Laguerre Expansions of $C-$regularized semigroups Functions.

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Abstract

The aim of this paper is to approximate the exponentially bounded $C-$regularized semigroups function by the Laguerre series, recalling the notions and the results used.

Keywords: Laguerre functions, $C-$regularized semigroup, C_0-semigroup.

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1 Introduction and preliminaries

The series expansion of Laguerre orthogonal polynomials have been an important tool in mathematical physics, in problems involving the integration of Helmholtz's equation in prabolic coordinates, in the theory of the Hydrogen atom, in the theory of propagation of electomagnetic waves along transmission lines terminated by a lumped inductance [8]. The study of sufficient conditions for the convergence of Laguerre series has been the subject of numerous works, for more details see [14], [3], [13], [8] and [2].
In 2014, Abadias and Miana studied in their article [1] the Laguerre expansion of C_0-semigroups and Resolvent Operators. In this work we will be interested in Laguerre expansion of C-regularized semigroups function, starting with reminding the notations, concepts and results used.
Throughout this paper E denotes a non-trivial complex Banach space, $\mathfrak{F}(E,F)$ denotes the set of all applications from E to another Banach space F, $B(E)$ denotes the space of all bounded linear operators from E into itself, and $L^1_{loc}(E)$ the set of all $f \in \mathfrak{F}(\mathbb{R},E)$ locally integrable. For a closed linear operator A on E, $\mathcal{D}(A)$, $R(A)$ and $\rho(A)$ denote its domain, range and resolvent set, respectively. $\mathcal{D}(A)$ equipped with the graph norm $\| x \|_{\mathcal{D}(A)}=\| x \|_E + \| Ax \|_E$ becomes a Banach space. Throughout this paper, $C \in B(E)$ will be an injective operator. The C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \; and \; \lambda I - A \; is \; injective\}$ and if $\lambda \in \rho_C(A)$ then we denote by $R_C(\lambda, A) = (\lambda I - A)^{-1}C$ the $C-$resolvent.

Laguerre functions and Laguerre expansions on Banach spaces

For all $n \in \mathbb{N}$, and arbitrary real $\alpha > -1$ the classical Laguerre polynomial is defined by Rodrigues formula:

$$(\forall x \in \mathbb{R}) \;\; \phi_{n,\alpha}(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}).$$

$\phi_{n,\alpha}$ is a polynomial with the degree n, the same parity as n, whose highest monomial degree is $\frac{(-1)^n}{n!}x^n$ and have real coefficients. Furthemore, they verify the following condition of orthogonality wiht respect to gamma density $x \longmapsto x^\alpha e^{-x}$ on $[0,+\infty[$:

$$\int_{\mathbb{R}^+} \phi_{n,\alpha}(x)\phi_{m,\alpha}(x)x^\alpha e^{-x}dx = \delta_{n,m}\frac{\Gamma(n+\alpha+1)}{n!},$$

where $\delta_{n,m}$ is the Kronecker delta. We also have recurrence relations, differential equations and the estimates:

$$(\forall x \in \mathbb{R}^+)\ \ (\exists c_x > 0)\ \ (\exists n_0 \in \mathbb{N})\ \ (\forall n \geq n_0), \mid \phi_{n,\alpha}(x) \mid \leq c_x n^{\frac{\alpha}{2}}. \tag{1}$$

For more details of the classical theory of orthogonal polynomials see [8], [14], [3], [13], [1] and [2].
The Laguerre functions on $\mathbb{R}^+$ are defined by:

$$\varphi_{n,\alpha}(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\phi_{n,\alpha}(x)x^{\frac{\alpha}{2}}e^{-\frac{x}{2}} = \frac{1}{\sqrt{n!\Gamma(n+\alpha+1)}}x^{\frac{-\alpha}{2}}e^{\frac{x}{2}}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}). \tag{2}$$

$(\varphi_{n,\alpha})_{n\in\mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^2(\mathbb{R}^+)$ and satisfies some recurrence relations, equalities and inequalities. For more details see [14], [3], [1], [13] and [2].
For $n \in \mathbb{N}$ and arbitrary real $\alpha > -1$, we denote by $\psi_{n,\alpha}$, the function on $\mathbb{R}^+$ defined by:

$$(\forall x \in \mathbb{R}^+)\ \ \psi_{n,\alpha}(x) = \frac{n!}{\Gamma(n+\alpha+1)}x^\alpha e^{-x}\phi_{n,\alpha}(x) = \frac{1}{\Gamma(n+\alpha+1)}\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}). \tag{3}$$

