Laguerre Expansions of $C-$regularized semigroups Functions.

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Abstract

The aim of this paper is to approximate the exponentially bounded $C-$regularized semigroups function by the Laguerre series, recalling the notions and the results used.

Keywords: Laguerre functions, $C-$regularized semigroup, C_0-semigroup.

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1 Introduction and preliminaries

The series expansion of Laguerre orthogonal polynomials have been an important tool in mathematical physics , in problems involving the integration of Helmholtz's equation in prabolic coordinates, in the theory of the Hydrogen atom, in the theory of propagation of electomagnetic waves a long transmission lines terminated by a lumped inductance [8]. The study of sufficient conditions for the convergence of Laguerre series has been the subject of numerous works, for more details see [14], [3], [13], [8] and [2].
In 2014, Abadias and Miana studied in their article [1], the Laguerre expansion of C_0-semigroups and Resolvent Operators. In this work we will be interested in Laguerre expansion of C-regularized semigroups function, starting with reminding the notations, concepts and results used.
Throughout this paper E denotes a non-trivial complex Banach space, $\mathfrak{F}(E,F)$ denotes the set of all applications from E to another Banach space F, $B(E)$ denotes the space of all bounded linear operators from E into itself, and $L^1_{loc}(E)$ the set of all $f \in \mathfrak{F}(\mathbb{R},E)$ locally integrable. For a closed linear operator A on E, $\mathcal{D}(A)$, $R(A)$ and $\rho(A)$ denote its domain, range and resolvent set, respectively. $\mathcal{D}(A)$ equipped with the graph norm $\| x \|_{\mathcal{D}(A)} = \| x \|_E + \| Ax \|_E$ become Banach space. Throughout this paper, $C \in B(E)$ will be an injective operator. The C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \;\; and \;\; \lambda I - A$ is injective $\}$ and if $\lambda \in \rho_C(A)$ then we denoted by $R_C(\lambda, A) = (\lambda I - A)^{-1}C$ the $C-$resolvent.

Laguerre functions and Laguerre expansions on Banach spaces

For all $n \in \mathbb{N}$, and arbitrary real $\alpha > -1$ the classical Laguerre polynomial, is defined by Rodrigues formula:

$$(\forall x \in \mathbb{R}) \;\; \phi_{n,\alpha}(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}).$$

$\phi_{n,\alpha}$ is a polynomial with the degree n, the same parity as n, whose highest monomial degree is $\frac{(-1)^n}{n!}x^n$ and have real coefficients. Furthemore, they verify the following condition of orthogonality wiht respect to gamma density $x \longmapsto x^\alpha e^{-x}$ on $[0, +\infty[$:

$$\int_{\mathbb{R}^+} \phi_{n,\alpha}(x)\phi_{m,\alpha}(x)x^\alpha e^{-x}dx = \delta_{n,m}\frac{\Gamma(n+\alpha+1)}{n!},$$

where $\delta_{n,m}$ is the Kronecker delta. We also have recurrence relations, differential equations and the estimates:

$$(\forall x \in \mathbb{R}^+)\ \ (\exists c_x > 0)\ \ (\exists n_0 \in \mathbb{N})\ \ (\forall n \geq n_0), \mid \phi_{n,\alpha}(x) \mid \leq c_x n^{\frac{\alpha}{2}}. \tag{1}$$

For more details of the classical theory of orthogonal polynomials see [8], [14], [3], [13], [1] and [2].
The Laguerre functions on $\mathbb{R}^+$ are defined by:

$$\varphi_{n,\alpha}(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\phi_{n,\alpha}(x)x^{\frac{\alpha}{2}}e^{-\frac{x}{2}} = \frac{1}{\sqrt{n!\Gamma(n+\alpha+1)}}x^{\frac{-\alpha}{2}}e^{\frac{x}{2}}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}). \tag{2}$$

$(\varphi_{n,\alpha})_{n\in\mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^2(\mathbb{R}^+)$ and satisfied some recurrence relations, equality and inequality. For more details see [14], [3], [1], [13] and [2].
For $n \in \mathbb{N}$ and arbitrary real $\alpha > -1$, we denote by $\psi_{n,\alpha}$, the function on $\mathbb{R}^+$ defined by :

$$(\forall x \in \mathbb{R}^+)\ \ \psi_{n,\alpha}(x) = \frac{n!}{\Gamma(n+\alpha+1)}x^\alpha e^{-x}\phi_{n,\alpha}(x) = \frac{1}{\Gamma(n+\alpha+1)}\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}). \tag{3}$$

