^d)$. Then the space $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$.*

Theorem 3.7. *Let $q \in \mathcal{P}^{log}(\Omega)$, $p \in \mathcal{P}^{log}(\Omega\times\Omega)$ and suppose that Ω is a $W^{s,q(.),p(.,.)}$-extension domain. Then the space $C^\infty(\overline{\Omega})$ is dense in $W^{s,q(.),p(.,.)}(\Omega)$.*

The following theorem is useful in computing the fractional Sobolev $(s, q(.), p(.,.))$-capacity of compact sets.

Theorem 3.8. *Let $s \in (0,1)$, $q \in \mathcal{P}^{log}(\mathbb{R}^d)$ and $p \in \mathcal{P}^{log}(\mathbb{R}^d \times \mathbb{R}^d)$. Assume that p satisfies (3.3). If $K \subset \mathbb{R}^d$ is compact, then*

$$C^s_{q(.),p(.,.)}(K) = \inf\left\{\rho^s_{q(.),p(.,.)}(u) : u \in C_0^\infty(\mathbb{R}^d), u \geq 1 \text{ a.e. on } K\right\}.$$

Proof. Let $K \subset \mathbb{R}^d$ be a compact set and let $u \in C_0^\infty(\mathbb{R}^d)$ be such that $u \geq 1$ a.e. on K. We choose a sequence of functions $\varphi_n \in C_0^\infty(\mathbb{R}^d)$ which converge to φ in $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$. Then there exists an open neighborhood of K such that $\varphi := \max\{0, \min\{1,u\}\} = 1$ a.e. on O. Let $\eta \in C_0^\infty(O)$ be such that $0 \leqslant \eta \leq 1$ and $\eta = 1$ on K. Hence $u_n = \eta\varphi + (1-\eta)\varphi_n^+$ converge to $\eta\varphi + (1-\eta)\varphi = \varphi \in W^{s,q(.),p(.,.)}(\mathbb{R}^d)$. Since $u_n \in C_0^\infty(\mathbb{R}^d)$, $u_n \geq 1$ on K and $\rho^s_{q(.),p(.,.)}(\varphi) \leqslant \rho^s_{q(.),p(.,.)}(u)$, we get that

$$C^s_{q(.),p(.,.)}(K) \geq \inf\left\{\rho^s_{q(.),p(.,.)}(u) : u \in C_0^\infty(\mathbb{R}^d), u \geq 1 \text{ a.e. on } K\right\}.$$

To prove the converse inequality, we set $u_n := (1 + \dfrac{1}{n})u$. Hence

$$u_n \in \left\{u \in C_0^\infty(\mathbb{R}^d) : \exists O \subset \mathbb{R}^d, O \text{ open contain } K \text{ and } u \geq 1 \text{ a.e. on } O\right\}$$

and

$$\begin{aligned} C^s_{q(.),p(.,.)}(K) &\leqslant \rho^s_{q(.),p(.,.)}(u_n) \\ &\leqslant \max\left\{(1+\frac{1}{n})^{q^+},(1+\frac{1}{n})^{p^+}\right\}\rho^s_{q(.),p(.,.)}(u). \end{aligned}$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{s,q(.),p(.,.)}(\mathbb{R}^d)$, we get that $\rho^s_{q(.),p(.,.)}(u)<\infty$. Hence

$$\max\left\{(1+\frac{1}{n})^{q^+},(1+\frac{1}{n})^{p^+}\right\}\rho^s_{q(.),p(.,.)}(u)\longrightarrow \rho^s_{q(.),p(.,.)}(u) \text{ as } n\longrightarrow\infty.$$

Consequently

$$C^s_{q(.),p(.,.)}(K)\leqslant \inf\left\{\rho^s_{q(.),p(.,.)}(u): u\in C_0^\infty(\mathbb{R}^d), u\geq 1 \text{ a.e. on } K\right\}$$

and this completes the proof. ☐

Now, we discuss an alternative to the fractional Sobolev $(s,q(.),p(.,.))-$capacity, in which the capacity of a set is taken relative to an open subset O. Recall that the fractional relative $(s,q(.),p(.,.))$-capacity of O, with respect to Ω, is the number

$$C^{s,\Omega}_{q(.),p(.,.)}(O):=\inf_{u\in\mathcal{R}^{s,\overline{\Omega}}_{q(.),p(.,.)}(O)}\rho^{s,\Omega}_{q(.),p(.,.)}(u),$$

where

$$\mathcal{R}^{s,\overline{\Omega}}_{q(.),p(.,.)}(O):=\left\{u\in\tilde{W}^{s,q(.),p(.,.)}(\Omega): u\geq 1 \ \text{ a.e. on } O\right\}.$$

For any set $E\subset\overline{\Omega}$,

$$C^{s,\Omega}_{q(.),p(.,.)}(E)=\inf\left\{C^{s,\Omega}_{q(.),p(.,.)}(O): O \text{ relatively open in } \overline{\Omega} \text{ containing } E\right\}.$$

Notice that as in Proposition 3.1 we can restrict ourselves in the definition of $(s,q(.),p(.,.))$-capacity with respect to Ω, to those admissible functions u for which $0\leqslant u\leqslant 1$.

In analogy with Theorem 3.4 and Corollary 3.5 we have:

Theorem 3.9. *(Choquet's capacitability theorem). Let $s\in(0,1), q\in\mathcal{P}(\Omega)$ and $p\in\mathcal{P}(\Omega\times\Omega)$. Then the fractional relative $(s,q(.),p(.,.))$-capacity with respect to Ω, is a Choquet capacity on $\overline{\Omega}$. In particular, all Borel sets $E\subset\overline{\Omega}$ are capacitable, that is,*

$$\begin{aligned} C^{s,\Omega}_{q(.),p(.,.)}(E) &= \inf\left\{C^{s,\Omega}_{q(.),p(.,.)}(O): O \text{ relatively open in } \overline{\Omega} \text{ containing } E\right\} \\ &= \sup\left\{C^{s,\Omega}_{q(.),p(.,.)}(K): K\subseteq E\subset\overline{\Omega},\ K \text{ compact}\right\}. \end{aligned}$$

In the same way in the Theorem 3.8, we get for a compact set $K\subset\overline{\Omega}$ the following useful theorem:

Theorem 3.10. *Let $s\in(0,1)$, $q\in\mathcal{P}^{log}(\Omega)$ and $p\in\mathcal{P}^{log}(\Omega\times\Omega)$ with p satisfy (3.3). Suppose that Ω is a $W^{s,q(.),p(.,.)}$-extension domain. If $K\subset\overline{\Omega}$ is a compact set. Then*

$$C^{s,\Omega}_{q(.),p(.,.)}(K)=\inf\left\{\rho^{s,\Omega}_{q(.),p(.,.)}(u): u\in C^\infty(\overline{\Omega}) \ \text{ and } u\geq 1 \text{ a.e. on } K\right\}.$$

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