$\psi_{n,\alpha}$ satisfies recurrence relations and differential equations, for example:

$$\left(\forall\,(n,m) \in \mathbb{N}^2\right)\ \psi_{n,\alpha}^{(m)} = \psi_{n+m,\alpha-m}. \tag{4}$$

And the following useful inequality:

$$(\forall n \geq 1)\ \ \ \| \psi_{n,\alpha} \|_1 \leq \frac{c_\alpha}{n^{\frac{\alpha}{2}}}. \tag{5}$$

For more details see [1] and [8].
The most important property of the family $(\psi_{n,\alpha})_{n\in\mathbb{N}}$ is that if $f : \mathbb{R}_+^* \to E$ be a differentiable function such that $\int_0^{+\infty} e^{-t}t^\alpha \parallel f(t) \parallel^2 dt < +\infty$, then the series $\sum_{n\in\mathbb{N}} c_n(f)\phi_{n,\alpha}(t)$ converges pointwise to f on $\mathbb{R}_+^*$, where

$$c_n(f) = \int_0^{+\infty} \psi_{n,\alpha}(t)f(t)dt.$$

For more details see [8], [3] and [13].

C-regularized semigroup

A family of operators $(T(t))_{t\geq 0}$ in $B(E)$ is called exponentially bounded $C-$regularized semigroup or exponentially bounded C-semigroup on E, if

1. $T(t+s)C = T(t)T(s)$ *for all* $t, s \in \mathbb{R}^+$.
2. $T(0) = C$.
3. The function $t \mapsto T(t)x$ is continuous on $\mathbb{R}^+$ for any $x \in E$.
4. $(\exists M \geq 0)$ $(\exists \omega \geq 0)$: $\| T(t) \| \leq Me^{\omega t}$, *for all* $t \geq 0$ (exponentially bounded condition).

Its generator W is defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s\to 0^+} \frac{T(s)x - Cx}{s} \text{ exists in R(C)} \right\}$$

and

$$(\forall x \in \mathcal{D}(W)) \quad Wx = C^{-1} \lim_{s\to 0^+} \frac{T(s)x - Cx}{s}.$$

In all that follows, $(T(t))_{t\geq 0} \subset B(E)$ is exponentially bounded $C-$regularized semigroup on E with generator $(W, \mathcal{D}(W))$ such that

$$(\exists M > 0) \quad (\exists \omega \geq 0) : \quad (\forall t \geq 0) \, \| T(t) \| \leq Me^{\omega t}. \tag{6}$$

We present some known facts about $C-$semigroup and its generator, which will be used in the sequel (see [7], [5], [6], [7], [9], [10], [11], [12], [15] and [16] for more details):

- By the property 1, we conclude that $T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$, this means that $T(t)x \in \mathcal{D}(W)$ and $WT(t)x = T(t)Wx$, for all $t \geq 0$ and $x \in D(W)$.
- $\int_0^t T(s)xds \in \mathcal{D}(W)$ and $W\int_0^t T(s)xds = T(t)x - Cx$ for every $x \in E$ and $t \geq 0$, which implies that for each $x \in \mathcal{D}(W)$, $u := T(.)x$ is of class $C^1(\mathbb{R}^+, E)$ and solves the Abstract Cauchy problem
$$((ACP(W, x, 0)_1) \begin{cases} u'(t) = Wu(t), & t \in \mathbb{R}^+ \\ u(0) = Cx. \end{cases}$$
- $\displaystyle\lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x$ for all $x \in E$ and $t \geq 0$, this means that W is a closed linear operator with $R(C) \subset \overline{D(W)}$.
- The $C-$resolvent operator $R_C(\lambda, W)$ is analytic in the $C-$resolvent set $\rho_C(W)$ and
$$\frac{d^n}{d\lambda^n}(R_C(\lambda, W)) = (-1)^n n!(\lambda I - W))^{-n-1}C \text{ for all } n \in \mathbb{N}. \tag{7}$$

