

## Laguerre Expansions of $C$ -regularized semigroups Functions.

*Y. Bajjou, A. Blali, A. El Amrani*

### Abstract

The aim of this paper is to approximate the exponentially bounded  $C$ -regularized semigroups function by the Laguerre series, recalling the notions and the results used.

**Keywords:** Laguerre functions,  $C$ -regularized semigroup,  $C_0$ -semigroup.

*MSC2010 : 47D03, 47D60 .*

## 1 Introduction and preliminaries

The series expansion of Laguerre orthogonal polynomials have been an important tool in mathematical physics , in problems involving the integration of Helmholtz's equation in prabolic coordinates, in the theory of the Hydrogen atom, in the theory of propagation of electromagnetic waves a long transmission lines terminated by a lumped inductance [8]. The study of sufficient conditions for the convergence of Laguerre series has been the subject of numerous works, for more details see [14], [3], [13], [8] and [2].

In 2014, Abadias and Miana studied in their article [1], the Laguerre expansion of  $C_0$ -semigroups and Resolvent Operators. In this work we will be interested in Laguerre expansion of  $C$ -regularized semigroups function, starting with reminding the notations, concepts and results used.

Throughout this paper  $E$  denotes a non-trivial complex Banach space,  $\mathfrak{F}(E, F)$  denotes the set of all applications from  $E$  to another Banach space  $F$ ,  $B(E)$  denotes the space of all bounded linear operators from  $E$  into itself, and  $L_{loc}^1(E)$  the set of all  $f \in \mathfrak{F}(\mathbb{R}, E)$  locally integrable. For a closed linear operator  $A$  on  $E$ ,  $\mathcal{D}(A)$ ,  $R(A)$  and  $\rho(A)$  denote its domain, range and resolvent set, respectively.  $\mathcal{D}(A)$  equipped with the graph norm  $\|x\|_{\mathcal{D}(A)} = \|x\|_E + \|Ax\|_E$  become Banach space. Throughout this paper,  $C \in B(E)$  will be an injective operator. The  $C$ -resolvent set of  $A$ , denoted by  $\rho_C(A)$ , is defined by  $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \text{ and } \lambda I - A \text{ is injective}\}$  and if  $\lambda \in \rho_C(A)$  then we denoted by  $R_C(\lambda, A) = (\lambda I - A)^{-1}C$  the  $C$ -resolvent.

### Laguerre functions and Laguerre expansions on Banach spaces

For all  $n \in \mathbb{N}$ , and arbitrary real  $\alpha > -1$  the classical Laguerre polynomial, is defined by Rodrigues formula:

$$(\forall x \in \mathbb{R}) \quad \phi_{n,\alpha}(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}).$$

$\phi_{n,\alpha}$  is a polynomial with the degree  $n$ , the same parity as  $n$ , whose highest monomial degree is  $\frac{(-1)^n}{n!}x^n$  and have real coefficients. Furthermore, they verify the following condition of orthogonality with respect to gamma density  $x \mapsto x^\alpha e^{-x}$  on  $[0, +\infty[$  :

$$\int_{\mathbb{R}^+} \phi_{n,\alpha}(x) \phi_{m,\alpha}(x) x^\alpha e^{-x} dx = \delta_{n,m} \frac{\Gamma(n + \alpha + 1)}{n!},$$

where  $\delta_{n,m}$  is the Kronecker delta. We also have recurrence relations, differential equations and the estimates:

$$(\forall x \in \mathbb{R}^+) (\exists c_x > 0) (\exists n_0 \in \mathbb{N}) (\forall n \geq n_0), |\phi_{n,\alpha}(x)| \leq c_x n^{\frac{\alpha}{2}}. \quad (1)$$

For more details of the classical theory of orthogonal polynomials see [8], [14], [3], [13], [1] and [2].

The Laguerre functions on  $\mathbb{R}^+$  are defined by:

$$\varphi_{n,\alpha}(x) = \sqrt{\frac{n!}{\Gamma(n + \alpha + 1)}} \phi_{n,\alpha}(x) x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} = \frac{1}{\sqrt{n! \Gamma(n + \alpha + 1)}} x^{\frac{-\alpha}{2}} e^{\frac{x}{2}} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}). \quad (2)$$

$(\varphi_{n,\alpha})_{n \in \mathbb{N}}$  is an orthonormal basis in the Hilbert space  $L^2(\mathbb{R}^+)$  and satisfied some recurrence relations, equality and inequality. For more details see [14], [3], [1], [13] and [2].