$\psi_{n,\alpha}$ satisfies recurrence relations and differential equations, for example:

$$\left(\forall\,(n,m) \in \mathbb{N}^2\right)\ \psi_{n,\alpha}^{(m)} = \psi_{n+m,\alpha-m}. \tag{4}$$

And the following useful inequality :

$$(\forall n \geq 1)\quad \parallel \psi_{n,\alpha} \parallel_1 \leq \frac{c_\alpha}{n^{\frac{\alpha}{2}}}. \tag{5}$$

For more details see [1] and [8].
The most important properties of the family $(\psi_{n,\alpha})_{n\in\mathbb{N}}$ is that if $f : \mathbb{R}_+^* \longrightarrow E$ be a differentiable function such that $\int_0^{+\infty} e^{-t}t^\alpha \parallel f(t) \parallel^2 dt < +\infty$, then the series $\sum_{n\in\mathbb{N}} c_n(f)\phi_{n,\alpha}(t)$ converges pointwise to f on $\mathbb{R}_+^*$, where

$$c_n(f) = \int_0^{+\infty} \psi_{n,\alpha}(t)f(t)dt.$$

For more details see [8], [3] and [13].

C-regularized semigroup

A family of operators $(T(t))_{t\geq 0}$ in $B(E)$ is called exponentially bounded $C-$regularized semigroup or exponentially bounded C-semigroup on E, if

1. $T(t+s)C = T(t)T(s)$ *for all* $t, s \in \mathbb{R}^+$.
2. $T(0) = C$.
3. The function $t \mapsto T(t)x$ is continuous on $\mathbb{R}^+$ for any $x \in E$.
4. $(\exists M \geq 0)\ (\exists \omega \geq 0) :\ \| T(t) \| \leq Me^{\omega t},\ $ *for all* $t \geq 0$ (exponentially bounded condition).

Its generator W is defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s\to 0^+} \frac{T(s)x - Cx}{s} \text{ exists in R(C) } \right\}$$

and

$$(\forall x \in \mathcal{D}(W)) \quad Wx = C^{-1} \lim_{s\to 0^+} \frac{T(s)x - Cx}{s}.$$

In all that follows, $(T(t))_{t\geq 0} \subset B(E)$ is exponentially bounded $C-$regularized semigroup on E with generator $(W, \mathcal{D}(W))$ such that

$$(\exists M > 0) \quad (\exists \omega \geq 0) : \quad (\forall t \geq 0)\ \| T(t) \| \leq Me^{\omega t}. \tag{6}$$

We present some known facts about $C-$semigroup and its generator, which will be used in the sequel (see [7], [5], [6], [7], [9], [10], [11], [12], [15] and [16] for more details):

- By the property 1, we conclude that $T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$, this means that $T(t)x \in \mathcal{D}(W)$ and $WT(t)x = T(t)Wx$, for all $t \geq 0$ and $x \in D(W)$.
- $\int_0^t T(s)xds \in \mathcal{D}(W)$ and $W\int_0^t T(s)xds = T(t)x - Cx$ for every $x \in E$ and $t \geq 0$, which implies that for each $x \in \mathcal{D}(W)$, $\ u := T(.)x$ is of class $C^1(\mathbb{R}^+, E)$ and solves the Abstract Cauchy problem
$$((ACP(W, x, 0)_1) \begin{cases} u'(t) = Wu(t), & t \in \mathbb{R}^+ \\ u(0) = Cx. \end{cases}$$
- $\displaystyle\lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x$ for all $x \in E$ and $t \geq 0$, this means that W is closed linear operator with $R(C) \subset \overline{D(W)}$.
- The $C-$resolvent operator $R_C(\lambda, W)$ is analytic in the $C-$resolvent set $\rho_C(W)$ and
$$\frac{d^n}{d\lambda^n}(R_C(\lambda, W)) = (-1)^n n!(\lambda I - W))^{-n-1}C \text{ for all } n \in \mathbb{N}. \tag{7}$$

