

Laguerre Expansions of C -regularized semigroups Functions.

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Abstract

The aim of this paper is to approximate the exponentially bounded C -regularized semigroups function by the Laguerre series, recalling the notions and the results used.

Keywords: Laguerre functions, C -regularized semigroup, C_0 -semigroup.

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1 Introduction and preliminaries

The series expansion of Laguerre orthogonal polynomials have been an important tool in mathematical physics , in problems involving the integration of Helmholtz's equation in parabolic coordinates, in the theory of the Hydrogen atom, in the theory of propagation of electromagnetic waves a long transmission lines terminated by a lumped inductance [8]. The study of sufficient conditions for the convergence of Laguerre series has been the subject of numerous works, for more details see [14], [3], [13], [8] and [2].

In 2014, Luciano and Pedro studied in their article [1], the Laguerre expansion of C_0 -semigroups and Resolvent Operators. in this work we will be interested in Laguerre expansion of C -regularized semigroups function, starting with reminding the notations, concepts and results used.

Throughout this paper E denotes a non-trivial complex Banach space, $\mathfrak{F}(E, F)$ denotes the set of all applications from E to another Banach space F , $B(E)$ denotes the space of all bounded linear operators from E into itself, and $L_{loc}^1(E)$ the set of all $f \in \mathfrak{F}(\mathbb{R}, E)$ locally integrable. For a closed linear operator A on E , $\mathcal{D}(A)$, $R(A)$ and $\rho(A)$ denote its domain, range and resolvent set, respectively. $\mathcal{D}(A)$ equipped with the graph norm $\|x\|_{\mathcal{D}(A)} = \|x\|_E + \|Ax\|_E$ become Banach space. Throughout this paper, $C \in B(E)$ will be an injective operator. The C -resolvent set of A , denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \text{ and } \lambda I - A \text{ is injective in } B(E)\}$ and if $\lambda \in \rho_C(A)$ then we denoted by $R_C(\lambda, A) = (\lambda I - A)^{-1}C$ the C -resolvent.

Laguerre functions and Laguerre expansions on Banach spaces

For all $n \in \mathbb{N}$, and arbitrary real $\alpha > -1$ the classical Laguerre polynomial, is defined by Rodrigues formula:

$$(\forall x \in \mathbb{R}) \quad \phi_{n,\alpha}(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}).$$

$\phi_{n,\alpha}$ is a polynomial with the degree n , the same parity as n , whose highest monomial degree is $\frac{(-1)^n}{n!} X^n$ and have real coefficients. Furthermore, they verify the following condition of orthogonality with respect to gamma density $x \rightarrow x^\alpha e^{-x}$ on $[0, +\infty[$:

$$\int_{\mathbb{R}^+} \phi_{n,\alpha}(x) \phi_{m,\alpha}(x) x^\alpha e^{-x} dx = \delta_{n,m} \frac{\Gamma(n + \alpha + 1)}{n!},$$

where $\delta_{n,m}$ is the Kronecker delta. We also have recurrence relations, differential equations and the estimates:

$$(\forall x \in \mathbb{R}^+) (\exists c_x > 0) (\exists n_0 \in \mathbb{N}) (\forall n \geq n_0), |\phi_{n,\alpha}(x)| \leq c_x n^{\frac{\alpha}{2}}. \quad (1)$$

For more details of the classical theory of orthogonal polynomials see [8], [14], [3], [13], [1] and [2].

The Laguerre functions on \mathbb{R}^+ are defined by:

$$\varphi_{n,\alpha}(x) = \sqrt{\frac{n!}{\Gamma(n + \alpha + 1)}} \phi_{n,\alpha}(x) x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} = \frac{1}{\sqrt{n! \Gamma(n + \alpha + 1)}} x^{\frac{-\alpha}{2}} e^{\frac{x}{2}} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}). \quad (2)$$

$(\varphi_{n,\alpha})_{n \in \mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^2(\mathbb{R}^+)$ and satisfied some recurrence relations, equality and inequality. For more details see [14], [3], [1], [13] and [2].