For  $n \in \mathbb{N}$  and arbitrary real  $\alpha > -1$ , we denote by  $\psi_{n,\alpha}$ , the function on  $\mathbb{R}^+$  defined by :

$$(\forall x \in \mathbb{R}^+) \psi_{n,\alpha}(x) = \frac{n!}{\Gamma(n + \alpha + 1)} x^\alpha e^{-x} \phi_{n,\alpha}(x) = \frac{1}{\Gamma(n + \alpha + 1)} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}). \quad (3)$$

$\psi_{n,\alpha}$  satisfies recurrence relations and differential equations, for example:

$$(\forall (n, m) \in \mathbb{N}^2) \psi_{n,\alpha}^{(m)} = \psi_{n+m,\alpha-m}. \quad (4)$$

And the following useful inequality :

$$(\forall n \geq 1) \|\psi_{n,\alpha}\|_1 \leq \frac{C_\alpha}{n^{\frac{\alpha}{2}}}. \quad (5)$$

For more details see [1] and [8].

The most important properties of the family  $(\psi_{n,\alpha})_{n \in \mathbb{N}}$  is that if  $f : \mathbb{R}_+^* \rightarrow E$  be a differentiable function such that  $\int_0^{+\infty} e^{-t} t^\alpha \|f(t)\|^2 dt < +\infty$ , then the series  $\sum_{n \in \mathbb{N}} c_n(f) \phi_{n,\alpha}(t)$  converges pointwise to  $f$  on  $\mathbb{R}_+^*$ , where

$$c_n(f) = \int_0^{+\infty} \psi_{n,\alpha}(t) f(t) dt.$$

For more details see [8], [3] and [13].

### C-regularized semigroup

A family of operators  $(T(t))_{t \geq 0}$  in  $B(E)$  is called exponentially bounded C-regularized semigroup or exponentially bounded C-semigroup on  $E$ , if

1.  $T(t+s)C = T(t)T(s)$  for all  $t, s \in \mathbb{R}^+$ .
2.  $T(0) = C$ .
3. The function  $t \mapsto T(t)x$  is continuous on  $\mathbb{R}^+$  for any  $x \in E$ .
4.  $(\exists M \geq 0) (\exists \omega \geq 0) : \|T(t)\| \leq Me^{\omega t}$ , for all  $t \geq 0$  (exponentially bounded condition).

Its generator  $W$  is defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s \rightarrow 0^+} \frac{T(s)x - Cx}{s} \text{ exists in } R(C) \right\}$$

and

$$(\forall x \in \mathcal{D}(W)) \quad Wx = C^{-1} \lim_{s \rightarrow 0^+} \frac{T(s)x - Cx}{s}.$$

In all that follows,  $(T(t))_{t \geq 0} \subset B(E)$  is exponentially bounded C-regularized semigroup on  $E$  with generator  $(W, \mathcal{D}(W))$  such that

$$(\exists M > 0) (\exists \omega \geq 0) : (\forall t \geq 0) \|T(t)\| \leq Me^{\omega t}. \quad (6)$$

We present some known facts about C-semigroup and its generator, which will be used in the sequel (see [7], [5], [6], [7], [9], [10], [11], [12], [15] and [16] for more details):

- By the property (1), we conclude that  $T(t)T(s) = T(s)T(t)$  for all  $t, s \geq 0$ , this means that  $T(t)x \in \mathcal{D}(W)$  and  $WT(t)x = T(t)Wx$ , for all  $t \geq 0$  and  $x \in \mathcal{D}(W)$ .
- $\int_0^t T(s)x ds \in \mathcal{D}(W)$  and  $W \int_0^t T(s)x ds = T(t)x - Cx$  for every  $x \in E$  and  $t \geq 0$ , which implies that for each  $x \in \mathcal{D}(W)$ ,  $u := T(\cdot)x$  is of class  $C^1(\mathbb{R}^+, E)$  and solves the Abstract Cauchy problem

$$((ACP(W, x, 0))_1) \begin{cases} u'(t) = Wu(t), & t \in \mathbb{R}^+ \\ u(0) = Cx. \end{cases}$$