- $W = C^{-1}WC$, $(\omega, +\infty) \subset \rho_C(W)$ *and* $R_C(\lambda, W)x = \int_0^{+\infty} e^{-\lambda t}T(t)xdt$ *for* $\lambda > \omega$ and $x \in E$. For every $\lambda > \omega$ and $n \in \mathbb{N}$, $R(C) \subset D((\lambda I - W)^{-n})$ and

$$(\lambda I - W)^{-n}Cx = \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)xdt, \tag{8}$$

which implies

$$\| (\lambda - \omega)^n (\lambda I - W)^{-n}C \| \leq M. \tag{9}$$

- Let $\alpha > 0$ and $\lambda > 0$. Like $\int_0^{+\infty} t^{\alpha-1}e^{-\lambda t}dt = \frac{\Gamma(\alpha)}{\lambda^\alpha}$ and the family of operators $(e^{-\omega t}T(t))_{t\geq 0}$ is uniformly bounded $C-$semigroup with the generator $W - \omega I$. Then we can define the fractional power of C-resolvent operator(see [5] for more details) as below :

$$(\lambda I - (W - \omega I))^{-\alpha}Cx = ((\lambda+\omega)I - W)^{-\alpha}Cx := \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}e^{-\lambda t}e^{-\omega t}T(t)xdt, \; for\, all\, x \in E. \tag{10}$$

With a simple verification, $(\lambda I - (W - \omega I))^{-\alpha}C$ is bounded linear operator.

2 Main results

Theorem 2.1 *Let* $(T(t))_{t\in\mathbb{R}^+}$ *be an exponentially bounded* $C-$*semigroup in Banach space* E *with generator* $(W, \mathcal{D}(W))$, $q \in \mathbb{N}$ *such that* $q > 2$ *and* $\alpha > -1$, *then*

1. *For any* $n \in \mathbb{N}$ *and* $x \in E$ *we have*

$$\int_0^{+\infty} \psi_{n,\alpha}(t)e^{-\omega t}T(t)xdt = (\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}Cx.$$

2. *For* $x \in \mathcal{D}(W)$ *we have :*

   - *(a)* $T(t)x = e^{\omega t}\sum_{n=0}^{+\infty}(\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}\phi_{n,\alpha}(t)Cx, \; for\ all\ t > 0.$
   - *(b) For each* $t > 0$ *there is* $n_0 \in \mathbb{N}$ *such that for all integer* n *with* $n \geq n_0$ *and* $x \in \mathcal{D}(W^q)$, *we have*

$$\| T(t)x - e^{\omega t}\sum_{m=0}^{n}(\omega I - W)^m((\omega+1)I - W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t) \| \leq \frac{c_{t,\alpha,q} \| (W - \omega I)^q x \|}{n^{\frac{q}{2}-1}}.$$

   Where $c_{t,\alpha,q}$ *is a constant which depends only on* $t > 0$, α *and* q.

Proof.

Throughout the proof, α is an arbitrary real such that $\alpha > -1$.

1. Let $n \in \mathbb{N},\ x \in E$. The function $t \mapsto \psi_{n,\alpha}(t)T(t)x$ is continuous on $\mathbb{R}^+$ and integrable in the sens of Bochner, if we posed $I := \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt$ then

$$\begin{aligned}
I &:= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \\
&= \frac{1}{\Gamma(n+\alpha+1)} \int_0^{+\infty} \frac{d^n}{dt^n}(t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))xdt \\
&= \frac{1}{\Gamma(n+\alpha+1)} \left([\frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))x]_0^{+\infty} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})\frac{d}{dt}(e^{-\omega t}T(t))xdt \right) \\
&= \frac{1}{\Gamma(n+\alpha+1)} \left(0 - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})(W-\omega I)(e^{-\omega t}T(t))xdt \right) \\
&= \frac{-1}{\Gamma(n+\alpha+1)} \left((W-\omega I) \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))xdt \right) \\
&= (-1)^n (W-\omega I)^n \left(\frac{1}{\Gamma(n+\alpha+1)} \int_0^{+\infty} (t^{n+\alpha}e^{-t})(e^{-\omega t}T(t))xdt \right) \\
&= (\omega I - W)^n \left(\frac{1}{\Gamma(n+\alpha+1)} \int_0^{+\infty} (t^{(n+\alpha+1)-1}e^{-t})(e^{-\omega t}T(t))xdt \right) \\
&= (\omega I - W)^n (I - (W-\omega I))^{-n-\alpha-1}Cx \\
&= (\omega I - W)^n ((\omega+1)I - W)^{-n-\alpha-1}Cx
\end{aligned}$$