- $W = C^{-1}WC$, $(\omega, +\infty) \subset \rho_C(W)$ *and* $R_C(\lambda, W)x = \int_0^{+\infty} e^{-\lambda t}T(t)xdt \;\; for \;\; \lambda > \omega$ and $x \in E$. For every $\lambda > \omega$ and $n \in \mathbb{N}$, $R(C) \subset D((\lambda I - W)^{-n})$ and

$$(\lambda I - W)^{-n}Cx = \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)xdt, \tag{8}$$

which implies

$$\| (\lambda - \omega)^n(\lambda I - W)^{-n}C \| \leq M. \tag{9}$$

- Let $\alpha > 0$ and $\lambda > 0$, we have $\int_0^{+\infty} t^{\alpha-1}e^{-\lambda t}dt = \frac{\Gamma(\alpha)}{\lambda^\alpha}$ and the family of operators $(e^{-\omega t}T(t))_{t\geq 0}$ is uniformly bounded $C-$semigroup with the generator $W-\omega I$. Then we can define the fractional power of C-resolvent operator(see [5] for more details) as below :

$$(\lambda I-(W-\omega I))^{-\alpha}Cx = ((\lambda+\omega)I-W)^{-\alpha}Cx := \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e^{-\lambda t}e^{-\omega t}T(t)xdt, \; for\, all\, x \in E. \tag{10}$$

With a simple verification, $(\lambda I - (W - \omega I))^{-\alpha}C$ is bounded linear operator.

2 Main results

Theorem 2.1 *Let* $(T(t))_{t\in\mathbb{R}^+}$ *be an exponentially bounded* $C-$*semigroup in a Banach space* E *with generator* $(W, \mathcal{D}(W))$, $q \in \mathbb{N}$ *such that* $q > 2$ *and* $\alpha > -1$. *Then*

1. *For any* $n \in \mathbb{N}$ *and* $x \in E$, *we have*

$$\int_0^{+\infty} \psi_{n,\alpha}(t)e^{-\omega t}T(t)xdt = (\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}Cx.$$

2. *For* $x \in \mathcal{D}(W)$, *we have :*

   - *(a)* $T(t)x = e^{\omega t}\sum_{n=0}^{+\infty}(\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}\phi_{n,\alpha}(t)Cx, \; for\; all\; t > 0.$
   - *(b) For each* $t > 0$ *there is* $n_0 \in \mathbb{N}$ *such that for all integer* n *with* $n \geq n_0$ *and* $x \in \mathcal{D}(W^q)$, *we have*

$$\| T(t)x - e^{\omega t}\sum_{m=0}^{n}(\omega I - W)^m((\omega+1)I - W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t) \| \leq \frac{c_{t,\alpha,q} \| (W-\omega I)^q x \|}{n^{\frac{q}{2}-1}},$$

   where $c_{t,\alpha,q}$ *is a constant which depends only on* $t > 0$, α *and* q.

Proof.

Throughout the proof, α is an arbitrary real such that $\alpha > -1$.

1. Let $n \in \mathbb{N},\ x \in E$. The function $t \mapsto \psi_{n,\alpha}(t)T(t)x$ is continuous on $\mathbb{R}^+$ and integrable in the sens of Bochner. Put $I := \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt$, then

$$
\begin{aligned}
I &:= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \\
&= \frac{1}{\Gamma(n+\alpha+1)} \int_0^{+\infty} \frac{d^n}{dt^n}(t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))xdt \\
&= \frac{1}{\Gamma(n+\alpha+1)} \left([\frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))x]_0^{+\infty} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})\frac{d}{dt}(e^{-\omega t}T(t))xdt \right) \\
&= \frac{1}{\Gamma(n+\alpha+1)} \left(0 - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})(W-\omega I)(e^{-\omega t}T(t))xdt \right) \\
&= \frac{-1}{\Gamma(n+\alpha+1)} \left((W-\omega I) \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))xdt \right) \\
&= (-1)^n (W-\omega I)^n \left(\frac{1}{\Gamma(n+\alpha+1)} \int_0^{+\infty} (t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))xdt \right) \\
&= (\omega I - W)^n \left(\frac{1}{\Gamma(n+\alpha+1)} \int_0^{+\infty} (t^{(n+\alpha+1)-1}e^{-t})(e^{-\omega t}T(t))xdt \right) \\
&= (\omega I - W)^n (I - (W-\omega I))^{-n-\alpha-1}Cx \\
&= (\omega I - W)^n ((\omega+1)I - W)^{-n-\alpha-1}Cx.
\end{aligned}
$$