- $\lim_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$  for all  $x \in E$  and  $t \geq 0$ , this means that  $W$  is closed linear operator with  $R(C) \subset \overline{D(W)}$ .
- The C-resolvent operator  $R_C(\lambda, W)$  is analytic in the C-resolvent set  $\rho_C(W)$  and

$$\frac{d^n}{d\lambda^n} (R_C(\lambda, W)) = (-1)^n n! (\lambda I - W)^{-n-1} C \text{ for all } n \in \mathbb{N}. \quad (7)$$

- $W = C^{-1}WC$ ,  $(\omega, +\infty) \subset \rho_C(W)$  and  $R_C(\lambda, W)x = \int_0^{+\infty} e^{-\lambda t} T(t)x dt$  for  $\lambda > \omega$  and  $x \in E$ . For every  $\lambda > \omega$  and  $n \in \mathbb{N}$ ,  $R(C) \subset D((\lambda I - W)^{-n})$  and

$$(\lambda I - W)^{-n} Cx = \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} T(t)x dt, \quad (8)$$

which implies

$$\|(\lambda - \omega)^n (\lambda I - W)^{-n} C\| \leq M. \quad (9)$$

- Let  $\alpha > 0$  and  $\lambda > 0$ . Like  $\int_0^{+\infty} t^{\alpha-1} e^{-\lambda t} dt = \frac{\Gamma(\alpha)}{\lambda^\alpha}$  and the family of operators  $(e^{-\omega t} T(t))_{t \geq 0}$  is uniformly bounded  $C$ -semigroup with the generator  $W - \omega I$ . Then we can define the fractional power of  $C$ -resolvent operator (see [5] for more details) as below :

$$(\lambda I - (W - \omega I))^{-\alpha} Cx = ((\lambda + \omega)I - W)^{-\alpha} Cx := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t} e^{-\omega t} T(t)x dt, \text{ for all } x \in E. \quad (10)$$

With a simple verification,  $(\lambda I - (W - \omega I))^{-\alpha} C$  is bounded linear operator.

## 2 Main results

**Theorem 2.1** *Let  $(T(t))_{t \in \mathbb{R}^+}$  be an exponentially bounded  $C$ -semigroup in Banach space  $E$  with generator  $(W, \mathcal{D}(W))$ ,  $q \in \mathbb{N}$  such that  $q > 2$  and  $\alpha > -1$ , then*

1. *For any  $n \in \mathbb{N}$  and  $x \in E$  we have*

$$\int_0^{+\infty} \psi_{n,\alpha}(t) e^{-\omega t} T(t)x dt = (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx.$$

2. *For  $x \in \mathcal{D}(W)$  we have :*

- (a)  $T(t)x = e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} \phi_{n,\alpha}(t) Cx$ , for all  $t > 0$ .
- (b) For each  $t > 0$  there is  $n_0 \in \mathbb{N}$  such that for all integer  $n$  with  $n \geq n_0$  and all  $x \in \mathcal{D}(W^q)$ , we have

$$\|T(t)x - e^{\omega t} \sum_{m=0}^n (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m,\alpha}(t)\| \leq \frac{c_{t,\alpha,q} \| (W - \omega I)^q x \|}{n^{\frac{q}{2}-1}}.$$

Where  $c_{t,\alpha,q}$  is a constant which depends only on  $t > 0$ ,  $\alpha$  and  $q$ .

Proof.

Throughout the proof,  $\alpha$  is an arbitrary real such that  $\alpha > -1$ .

1. Let  $n \in \mathbb{N}$ ,  $x \in E$ . The function  $t \mapsto \psi_{n,\alpha}(t)T(t)x$  is continuous on  $\mathbb{R}^+$  and integrable in the sens of Bochner, if we posed  $I := \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt$  then

$$\begin{aligned}
I &:= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt \\
&= \frac{1}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} \frac{d^n}{dt^n} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x dt \\
&= \frac{1}{\Gamma(n + \alpha + 1)} \left( \left[ \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x \right]_0^{+\infty} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t}) \frac{d}{dt} (e^{-\omega t}T(t))x dt \right) \\
&= \frac{1}{\Gamma(n + \alpha + 1)} \left( 0 - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t})(W - \omega I)(e^{-\omega t}T(t))x dt \right) \\
&= \frac{-1}{\Gamma(n + \alpha + 1)} \left( (W - \omega I) \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x dt \right) \\
&= (-1)^n (W - \omega I)^n \left( \frac{1}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x dt \right) \\
&= (\omega I - W)^n \left( \frac{1}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} (t^{(n+\alpha+1)-1} e^{-t})(e^{-\omega t}T(t))x dt \right) \\
&= (\omega I - W)^n (I - (W - \omega I))^{-n-\alpha-1} Cx \\
&= (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx
\end{aligned}$$