2. Let $x \in \mathcal{D}(W)$.
   - (a) Let's remember that $e^{-\omega(.)}T(.)x : \mathbb{R}^+ \to E$ is in $C^1(\mathbb{R}^+, E)$ and like

$$\begin{aligned}
\int_0^{+\infty} e^{-t}t^\alpha \parallel e^{-\omega t}T(t)x \parallel^2 dt &\le \int_0^{+\infty} e^{-t}t^\alpha \parallel e^{-\omega t}T(t) \parallel^2 \parallel x \parallel^2 dt \\
&\le M^2 \parallel x \parallel^2 \int_0^{+\infty} e^{-t}t^\alpha dt \\
&< +\infty,
\end{aligned}$$

   then, we apply the most important properties of $(\psi_{n,\alpha})_{n\in\mathbb{N}}$ to get the series $\sum_{n\in\mathbb{N}} c_n(e^{-\omega(.)}T(.)x)\phi_{n,\alpha}(.)$ with

$$\begin{aligned}
c_n((e^{-\omega(.)}T(.))x) &= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \\
&= (\omega I - W)^n ((\omega+1)I - W)^{-n-\alpha-1}Cx
\end{aligned}$$

converges pointwise to $e^{-\omega(.)}T(.)x$ for $t \in \mathbb{R}^+$, it is

$$\begin{aligned} T(t)x &= e^{\omega t}(e^{-\omega t}T(t)x) \\ &= e^{\omega t}\sum_{n=0}^{+\infty}(\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}Cx\phi_{n,\alpha}(t). \end{aligned}$$

- (b) Let $t \geq 0$ and $q \in \mathbb{N}$ such that $q > 2$.
  Like $\lim_{n\to+\infty}(\frac{n}{n-q})^{\frac{\alpha}{2}} = 1$ then

$$(\exists N_1 \in \mathbb{N})\ (\forall n \geq max(q+1, N_1)),\ (\frac{n}{n-q})^{\frac{\alpha}{2}} \leq 2.$$

The Riemann serie $\sum_{m\geq q+1}\frac{1}{(m-q)^{\frac{q}{2}}}$ is convergent then

$$\sum_{m=n+1}^{+\infty}\frac{1}{(m-q)^{\frac{q}{2}}} \sim_{n\to+\infty} \frac{1}{(\frac{q}{2}-1)}\frac{1}{(n-q)^{\frac{q}{2}-1}}$$

but

$$\frac{1}{(\frac{q}{2}-1)}\frac{1}{(n-q)^{\frac{q}{2}-1}} \sim_{n\to+\infty} \frac{1}{(\frac{q}{2}-1)}\frac{1}{n^{\frac{q}{2}-1}}$$

thus

$$(\exists N_2 \in \mathbb{N})\ (\forall n \geq max(q+1, N_1, N_2)),\ \sum_{m=n+1}^{+\infty}\frac{1}{m^{\frac{q}{2}}} \leq \frac{1}{(\frac{q}{2}-1)}\frac{1}{n^{\frac{q}{2}-1}}.$$

By the inequation estimates (1),

$$(\exists\ N_3 \in \mathbb{N})\ (\exists\ c_t > 0):\ (\forall n \geq max(N_1, N_2, N_3, q+1))\ \mid \phi_{n,\alpha}(t) \mid\leq c_t n^{\frac{\alpha}{2}}.$$

If we posed $n_0 = max(N_1, N_2, N_3, q+1)$, then for $n \geq n_0$ and $x \in D(W^q)$

$$\begin{aligned} J &:= \int_0^{+\infty}\psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \\ &= \int_0^{+\infty}\frac{d^q}{dt^q}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))xdt\ \ (according\ \ to\ \ (4)) \\ &= [\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))x]_0^{+\infty} - \int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))\frac{d}{dt}(e^{-\omega t}T(t))xdt \end{aligned}$$

$$
\begin{aligned}
&= 0-\int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(W-\omega I)(e^{-\omega t}T(t))xdt\\
&= -(W-\omega I)\int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))xdt\\
&= (\omega I-W)\int_0^{+\infty}\frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))xdt\\
&= (\omega I-W)^q\int_0^{+\infty}\psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))xdt\\
&= \int_0^{+\infty}\psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))(\omega I-W)^q xdt.
\end{aligned}
$$