2. Let $x \in \mathcal{D}(W)$.

   - (a) Let's remember that $e^{-\omega(.)}T(.)x : \mathbb{R}^+ \to E$ is in $C^1(\mathbb{R}^+, E)$ and

$$
\begin{aligned}
\int_0^{+\infty} e^{-t}t^\alpha \parallel e^{-\omega t}T(t)x \parallel^2 dt &\leq \int_0^{+\infty} e^{-t}t^\alpha \parallel e^{-\omega t}T(t) \parallel^2 \parallel x \parallel^2 dt \\
&\leq M^2 \parallel x \parallel^2 \int_0^{+\infty} e^{-t}t^\alpha dt \\
&< +\infty.
\end{aligned}
$$

   Now, we apply the most important properties of $(\psi_{n,\alpha})_{n\in\mathbb{N}}$ to get the series $\sum_{n\in\mathbb{N}} c_n(e^{-\omega(.)}T(.)x)\phi_{n,\alpha}(.)$, where

$$
\begin{aligned}
c_n((e^{-\omega(.)}T(.))x) &= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \\
&= (\omega I - W)^n ((\omega+1)I - W)^{-n-\alpha-1}Cx
\end{aligned}
$$

converges pointwise to $e^{-\omega(.)}T(.)x$ for $t \in \mathbb{R}^+$. Therefore,

$$\begin{aligned} T(t)x &= e^{\omega t}(e^{-\omega t}T(t)x) \\ &= e^{\omega t}\sum_{n=0}^{+\infty}(\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}Cx\phi_{n,\alpha}(t). \end{aligned}$$

- (b) Let $t \geq 0$ and $q \in \mathbb{N}$ such that $q > 2$.
  Since $\lim_{n\to+\infty}(\frac{n}{n-q})^{\frac{\alpha}{2}} = 1$, we have

$$(\exists N_1 \in \mathbb{N})\ (\forall n \geq max(q+1, N_1)),\ (\frac{n}{n-q})^{\frac{\alpha}{2}} \leq 2.$$

The Riemann series $\sum_{m\geq q+1}\frac{1}{(m-q)^{\frac{q}{2}}}$ is convergent, hence,

$$\sum_{m=n+1}^{+\infty}\frac{1}{(m-q)^{\frac{q}{2}}} \sim_{n\to+\infty} \frac{1}{(\frac{q}{2}-1)}\frac{1}{(n-q)^{\frac{q}{2}-1}}.$$

We have

$$\frac{1}{(\frac{q}{2}-1)}\frac{1}{(n-q)^{\frac{q}{2}-1}} \sim_{n\to+\infty} \frac{1}{(\frac{q}{2}-1)}\frac{1}{n^{\frac{q}{2}-1}},$$

thus

$$(\exists N_2 \in \mathbb{N})\ (\forall n \geq max(q+1, N_1, N_2)),\ \sum_{m=n+1}^{+\infty}\frac{1}{m^{\frac{q}{2}}} \leq \frac{1}{(\frac{q}{2}-1)}\frac{1}{n^{\frac{q}{2}-1}}.$$

By (1),

$$(\exists\ N_3 \in \mathbb{N})\ (\exists\ c_t > 0):\ (\forall n \geq max(N_1, N_2, N_3, q+1))\ \mid \phi_{n,\alpha}(t) \mid\leq c_t n^{\frac{\alpha}{2}}.$$

Put $n_0 = max(N_1, N_2, N_3, q+1)$, then for $n \geq n_0$ and $x \in D(W^q)$

$$\begin{aligned} J &:= \int_0^{+\infty}\psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \\ &= \int_0^{+\infty}\frac{d^q}{dt^q}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))xdt\ \ (according\ \ to\ \ (4)) \\ &= [\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))x]_0^{+\infty} - \int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))\frac{d}{dt}(e^{-\omega t}T(t))xdt \end{aligned}$$

$$
\begin{aligned}
&= 0-\int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(W-\omega I)(e^{-\omega t}T(t))xdt\\
&= -(W-\omega I)\int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))xdt\\
&= (\omega I-W)\int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))xdt\\
&= (\omega I-W)^q\int_0^{+\infty}\psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))xdt\\
&= \int_0^{+\infty}\psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))(\omega I-W)^q xdt.
\end{aligned}
$$