2. • (a) Let  $x \in \mathcal{D}(W)$ . Let's remember that  $e^{-\omega(\cdot)}T(\cdot)x : \mathbb{R}^+ \rightarrow E$  is in  $C^1(\mathbb{R}^+, E)$  and like

$$\begin{aligned}
\int_0^{+\infty} e^{-t} t^\alpha \| e^{-\omega t} T(t)x \|^2 dt &\leq \int_0^{+\infty} e^{-t} t^\alpha \| e^{-\omega t} T(t) \|^2 \| x \|^2 dt \\
&\leq M^2 \| x \|^2 \int_0^{+\infty} e^{-t} t^\alpha dt \\
&< +\infty,
\end{aligned}$$

then, we apply the most important properties of  $(\psi_{n,\alpha})_{n \in \mathbb{N}}$  to get the series  $\sum_{n \in \mathbb{N}} c_n(e^{-\omega(\cdot)}T(\cdot)x)\phi_{n,\alpha}(\cdot)$  with

$$\begin{aligned}
c_n((e^{-\omega(\cdot)}T(\cdot))x) &= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt \\
&= (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx
\end{aligned}$$

converges pointwise to  $e^{-\omega(\cdot)}T(\cdot)x$  for  $t \in \mathbb{R}^+$ , it is

$$\begin{aligned} T(t)x &= e^{\omega t}(e^{-\omega t}T(t)x) \\ &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx \phi_{n,\alpha}(t). \end{aligned}$$

- (b) Let  $t \geq 0$  and  $q \in \mathbb{N}$  such that  $q > 2$ .

Like  $\lim_{n \rightarrow +\infty} \left(\frac{n}{n-q}\right)^{\frac{\alpha}{2}} = 1$  then

$$(\exists N_1 \in \mathbb{N}) (\forall n \geq \max(q+1, N_1)), \left(\frac{n}{n-q}\right)^{\frac{\alpha}{2}} \leq 2.$$

The Riemann serie  $\sum_{m \geq q+1} \frac{1}{(m-q)^{\frac{q}{2}}}$  is convergent then

$$\sum_{m=n+1}^{+\infty} \frac{1}{(m-q)^{\frac{q}{2}}} \sim_{n \rightarrow +\infty} \frac{1}{(\frac{q}{2}-1)} \frac{1}{(n-q)^{\frac{q}{2}-1}}$$

but

$$\frac{1}{(\frac{q}{2}-1)} \frac{1}{(n-q)^{\frac{q}{2}-1}} \sim_{n \rightarrow +\infty} \frac{1}{(\frac{q}{2}-1)} \frac{1}{n^{\frac{q}{2}-1}}$$

thus

$$(\exists N_2 \in \mathbb{N}) (\forall n \geq \max(q+1, N_1, N_2)), \sum_{m=n+1}^{+\infty} \frac{1}{m^{\frac{q}{2}}} \leq \frac{1}{(\frac{q}{2}-1)} \frac{1}{n^{\frac{q}{2}-1}}.$$

By the inequation estimates (1),

$$(\exists N_3 \in \mathbb{N}) (\exists c_t > 0) : (\forall n \geq \max(N_1, N_2, N_3, q+1)) \mid \phi_{n,\alpha}(t) \mid \leq c_t n^{\frac{\alpha}{2}}.$$