So

$$
\begin{aligned}
\| (\omega I-W)^n((\omega+1)I-W)^{-n-\alpha-1}Cx \| &= \| \int_0^{+\infty}\psi_{n,\alpha}(t)(e^{-\omega t}T(t))xdt \|\\
&= \| \int_0^{+\infty}(\psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))(W-\omega I)^q xdt \|\\
&\leq \int_0^{+\infty} | \psi_{n-q,\alpha+q}(t) | \, \| e^{-\omega t}T(t) \| \, dt \, \| (W-\omega I)^q x \|\\
&\leq \| (W-\omega I)^q x \| M \| \psi_{n-q,\alpha+q} \|_1\\
&\leq \| (W-\omega I)^q x \| \frac{c_{\alpha+q}}{(n-q)^{\frac{\alpha+q}{2}}} \ (according\ to\ (5)).
\end{aligned}
$$

On the other hand, we are looking to increase the quantity

$$
e^{\omega t} \| \sum_{m=0}^{+\infty}(\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t)-\sum_{m=0}^{n}(\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t) \| \tag{11}
$$

We have

$$
\begin{aligned}
(11) &\leq \sum_{m=n+1}^{+\infty} \| e^{\omega t}(\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx\phi_{m,\alpha}(t) \|\\
&\leq e^{\omega t}\sum_{m=n+1}^{+\infty} \| (\omega I-W)^m((\omega+1)I-W)^{-m-\alpha-1}Cx \| \quad | \phi_{m,\alpha}(t) |\\
&\leq e^{\omega t}\sum_{m=n+1}^{+\infty} \| (W-\omega I)^q x \| \frac{c_{\alpha+q}}{(m-q)^{\frac{\alpha+q}{2}}} c_t m^{\frac{\alpha}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \parallel (W-\omega I)^q x \parallel e^{\omega t} c_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \\
&\leq \parallel (W-\omega I)^q x \parallel c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha}{2}}} \\
&\leq \parallel (W-\omega I)^q x \parallel c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} (\frac{m}{m-q})^{\frac{\alpha}{2}} \\
&\leq \parallel (W-\omega I)^q x \parallel c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{1}{(m-q)^{\frac{q}{2}}} \times 2 \\
&\leq \parallel (W-\omega I)^q x \parallel c'_t c_{\alpha+q} \frac{1}{(\frac{q}{2}-1)} \frac{1}{n^{\frac{q}{2}-1}}
\end{aligned}
$$

where

$$
\parallel T(t)x - e^{\omega t} \sum_{m=0}^{n} (\omega I - W)^m ((\omega+1)I - W)^{-m-\alpha-1} C x \phi_{m,\alpha}(t) \parallel \leq \frac{c_{t,\alpha,q} \parallel (W-\omega I)^q x \parallel}{n^{\frac{q}{2}-1}}.
$$

Theorem 2.2 *Let* $(T(t))_{t\in\mathbb{R}^+}$ *be an exponentially bounded* $C-$*semigroup in Banach space* E *with generator* $(W, \mathcal{D}(W))$ *and* $\alpha \in \mathbb{R}$.

1. *For* $x \in E$, $\alpha > 0$ *and* $n \in \mathbb{N}$, *we have*

$$
\int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W-\omega I) x dt = \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds.
$$

2. *For* $x \in E$ *and* $\alpha > 1$, *we have*

$$
R_C(t, W-\omega I)x = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s) x ds \right) \phi_{n,\alpha}(t), \ for\ all\ t > 0.
$$

Proof.

1. Let $x \in E$, $\alpha > 0$ and $n \in \mathbb{N}$. For all $t > 0$, $t + \omega \in \rho_C(W)$, then the function $t \longmapsto R_C(t, W-\omega I) = R_C(t+\omega, W)$ is analytic in $\mathbb{R}_+^*$.

In the other hand by equation (9), we know that $\| R_C(t+\omega, W) \| \leq \frac{M}{t}$ for all $t > 0$, so

$$\begin{aligned}
\int_0^{+\infty} \| \psi_{n,\alpha}(t) R_C(t, W-\omega I)x \| \, dt &= \int_0^{+\infty} | \psi_{n,\alpha}(t) | \| (R_C(t, W-\omega I)x \| \, dt \\
&\leq \int_0^{+\infty} \frac{n!M}{\Gamma(n+\alpha+1)t} e^{-t} t^{\alpha} | \phi_{n,\alpha}(t) | \, dt \\
&\leq \frac{n!M}{\Gamma(n+\alpha+1)} \int_0^{+\infty} e^{-t} t^{\alpha-1} | \phi_{n,\alpha}(t) | \, dt \\
&< +\infty.
\end{aligned}$$