For $n \in \mathbb{N}$ and arbitrary real $\alpha > -1$, we denote by $\psi_{n,\alpha}$, the function on \mathbb{R}^+ defined by :

$$(\forall x \in \mathbb{R}^+) \psi_{n,\alpha}(x) = \frac{n!}{\Gamma(n + \alpha + 1)} x^\alpha e^{-x} \phi_{n,\alpha}(x) = \frac{1}{\Gamma(n + \alpha + 1)} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}). \quad (3)$$

$\psi_{n,\alpha}$ satisfies recurrence relations and differential equations, for example:

$$(\forall (n, m) \in \mathbb{N}^2) \psi_{n,\alpha}^{(m)} = \psi_{n+m,\alpha-m}. \quad (4)$$

And the following useful inequality :

$$(\forall n \geq 1) \|\psi_{n,\alpha}\|_1 \leq \frac{C_\alpha}{n^{\frac{\alpha}{2}}}. \quad (5)$$

For more details see [1] and [8].

The most important properties of the family $(\psi_{n,\alpha})_{n \in \mathbb{N}}$ is that if $f : \mathbb{R}_+^* \rightarrow E$ be a differentiable function such that $\int_0^{+\infty} e^{-t} t^\alpha \|f(t)\|^2 dt < +\infty$, then the series $\sum_{n \in \mathbb{N}} c_n(f) \phi_{n,\alpha}(t)$ converges pointwise to f on \mathbb{R}_+^* , where

$$c_n(f) = \int_0^{+\infty} \psi_{n,\alpha}(t) f(t) dt.$$

For more details see [8], [3] and [13].

C-regularized semigroup

A family of operators $(T(t))_{t \geq 0}$ in $B(E)$ is called exponentially bounded C-regularized semigroup or exponentially bounded C-semigroup on E , if

1. $T(t+s)C = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$.
2. $T(0) = C$.
3. The function $t \mapsto T(t)x$ is continuous on \mathbb{R}^+ for any $x \in E$.
4. $(\exists M \geq 0) (\exists \omega \geq 0) : \|T(t)\| \leq Me^{\omega t}$, for all $t \geq 0$ (exponentially bounded condition).

Its generator W is defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s \rightarrow 0^+} \frac{T(s)x - Cx}{s} \text{ exists in } R(C) \right\}$$

and

$$(\forall x \in \mathcal{D}(W)) \quad Wx = C^{-1} \lim_{s \rightarrow 0^+} \frac{T(s)x - Cx}{s}.$$

In all that follows, $(T(t))_{t \geq 0} \subset B(E)$ is exponentially bounded C-regularized semigroup on E with generator $(W, \mathcal{D}(W))$ such that

$$(\exists M > 0) (\exists \omega \geq 0) : (\forall t \geq 0) \|T(t)\| \leq Me^{\omega t}. \quad (6)$$

We present some known facts about C-semigroup and its generator, which will be used in the sequel (see [7], [5], [6], [7], [9], [10], [11], [12], [15] and [16] for more details):

- By the equality (1), we conclude that $T(t)T(s) = T(s)T(t)$ for all $t, s \geq 0$, this means that $T(t)x \in \mathcal{D}(W)$ and $WT(t)x = T(t)Wx$, for all $t \geq 0$ and $x \in \mathcal{D}(W)$.
- $\int_0^t T(s)x ds \in \mathcal{D}(W)$ and $W \int_0^t T(s)x ds = T(t)x - Cx$ for every $x \in E$ and $t \geq 0$, which implies that for each $x \in \mathcal{D}(W)$, $u := T(\cdot)x$ is of class $C^1(\mathbb{R}^+, E)$ and solves the Abstract Cauchy problem

$$((ACP(W, x, 0))_1) \begin{cases} u'(t) = Wu(t), & t \in \mathbb{R}^+ \\ u(0) = Cx. \end{cases}$$

- $\lim_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$ for all $x \in E$ and $t \geq 0$, this means that W is closed linear operator with $R(C) \subset \overline{D(W)}$.