If we posed  $n_0 = \max(N_1, N_2, N_3, q+1)$ , then for  $n \geq n_0$  and  $x \in D(W^q)$

$$\begin{aligned} J &:= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt \\ &= \int_0^{+\infty} \frac{d^q}{dt^q}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))x dt \quad (\text{according to (4)}) \\ &= \left[ \frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))x \right]_0^{+\infty} - \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t)) \frac{d}{dt}(e^{-\omega t}T(t))x dt \end{aligned}$$

$$\begin{aligned}
&= 0 - \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}} (\psi_{n-q, \alpha+q}(t)(W - \omega I)(e^{-\omega t} T(t))x) dt \\
&= -(W - \omega I) \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}} \psi_{n-q, \alpha+q}(t)(e^{-\omega t} T(t))x dt \\
&= (\omega I - W) \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}} \psi_{n-q, \alpha+q}(t)(e^{-\omega t} T(t))x dt \\
&= (\omega I - W)^q \int_0^{+\infty} \psi_{n-q, \alpha+q}(t)(e^{-\omega t} T(t))x dt \\
&= \int_0^{+\infty} \psi_{n-q, \alpha+q}(t)(e^{-\omega t} T(t))(\omega I - W)^q x dt.
\end{aligned}$$

So

$$\begin{aligned}
\| (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx \| &= \left\| \int_0^{+\infty} \psi_{n, \alpha}(t)(e^{-\omega t} T(t))x dt \right\| \\
&= \left\| \int_0^{+\infty} (\psi_{n-q, \alpha+q}(t)(e^{-\omega t} T(t))(W - \omega I)^q x) dt \right\| \\
&\leq \int_0^{+\infty} | \psi_{n-q, \alpha+q}(t) | \| e^{-\omega t} T(t) \| dt \| (W - \omega I)^q x \| \\
&\leq \| (W - \omega I)^q x \| M \| \psi_{n-q, \alpha+q} \|_1 \\
&\leq \| (W - \omega I)^q x \| \frac{C_{\alpha+q}}{(n-q)^{\frac{\alpha+q}{2}}} \text{ (according to (5)).}
\end{aligned}$$

On the other hand, we are looking to increase the quantity

$$e^{\omega t} \left\| \sum_{m=0}^{+\infty} (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m, \alpha}(t) - \sum_{m=0}^n (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m, \alpha}(t) \right\| \quad (11)$$

We have

$$\begin{aligned}
(11) &\leq \sum_{m=n+1}^{+\infty} \| e^{\omega t} (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m, \alpha}(t) \| \\
&\leq e^{\omega t} \sum_{m=n+1}^{+\infty} \| (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \| | \phi_{m, \alpha}(t) | \\
&\leq e^{\omega t} \sum_{m=n+1}^{+\infty} \| (W - \omega I)^q x \| \frac{C_{\alpha+q}}{(m-q)^{\frac{\alpha+q}{2}}} c_t m^{\frac{\alpha}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \| (W - \omega I)^q x \| e^{\omega t} c_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha}{2}}} \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \left(\frac{m}{m-q}\right)^{\frac{\alpha}{2}} \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{1}{(m-q)^{\frac{q}{2}}} \times 2 \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \frac{1}{(\frac{q}{2}-1)} \frac{1}{n^{\frac{q}{2}-1}}
\end{aligned}$$

where

$$\| T(t)x - e^{\omega t} \sum_{m=0}^n (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m,\alpha}(t) \| \leq \frac{c_{t,\alpha,q} \| (W - \omega I)^q x \|}{n^{\frac{q}{2}-1}}.$$

**Theorem 2.2** Let  $(T(t))_{t \in \mathbb{R}^+}$  be an exponentially bounded  $C$ -semigroup in Banach space  $E$  with generator  $(W, \mathcal{D}(W))$ ,  $\alpha \in \mathbb{R}$  such that  $\alpha > 0$ .

1. For  $x \in E$  and  $n \in \mathbb{N}$ , we have

$$\int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I) x dt = \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds.$$

2. For  $x \in E$  we have

$$R_C(t, W - \omega I) x = \sum_{n=0}^{+\infty} \left( \int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s) x ds \right) \phi_{n,\alpha}(t), \text{ for all } t > 0.$$

Proof.

1. For all  $t > 0$ ,  $t + \omega \in \rho_C(W)$ , then the function  $t \mapsto R_C(t, W - \omega I) = R_C(t + \omega, W)$  is analytic in  $\mathbb{R}_+^*$ .