So

$$
\begin{aligned}
\| (\omega I-W)^n((\omega+1)I-W)^{-n-\alpha-1}Cx \| &= \| \int_0^{+\infty}\psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \|\\
&= \| \int_0^{+\infty}(\psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))(W-\omega I)^q xdt \|\\
&\leq \int_0^{+\infty} | \psi_{n-q,\alpha+q}(t) | \| e^{-\omega t}T(t) \| dt \| (W-\omega I)^q x \|\\
&\leq \| (W-\omega I)^q x \| M \| \psi_{n-q,\alpha+q} \|_1\\
&\leq \| (W-\omega I)^q x \| \frac{c_{\alpha+q}}{(n-q)^{\frac{\alpha+q}{2}}} \ (according\ to\ (5)).
\end{aligned}
$$

On the other hand, we are looking to estimate the quantity

$$
e^{\omega t} \| \sum_{m=0}^{+\infty}(\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t)-\sum_{m=0}^{n}(\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t) \| . \tag{11}
$$

We have

$$
\begin{aligned}
(11) &\leq \sum_{m=n+1}^{+\infty} \| e^{\omega t}(\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t) \|\\
&\leq e^{\omega t}\sum_{m=n+1}^{+\infty} \| (\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx \| \quad | \phi_{m,\alpha}(t) |\\
&\leq e^{\omega t}\sum_{m=n+1}^{+\infty} \| (W-\omega I)^q x \| \frac{c_{\alpha+q}}{(m-q)^{\frac{\alpha+q}{2}}}c_t m^{\frac{\alpha}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \ \| (W-\omega I)^q x \| e^{\omega t} c_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \\
&\leq \ \| (W-\omega I)^q x \| c_t' c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha}{2}}} \\
&\leq \ \| (W-\omega I)^q x \| c_t' c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \left(\frac{m}{m-q}\right)^{\frac{\alpha}{2}} \\
&\leq \ \| (W-\omega I)^q x \| c_t' c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{1}{(m-q)^{\frac{q}{2}}} \times 2 \\
&\leq \ \| (W-\omega I)^q x \| c_t' c_{\alpha+q} \frac{1}{(\frac{q}{2}-1)} \frac{1}{n^{\frac{q}{2}-1}},
\end{aligned}
$$

where

$$
\| T(t)x - e^{\omega t} \sum_{m=0}^{n} (\omega I - W)^m ((\omega+1)I - W)^{-m-\alpha-1} C x \phi_{m,\alpha}(t) \| \leq \frac{c_{t,\alpha,q} \| (W-\omega I)^q x \|}{n^{\frac{q}{2}-1}}.
$$

Theorem 2.2 *Let $(T(t))_{t\in\mathbb{R}^+}$ be an exponentially bounded $C-$semigroup in a Banach space E with generator $(W, \mathcal{D}(W))$ and $\alpha \in \mathbb{R}$.*

1. *For $x \in E$, $\alpha > 0$ and $n \in \mathbb{N}$, we have*

$$
\int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W-\omega I) x dt = \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds.
$$

2. *For $x \in E$ and $\alpha > 1$, we have*

$$
R_C(t, W-\omega I)x = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s) x ds \right) \phi_{n,\alpha}(t), \ for\ all\ t > 0.
$$

Proof.

1. Let $x \in E$, $\alpha > 0$ and $n \in \mathbb{N}$. For all $t > 0$, $t + \omega \in \rho_C(W)$, so the function $t \longmapsto R_C(t, W-\omega I) = R_C(t+\omega, W)$ is analytic in $\mathbb{R}_+^*$.

In the other hand, by equation (9), we know that $\| R_C(t+\omega, W) \| \leq \frac{M}{t}$ for all $t > 0$, so

$$\begin{aligned}
\int_0^{+\infty} \| \psi_{n,\alpha}(t) R_C(t, W-\omega I)x \| \, dt &= \int_0^{+\infty} | \psi_{n,\alpha}(t) | \| (R_C(t, W-\omega I)x \| \, dt \\
&\leq \int_0^{+\infty} \frac{n!M}{\Gamma(n+\alpha+1)t} e^{-t} t^{\alpha} \mid \phi_{n,\alpha}(t) \mid dt \\
&\leq \frac{n!M}{\Gamma(n+\alpha+1)} \int_0^{+\infty} e^{-t} t^{\alpha-1} \mid \phi_{n,\alpha}(t) \mid dt \\
&< +\infty.
\end{aligned}$$