- The C -resolvent operator $R_C(\lambda, W)$ is analytic in the C -resolvent set $\rho_C(W)$ and

$$\frac{d^n}{d\lambda^n}(R_C(\lambda, W)) = (-1)^n n! (R_C(\lambda, W))^{-n-1} \text{ for all } n \in \mathbb{N}. \quad (7)$$

- $W = C^{-1}WC$, $(\omega, +\infty) \subset \rho_C(W)$ and $R_C(\lambda, W)x = \int_0^{+\infty} e^{-\lambda t} T(t)x dt$ for $\lambda > \omega$ and $x \in E$. For every $\lambda > \omega$ and $n \in \mathbb{N}$, $R_C \subset D((\lambda I - W)^{-n})$ and

$$(\lambda I - W)^{-n} Cx = \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} T(t)x dt. \quad (8)$$

which implies

$$\|(\lambda - \omega)^n (\lambda I - W)^{-n} C\| \leq M. \quad (9)$$

- Let $\alpha > 0$ and $\lambda > 0$. Like $\int_0^{+\infty} t^{\alpha-1} e^{-\lambda t} dt = \frac{\Gamma(\alpha)}{\lambda^\alpha}$ and the family of operators $(e^{-\omega t} T(t))_{t \geq 0}$ is uniformly bounded C -semigroup with the generator $W - \omega I$. Then we can define the fractional power of C -resolvent operator (see [5] for more details) as below :

$$(\lambda I - (W - \omega I))^{-\alpha} Cx = ((\lambda + \omega)I - W)^{-\alpha} Cx := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t} e^{-\omega t} T(t)x dt, \text{ for all } x \in E. \quad (10)$$

With a simple verification, $(\lambda I - (W - \omega I))^{-\alpha} C$ is bounded linear operator.

2 Main results

Theorem 2.1 *Let $(T(t))_{t \in \mathbb{R}^+}$ be an exponentially bounded C -semigroup in Banach space E with generator $(W, \mathcal{D}(W))$, $q \in \mathbb{N}$ such that $q > 2$ and $\alpha > 0$, then*

1. For any $n \in \mathbb{N}$ and $x \in E$ we have

$$\int_0^{+\infty} \psi_{n,\alpha}(t) e^{-\omega t} T(t)x dt = (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx.$$

2. For $x \in \mathcal{D}(W)$ we have :

- (a) $T(t)x = e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} \phi_{n,\alpha}(t)x$, for all $t > 0$.
- (b) For each $t > 0$ there is $n_0 \in \mathbb{N}$ such that for all integer n with $n \geq n_0$ and all $x \in \mathcal{D}(W^q)$, we have

$$\|T(t)x - \sum_{m=0}^n (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m,\alpha}(t)\| \leq \frac{c_{t,\alpha,q} \| (W - \omega I)^q x \|}{n^{\frac{q}{2}-1}}.$$

Where $c_{t,\alpha,q}$ is a constant which depends only on $t > 0$, α and q .

Proof.

Throughout the proof, α is an arbitrary real such that $\alpha > -1$.

1. Let $n \in \mathbb{N}$, $x \in E$. The function $t \mapsto \psi_{n,\alpha}(t)T(t)x$ is continuous on \mathbb{R}^+ and integrable in the sens of Bochner, if we posed $I := \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt$ then

$$\begin{aligned}
I &:= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt \\
&= \frac{1}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} \frac{d^n}{dt^n} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x dt \\
&= \frac{1}{\Gamma(n + \alpha + 1)} \left(\left[\frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x \right]_0^{+\infty} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t}) \frac{d}{dt} (e^{-\omega t}T(t))x dt \right) \\
&= \frac{1}{\Gamma(n + \alpha + 1)} \left(0 - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t})(W - \omega I)(e^{-\omega t}T(t))x dt \right) \\
&= \frac{-1}{\Gamma(n + \alpha + 1)} \left((W - \omega I) \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x dt \right) \\
&= (-1)^n (W - \omega I)^n \left(\frac{1}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} (t^{n+\alpha} e^{-t})(e^{-\omega t}T(t))x dt \right) \\
&= (\omega I - W)^n \left(\frac{1}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} (t^{(n+\alpha+1)-1} e^{-t})(e^{-\omega t}T(t))x dt \right) \\
&= (\omega I - W)^n (I - (W - \omega I))^{-n-\alpha-1} Cx \\
&= (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx
\end{aligned}$$