In the other hand by equation (9), we know that  $\| R_C(t + \omega, W) \| \leq \frac{M}{t}$  for all  $t > 0$ , so

$$\begin{aligned} \int_0^{+\infty} \| \psi_{n,\alpha}(t) R_C(t, W - \omega I) x \| dt &= \int_0^{+\infty} | \psi_{n,\alpha}(t) | \| (R_C(t, W - \omega I) x \| dt \\ &\leq \int_0^{+\infty} \frac{n!M}{\Gamma(n + \alpha + 1)t} e^{-t} t^\alpha | \phi_{n,\alpha}(t) | dt \\ &\leq \frac{n!M}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} e^{-t} t^{\alpha-1} | \phi_{n,\alpha}(t) | dt \\ &< +\infty. \end{aligned}$$

If we posed

$$H := \Gamma(n + \alpha + 1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I) x dt,$$

then

$$\begin{aligned} H &= \Gamma(n + \alpha + 1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I) x dt \\ &= \int_0^{+\infty} \frac{d^n}{dt^n} (e^{-t} t^{n+\alpha}) R_C(t, W - \omega I) x dt \\ &= \underbrace{\left[ \frac{d^{n-1}}{dt^{n-1}} (e^{-t} t^{n+\alpha}) R_C(t, W - \omega I) x \right]_0^{+\infty}}_{=0} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (e^{-t} t^{n+\alpha}) \frac{d}{dt} (R_C(t, W - \omega I) x) dt \\ &= \left[ -\frac{d^{n-2}}{dt^{n-2}} (e^{-t} t^{n+\alpha}) \frac{d}{dt} (R_C(t, W - \omega I) x) \right]_0^{+\infty} \\ &\quad + (-1)^2 \int_0^{+\infty} \frac{d^{n-2}}{dt^{n-2}} (e^{-t} t^{n+\alpha}) \frac{d^2}{dt^2} (R_C(t, W - \omega I) x) dt \\ &= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} \frac{d^n}{dt^n} (R_C(t, W - \omega I) x) dt \quad (\text{integration by parts}) \\ &= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} (-1)^n n! ((t + \omega)I - W)^{-n-1} C x dt \\ &= \int_0^{+\infty} e^{-t} t^{n+\alpha} (n!) \frac{1}{n!} \int_0^{+\infty} s^n e^{-ts} e^{-\omega s} T(s) x ds dt \\ &= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left( \int_0^{+\infty} t^{n+\alpha} e^{-t} e^{-ts} dt \right) ds \quad (\text{Fubini's Theorem}) \end{aligned}$$

$$\begin{aligned}
&= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left( \int_0^{+\infty} t^{n+\alpha} e^{-(s+1)t} dt \right) ds \\
&= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left( \int_0^{+\infty} \frac{u^{n+\alpha}}{(s+1)^{n+\alpha}} e^{-u} \frac{du}{s+1} \right) ds \quad (u = (s+1)t) \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x \left( \int_0^{+\infty} u^{n+\alpha} e^{-u} du \right) ds \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x \Gamma(n+\alpha+1) ds \\
&= \Gamma(n+\alpha+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds,
\end{aligned}$$

hence the result.

2. Let  $x \in E$ .

The function  $t \mapsto R_C(t, W - \omega I)x = R_C(t + \omega, W)x$  is differentiable in  $\mathbb{R}_+^*$  (because it's analytic in  $\mathbb{R}_+^*$ ), and

$$\begin{aligned}
\int_0^{+\infty} t^\alpha e^{-t} \| R_C(t, W - \omega I)x \|^2 dt &= \int_0^{+\infty} t^\alpha e^{-t} \| ((t + \omega)I - W)^{-1} Cx \|^2 dt \\
&\leq \int_0^{+\infty} t^\alpha e^{-t} \frac{M^2}{t^2} \| x \|^2 dt \quad (\text{equation (9)}) \\
&\leq M^2 \| x \|^2 \int_0^{+\infty} t^{\alpha-2} e^{-t} dt \\
&= M^2 \| x \|^2 \Gamma(\alpha - 1) \quad (\text{because } \alpha > 1) \\
&< +\infty.
\end{aligned}$$

Thus, the series  $\sum_{n \in \mathbb{N}} c_n(R_C(\cdot, W - \omega I)x) \phi_{n,\alpha}$  converges pointwise to  $R_C(\cdot, W - \omega I)x$  on  $\mathbb{R}_+^*$ , where

$$\begin{aligned}
c_n(R_C(\cdot, W - \omega I)x) &= \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I)x dt \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds \quad (\text{according to (a) of theorem 2.2}).
\end{aligned}$$