If we posed

$$H := \Gamma(n+\alpha+1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W-\omega I)x dt,$$

then

$$\begin{aligned}
H &= \Gamma(n+\alpha+1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W-\omega I)x dt \\
&= \int_0^{+\infty} \frac{d^n}{dt^n}(e^{-t} t^{n+\alpha}) R_C(t, W-\omega I)x dt \\
&= \underbrace{[\frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^{n+\alpha}) R_C(t, W-\omega I)x]_0^{+\infty}}_{=0} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^{n+\alpha}) \frac{d}{dt}(R_C(t, W-\omega I)x) dt \\
&= [-\frac{d^{n-2}}{dt^{n-2}}(e^{-t} t^{n+\alpha}) \frac{d}{dt}(R_C(t, W-\omega I)x)]_0^{+\infty} \\
&+ (-1)^2 \int_0^{+\infty} \frac{d^{n-2}}{dt^{n-2}}(e^{-t} t^{n+\alpha}) \frac{d^2}{dt^2}(R_C(t, W-\omega I)x) dt \\
&= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} \frac{d^n}{dt^n}(R_C(t, W-\omega I)x) dt \;\; (\textit{integtration by parts}) \\
&= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} (-1)^n n! ((t+\omega)I - W)^{-n-1} C x dt \\
&= \int_0^{+\infty} e^{-t} t^{n+\alpha} (n!) \frac{1}{n!} \int_0^{+\infty} s^n e^{-ts} e^{-\omega s} T(s) x ds dt \\
&= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left(\int_0^{+\infty} t^{n+\alpha} e^{-t} e^{-ts} dt \right) ds \; (\textit{Fubini's Theorem})
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{+\infty} s^n e^{-\omega s} T(s)x \left(\int_0^{+\infty} t^{n+\alpha} e^{-(s+1)t} dt \right) ds \\
&= \int_0^{+\infty} s^n e^{-\omega s} T(s)x \left(\int_0^{+\infty} \frac{u^{n+\alpha}}{(s+1)^{n+\alpha}} e^{-u} \frac{du}{s+1} \right) ds \ \ (u = (s+1)t) \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x \left(\int_0^{+\infty} u^{n+\alpha} e^{-u} du \right) ds \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x \Gamma(n+\alpha+1) ds \\
&= \Gamma(n+\alpha+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x ds,
\end{aligned}$$

hence the result.

2. Let $x \in E$ and $\alpha > 1$.
The function $t \longmapsto R_C(t, W - \omega I)x = R_C(t + \omega, W)x$ is differentiable in $\mathbb{R}_+^*$ (because it's analytic in $\mathbb{R}_+^*$), and

$$\begin{aligned}
\int_0^{+\infty} t^\alpha e^{-t} \parallel R_C(t, W - \omega I)x \parallel^2 dt &= \int_0^{+\infty} t^\alpha e^{-t} \parallel ((t+\omega)I - W)^{-1} Cx \parallel^2 dt \\
&\leq \int_0^{+\infty} t^\alpha e^{-t} \frac{M^2}{t^2} \parallel x \parallel^2 dt \ \ (equation \ \ (9)) \\
&\leq M^2 \parallel x \parallel^2 \int_0^{+\infty} t^{\alpha-2} e^{-t} dt \\
&= M^2 \parallel x \parallel^2 \Gamma(\alpha - 1) \ (because \ \ \alpha > 1) \\
&< +\infty.
\end{aligned}$$

Thus, the series $\sum_{n \in \mathbb{N}} c_n(R_C(., W - \omega I)x)\phi_{n,\alpha}$ converges pointwise to $R_C(., W - \omega I)x$ on $\mathbb{R}_+^*$, where

$$\begin{aligned}
c_n(R_C(., W - \omega I)x) &= \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I)x dt \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s)x ds \ \ (according \ to \ (a) \ of \ theorem \ 2.2).
\end{aligned}$$

Therefore

$$(\forall t > 0) \ \ R_C(t, W - \omega I)x = ((t+\omega)I - W)^{-1} Cx = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s)x ds \right) \phi_{n,\alpha}(t).$$