Put

$$H := \Gamma(n+\alpha+1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W-\omega I)x dt,$$

then

$$\begin{aligned}
H &= \Gamma(n+\alpha+1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W-\omega I)x dt \\
&= \int_0^{+\infty} \frac{d^n}{dt^n}(e^{-t} t^{n+\alpha}) R_C(t, W-\omega I)x dt \\
&= \underbrace{[\frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^{n+\alpha}) R_C(t, W-\omega I)x]_0^{+\infty}}_{=0} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^{n+\alpha}) \frac{d}{dt}(R_C(t, W-\omega I)x) dt \\
&= [-\frac{d^{n-2}}{dt^{n-2}}(e^{-t} t^{n+\alpha}) \frac{d}{dt}(R_C(t, W-\omega I)x)]_0^{+\infty} \\
&+ (-1)^2 \int_0^{+\infty} \frac{d^{n-2}}{dt^{n-2}}(e^{-t} t^{n+\alpha}) \frac{d^2}{dt^2}(R_C(t, W-\omega I)x) dt \\
&= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} \frac{d^n}{dt^n}(R_C(t, W-\omega I)x) dt \ \ (\textit{integtration by parts}) \\
&= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} (-1)^n n! ((t+\omega)I - W)^{-n-1} C x dt \\
&= \int_0^{+\infty} e^{-t} t^{n+\alpha} (n!) \frac{1}{n!} \int_0^{+\infty} s^n e^{-ts} e^{-\omega s} T(s) x ds dt \\
&= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left(\int_0^{+\infty} t^{n+\alpha} e^{-t} e^{-ts} dt \right) ds \ (\textit{Fubini's Theorem})
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} s^n e^{-\omega s} T(s)x \left(\int_0^{+\infty} t^{n+\alpha} e^{-(s+1)t} dt \right) ds \\
&= \int_0^{+\infty} s^n e^{-\omega s} T(s)x \left(\int_0^{+\infty} \frac{u^{n+\alpha}}{(s+1)^{n+\alpha}} e^{-u} \frac{du}{s+1} \right) ds \ \ (u = (s+1)t) \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x \left(\int_0^{+\infty} u^{n+\alpha} e^{-u} du \right) ds \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x \Gamma(n+\alpha+1) ds \\
&= \Gamma(n+\alpha+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x ds,
\end{aligned}
$$

hence the result.

2. Let $x \in E$ and $\alpha > 1$.
The function $t \longmapsto R_C(t, W - \omega I)x = R_C(t + \omega, W)x$ is differentiable in $\mathbb{R}_+^*$ (because it's analytic in $\mathbb{R}_+^*$), and

$$
\begin{aligned}
\int_0^{+\infty} t^\alpha e^{-t} \parallel R_C(t, W - \omega I)x \parallel^2 dt &= \int_0^{+\infty} t^\alpha e^{-t} \parallel ((t+\omega)I - W)^{-1} Cx \parallel^2 dt \\
&\leq \int_0^{+\infty} t^\alpha e^{-t} \frac{M^2}{t^2} \parallel x \parallel^2 dt \ \ (\textit{equation} \ \ (9)) \\
&\leq M^2 \parallel x \parallel^2 \int_0^{+\infty} t^{\alpha-2} e^{-t} dt \\
&= M^2 \parallel x \parallel^2 \Gamma(\alpha - 1) \ (\textit{because} \ \ \alpha > 1) \\
&< +\infty.
\end{aligned}
$$

Thus, the series $\sum_{n \in \mathbb{N}} c_n(R_C(., W - \omega I)x)\phi_{n,\alpha}$ converges pointwise to $R_C(., W - \omega I)x$ on $\mathbb{R}_+^*$, where

$$
\begin{aligned}
c_n(R_C(., W - \omega I)x) &= \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I)x dt \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x ds \ \ (\textit{according to Theorem} 2.2(a)).
\end{aligned}
$$

Therefore,

$$
(\forall t > 0) \ \ R_C(t, W - \omega I)x = ((t+\omega)I - W)^{-1} Cx = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s)x ds \right) \phi_{n,\alpha}(t).
$$