Therefore

$$(\forall t > 0) \quad R_C(t, W - \omega I)x = ((t + \omega)I - W)^{-1} Cx = \sum_{n=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s) x ds \right\} \phi_{n,\alpha}(t).$$

**Example 2.3** Let  $m : \mathbb{R} \rightarrow \mathbb{R}^-$  be an even measurable function. In the Banach space  $L^1(\mathbb{R})$ , we consider the family  $T := (T(t))_{t \geq 0} \subset \mathfrak{F}(L^1(\mathbb{R}))$  defined by  $\forall t \geq 0, T(t) : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}), f \mapsto T(t)(f) : \mathbb{R} \rightarrow \mathbb{R}, s \mapsto T(t)(f)(s) = e^{t.m(s)} f(-s)$ . Clearly,  $(T(t))_{t \geq 0} \subset B(L^1(\mathbb{R}))$ . If we put,  $T(0) = C$ , then, A family of operators  $(T(t))_{t \geq 0}$  is uniformly bounded  $C$ -regularized semigroup with generator  $(W, \mathcal{D}(W))$  defined by

$$W : \mathcal{D}(W) = \{f \in L^1(\mathbb{R}) / m.f \in L^1(\mathbb{R})\} \rightarrow L^1(\mathbb{R}), f \mapsto W(f) = m.f.$$

The theorem 2.1 give for  $f \in \mathcal{D}(W)$  and  $s, t \in \mathbb{R}^+$  :

$$\begin{aligned} T(t)(f)(s) &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t) \\ &= e^{0 \times t} \sum_{n=0}^{+\infty} (0 \times I - W)^n ((0 + 1)I - W)^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} (-m(s))^n (1 - m(s))^{-n-\alpha-1} C(f)(s) \phi_{n,\alpha}(t). \end{aligned}$$

So  $T(t)(f) = \sum_{n=0}^{+\infty} \phi_{n,\alpha}(t) (-m)^n (1 - m)^{-n-\alpha-1} C(f)$ .

And Theorem 2.2 gives

$$R_C(t, W)(f)(.) = \sum_{n=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{sm(.)} ds \right\} C(f)(.) \phi_{n,\alpha}(t).$$

**Example 2.4** The space  $X = c_0 = \left\{ (x_k)_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \text{ tq } \lim_{k \rightarrow +\infty} x_k = 0 \right\}$ , equipped with the norm  $\| (x_k)_{k \in \mathbb{N}} \|_{\infty} = \max_{k \in \mathbb{N}} |x_k|$  becomes a Banach space. For each  $n \in \mathbb{N}$  let  $e_n = (\delta_{n,k})_{k \in \mathbb{N}}$  be element of

$X$ . Like for all  $x = (x_k)_{k \in \mathbb{N}} \in X$ ,  $x = \sum_{k=0}^{+\infty} x_k e_k$  then  $X = \text{span}\{e_n / n \in \mathbb{N}\}$ . Considering the family of operators  $(T(t))_{t \geq 0}$  defined by :

$$\text{for all } t \in \mathbb{R}^+ \text{ for all } x \left( = \sum_{k=0}^{+\infty} x_k e_k \right) \in X, T(t)x = \sum_{k=0}^{+\infty} e^{-k^2 t} x_k e_k$$

$(T(t))_{t \geq 0}$  is a uniformly bounded  $C_0$ -semigroup ( $\|T(t)\| \leq 1$ ) with generator  $(W, \mathcal{D}(W))$  such that  $\mathcal{D}(W) = \{x = (x_k)_{k \in \mathbb{N}} \in X / (k^2 x_k)_{k \in \mathbb{N}} \in X\}$  and  $(\forall x \in \mathcal{D}(W)) Wx = \sum_{k=0}^{+\infty} -k^2 x_k e_k$ . The theorem 2.1

give for  $x = (x_k)_{k \in \mathbb{N}} \in \mathcal{D}(W)$  and for all  $t \in \mathbb{R}^+$  :

$$\begin{aligned}
 T(t)(x) &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} (x) \phi_{n,\alpha}(t) \\
 &= e^{0 \times t} \sum_{n=0}^{+\infty} (0 \times I - W)^n ((0 + 1)I - W)^{-n-\alpha-1} (x) \phi_{n,\alpha}(t) \\
 &= \sum_{n=0}^{+\infty} (k^{2n} (1 + k^2)^{-n-\alpha-1} x_k)_{k \in \mathbb{N}} \phi_{n,\alpha}(t) \\
 &= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} k^{2n} (1 + k^2)^{-n-\alpha-1} x_k e_k \phi_{n,\alpha}(t).
 \end{aligned}$$