Example 2.3 *Let $m : \mathbb{R} \to \mathbb{R}^-$ be an even measurable function. In the Banach space $L^1(\mathbb{R})$, we consider the family $T := (T(t))_{t\geq 0} \subset \mathfrak{F}(L^1(\mathbb{R}))$ defined by $\forall t \geq 0,\ T(t) : L^1(\mathbb{R}) \to L^1(\mathbb{R}), f \mapsto T(t)(f) : \mathbb{R} \to \mathbb{R}, s \mapsto T(t)(f)(s) = e^{t.m(s)} f(-s)$. Clearly, $(T(t))_{t\geq 0} \subset B(L^1(\mathbb{R}))$. If we put, $T(0) = C$, then, A family of operators $(T(t))_{t\geq 0}$ is uniformly bounded C-regularized semigroup with generator $(W, \mathcal{D}(W))$ defined by*

$$W : \mathcal{D}(W) = \{f \in L^1(\mathbb{R})/\ m.f \in L^1(\mathbb{R})\} \to L^1(\mathbb{R}),\ f \mapsto W(f) = m.f.$$

The theorem 2.1 give for $f \in \mathcal{D}(W)$ and $s, t \in \mathbb{R}^+$:

$$\begin{aligned} T(t)(f)(s) &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega+1)I - W)^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t) \\ &= e^{0\times t} \sum_{n=0}^{+\infty} (0 \times I - W)^n ((0+1)I - W)^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} (-m(s))^n (1 - m(s))^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t). \end{aligned}$$

So $T(t)(f) = \sum_{n=0}^{+\infty} \phi_{n,\alpha}(t)(-m)^n (1-m)^{-n-\alpha-1} C(f)$.
And Theorem 2.2 gives

$$R_C(t, W)(f)(.) = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{sm(.)} ds \right) C(f)(.) \phi_{n,\alpha}(t).$$

Example 2.4 *The space $X = c_0 = \left\{ (x_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}\ \ tq \lim_{k\mapsto+\infty} x_k = 0 \right\}$, equipped wiht the norm $\| (x_k)_{k\in\mathbb{N}} \|_\infty = \max_{k\in\mathbb{N}} | x_k |$ becomes a Banach space. For each $n \in \mathbb{N}$ let $e_n = (\delta_{n,k})_{k\in\mathbb{N}}$ be element of X. Like for all $x = (x_k)_{k\in\mathbb{N}} \in X,\ x = \sum_{k=0}^{+\infty} x_k e_k$ then $X = span\{e_n / n \in \mathbb{N}\}$. Considering the family of operators $(T(t))_{t\geq 0}$ defined by :*

$$for\ all\ t \in \mathbb{R}^+\ \ for\ all\ x \left(= \sum_{k=0}^{+\infty} x_k e_k \right) \in X,\ T(t)x = \sum_{k=0}^{+\infty} e^{-k^2 t} x_k e_k$$

$(T(t))_{t\geq 0}$ is a unifomly bounded C_0−semigroup ($\| T(t) \| \leq 1$) with generator $(W, \mathcal{D}(W))$ such that $\mathcal{D}(W) = \{x = (x_k)_{k\in\mathbb{N}} \in X / (k^2 x_k)_{k\in\mathbb{N}} \in X\}$ and $(\forall x \in \mathcal{D}(W))\ \ Wx = \sum_{k=0}^{+\infty} -k^2 x_k e_k$. The theorem 2.1

give for $x = (x_k)_{k\in\mathbb{N}} \in \mathcal{D}(W)$ *and for all* $t \in \mathbb{R}^+$:

$$\begin{aligned} T(t)(x) &= e^{\omega t}\sum_{n=0}^{+\infty}(\omega I - W)^n((\omega+1)I - W)^{-n-\alpha-1}(x)\phi_{n,\alpha}(t) \\ &= e^{0\times t}\sum_{n=0}^{+\infty}(0\times I - W)^n((0+1)I - W)^{-n-\alpha-1}(x)\phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty}(k^{2n}(1+k^2)^{-n-\alpha-1}x_k)_{k\in\mathbb{N}}\phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}k^{2n}(1+k^2)^{-n-\alpha-1}x_k e_k\phi_{n,\alpha}(t). \end{aligned}$$

And Theorem 2.2 gives

$$R_C(t,W)x = \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\left(\int_0^{+\infty}\frac{s^n}{(s+1)^{n+\alpha+1}}e^{-k^2 s}ds\right)x_k e_k\phi_{n,\alpha}(t).$$

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