Example 2.3 *Let $m : \mathbb{R} \to \mathbb{R}^-$ be an even measurable function. In the Banach space $L^1(\mathbb{R})$, we consider the family $T := (T(t))_{t\geq 0} \subset \mathfrak{F}(L^1(\mathbb{R}))$ defined by $\forall t \geq 0,\ T(t) : L^1(\mathbb{R}) \to L^1(\mathbb{R}), f \mapsto T(t)(f) : \mathbb{R} \to \mathbb{R}, s \mapsto T(t)(f)(s) = e^{t.m(s)} f(-s)$. Clearly, $(T(t))_{t\geq 0} \subset B(L^1(\mathbb{R}))$. If we put $T(0) = C$, then the family of operators $(T(t))_{t\geq 0}$ is uniformly bounded C-regularized semigroup with generator $(W, \mathcal{D}(W))$ defined by*

$$W : \mathcal{D}(W) = \{f \in L^1(\mathbb{R}) /\ m.f \in L^1(\mathbb{R})\} \to L^1(\mathbb{R}),\ f \mapsto W(f) = m.f.$$

Theorem 2.1 gives for $f \in \mathcal{D}(W)$ and $s, t \in \mathbb{R}^+$:

$$\begin{aligned} T(t)(f)(s) &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t) \\ &= e^{0\times t} \sum_{n=0}^{+\infty} (0 \times I - W)^n ((0 + 1)I - W)^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} (-m(s))^n (1 - m(s))^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t). \end{aligned}$$

So $T(t)(f) = \sum_{n=0}^{+\infty} \phi_{n,\alpha}(t)(-m)^n (1 - m)^{-n-\alpha-1} C(f)$.
Theorem 2.2 gives

$$R_C(t, W)(f)(.) = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{sm(.)} ds \right) C(f)(.) \phi_{n,\alpha}(t).$$

Example 2.4 *The space $X = c_0 = \left\{ (x_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}\ tq\ \lim_{k\mapsto+\infty} x_k = 0 \right\}$, equipped with the norm $\| (x_k)_{k\in\mathbb{N}} \|_\infty = \max_{k\in\mathbb{N}} | x_k |$ becomes a Banach space. For each $n \in \mathbb{N}$, let $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$ be an element of X. Since for all $x = (x_k)_{k\in\mathbb{N}} \in X,\ x = \sum_{k=0}^{+\infty} x_k e_k$, we have $X = span\{e_n / n \in \mathbb{N}\}$. Considering the family of operators $(T(t))_{t\geq 0}$ defined by :*

$$for\ all\ t \in \mathbb{R}^+,\ for\ all\ x \left(= \sum_{k=0}^{+\infty} x_k e_k \right) \in X,\ T(t)x = \sum_{k=0}^{+\infty} e^{-k^2 t} x_k e_k$$

$(T(t))_{t\geq 0}$ is a unifomly bounded C_0-semigroup ($\| T(t) \| \leq 1$) with generator $(W, \mathcal{D}(W))$ such that $\mathcal{D}(W) = \{x = (x_k)_{k\in\mathbb{N}} \in X / (k^2 x_k)_{k\in\mathbb{N}} \in X\}$ and $(\forall x \in \mathcal{D}(W))\ \ Wx = \sum_{k=0}^{+\infty} -k^2 x_k e_k$. Theorem 2.1

gives for $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{D}(W)$ *and for all* $t \in \mathbb{R}^+$:

$$\begin{aligned} T(t)(x) &= e^{\omega t}\sum_{n=0}^{+\infty}(\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}(x)\phi_{n,\alpha}(t) \\ &= e^{0\times t}\sum_{n=0}^{+\infty}(0\times I - W)^n((0+1)I - W)^{-n-\alpha-1}(x)\phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty}(k^{2n}(1+k^2)^{-n-\alpha-1}x_k)_{k\in\mathbb{N}}\phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}k^{2n}(1+k^2)^{-n-\alpha-1}x_k e_k\phi_{n,\alpha}(t). \end{aligned}$$

Theorem 2.2 gives

$$R_C(t,W)x = \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\left(\int_0^{+\infty}\frac{s^n}{(s+1)^{n+\alpha+1}}e^{-k^2 s}ds\right)x_k e_k\phi_{n,\alpha}(t).$$

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