And Theorem 2.2 gives

$$R_C(t, W)x = \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \left( \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-k^2 s} ds \right) x_k e_k \phi_{n,\alpha}(t).$$

**Acknowledgments** The authors are greatly indebted to the editor and the referee for many valuable comments and suggestions which improved the initial draft of this paper.

## References

- [1] L. ABADIAS AND P. J. MIANA, *C<sub>0</sub>- semigroups and resolvent operators approximated by Laguerre expansions*, Journal of Approximation Theory 213 (2017), 1-22.
- [2] R. ASKEY, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. 87 (1965), 695-708.
- [3] R. DELAUBENFELS, *C-semigroups and the Cauchy problem*, J. Funct. Anal. 111 (1993), 44-61.
- [4] R. DELAUBENFELS, *Existence families, functional calculi and evolution equations*, Lecture Notes in Math, 1570, Springer-Verlag, 1994.
- [5] R. O. FATTORINI, *A note on fractional derivatives of semigroups and cosine functions*, Pacific journal of mathematics, Vol. 109, No 2, 1983.
- [6] M. KALTENBACK, F. L. SCHWENNINGER, *Generalisations of Semigroups of Operators in the View of Linear Relations*, Analysis und Scientific Computing der Technischen Universitat Wien, Matr. Nr. 0625539 Wahringer Strasse 125/15-1180 Wien. Vienna University of Technology.

- [7] M. KOSTIC,  *$(a, k)$ -Regularized  $C$ -Resolvent Families: Regularity and Local Properties*, Hindawi Publishing Corporation Abstract and Applied Analysis, Volume 2009, Article ID 858242, 27 pages doi:10. 1155/2009/858242.
- [8] N. N. LEBEDEV, *Special function and their applications*, Prentice-hall, INC, 1965.
- [9] M. LI, Q. ZHENG, J. ZHANG, *Regulrized resolvent famillies*, Taiwanese Journal of Mathematics, vol.11, no.1 (2007), 117-133.
- [10] C. MARTINEZ, M. SANZ, L. MARCO, *Fractional powers of operators*, J. Math. Soc. Japan, Vol. 40, No. 2, 1988.
- [11] M. MOSALLANEZHAD, M. JANFADA, *On Mixed  $C$ -Semigroups of Operators on Banach Spaces*, Filomat 30:10 (2016), 2673-2682.
- [12] P. PREDA, A. POGAN, C. PREDA, *The Perron Problem For  $C$ -semigroups*, Math. J. Okayama Univ. 46 (2004), 141-151.
- [13] J. A. SHOHAT, *On the developement of functions in series of orthogonal polynomials*, The university of Pennsyvania, 1935.
- [14] G. SZEGO, *Orthogonal polynomials*, PubLished by the American Mathematical Society Providence, Rhode Island, 1939.
- [15] Q. ZHENG, *Controllability of a class of linear systems in Banach spaces*, Proc. Amer. Math. Soc. 123 (1995), 1241-1251. MR 95e:93011. Analysis und Scientific Computing der Technischen Universitat Wien.
- [16] Q. ZHENG, Y. ZHAO, *An analyticity criterion for regularezed semigroups*, Proceedings of the american Mathematical society, Volume 130, Number 8, S0002-9939(02)06219-6 (2002).

Abdelkhalek El Amrani  
Sidi Mohamed Ben Abdellah University  
Dhar El mahraz Faculty of Science  
Department of mathematics and computer science,  
Fes, Morocco  
e-mail: abdelkhalek.elamrani@usmba.ac.ma

Youssef Bajjou  
Sidi Mohamed Ben Abdellah University  
Dhar El mahraz Faculty of Science  
Department of mathematics and computer science,  
Fes, Morocco  
e-mail: youssefbajjou2017@gmail.com

Aziz Blali  
Sidi Mohamed Ben Abdellah University  
Higher normal school  
Department of mathematics,  
Fes, Morocco  
e-mail: aziz.blali@usmba.ac.ma