2. • (a) Let $x \in \mathcal{D}(W)$. Let's remember that $e^{-\omega(\cdot)}T(\cdot)x : \mathbb{R}^+ \rightarrow E$ is in $C^1(\mathbb{R}^+, E)$ and like

$$\begin{aligned}
\int_0^{+\infty} e^{-t} t^\alpha \| e^{-\omega t} T(t)x \|^2 dt &\leq \int_0^{+\infty} e^{-t} t^\alpha \| e^{-\omega t} T(t) \|^2 \| x \|^2 dt \\
&\leq M^2 \| x \|^2 \int_0^{+\infty} e^{-t} t^\alpha dt \\
&< +\infty,
\end{aligned}$$

then, we apply the most important properties of $(\psi_{n,\alpha})_{n \in \mathbb{N}}$ to get the series $\sum_{n \in \mathbb{N}} c_n(e^{-\omega(\cdot)}T(\cdot)x)\phi_{n,\alpha}(\cdot)$ with

$$\begin{aligned}
c_n((e^{-\omega(\cdot)}T(\cdot))x) &= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt \\
&= (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx
\end{aligned}$$

converges pointwise to $e^{-\omega(\cdot)}T(\cdot)x$ for $t \in \mathbb{R}^+$, it is

$$\begin{aligned} T(t)x &= e^{\omega t}(e^{-\omega t}T(t)x) \\ &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx \phi_{n,\alpha}(t) \\ &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx \phi_{n,\alpha}(t). \end{aligned}$$

- (b) Let $t \geq 0$ and $q \in \mathbb{N}$ such that $q > 2$.

Like $\lim_{n \rightarrow +\infty} \left(\frac{n}{n-q}\right)^{\frac{\alpha}{2}} = 1$ then

$$(\exists N_1 \in \mathbb{N}) (\forall n \geq \max(q+1, N_1)), \left(\frac{n}{n-q}\right)^{\frac{\alpha}{2}} \leq 2.$$

The Riemann serie $\sum_{m \geq q+1} \frac{1}{(m-q)^{\frac{q}{2}}}$ is convergent then

$$\sum_{m=n+1}^{+\infty} \frac{1}{(m-q)^{\frac{q}{2}}} \sim_{n \rightarrow +\infty} \frac{1}{\left(\frac{q}{2}-1\right)} \frac{1}{(n-q)^{\frac{q}{2}-1}}$$

but

$$\frac{1}{\left(\frac{q}{2}-1\right)} \frac{1}{(n-q)^{\frac{q}{2}-1}} \sim_{n \rightarrow +\infty} \frac{1}{\left(\frac{q}{2}-1\right)} \frac{1}{n^{\frac{q}{2}-1}}$$

thus

$$(\exists N_2 \in \mathbb{N}) (\forall n \geq \max(q+1, N_1, N_2)), \sum_{m=n+1}^{+\infty} \frac{1}{n^{\frac{q}{2}}} \leq \frac{1}{\left(\frac{q}{2}-1\right)} \frac{1}{n^{\frac{q}{2}-1}}.$$

By the inequation estimates (1),

$$(\exists N_3 \in \mathbb{N}) (\exists c_t > 0) : (\forall n \geq \max(N_1, N_2, N_3, q+1)) \mid \phi_{n,\alpha}(t) \mid \leq c_t n^{\frac{\alpha}{2}}.$$

If we posed $n_0 = \max(N_1, N_2, N_3, q+1)$, then for $n \geq n_0$ and $x \in D(W^q)$

$$\begin{aligned} J &:= \int_0^{+\infty} \psi_{n,\alpha}(t)(e^{-\omega t}T(t))x dt \\ &= \int_0^{+\infty} \frac{d^q}{dt^q}(\psi_{n-q,\alpha+q}(t))(e^{-\omega t}T(t))x dt \quad (\text{according to (4)}) \\ &= \left[\frac{d^{q-1}}{dt^{q-1}} \psi_{n-q,\alpha+q}(t)(e^{-\omega t}T(t))x \right]_0^{+\infty} - \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}}(\psi_{n-q,\alpha+q}(t))x \frac{d}{dt}((e^{-\omega t}T(t))x) dt \end{aligned}$$

$$\begin{aligned}
&= 0 - \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}} (\psi_{n-q, \alpha+q}(t) (\omega I - W) (e^{-\omega t} T(t))) x dt \\
&= -(\omega I - W) \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}} \psi_{n-q, \alpha+q}(t) (e^{-\omega t} T(t)) x dt \\
&= (W - \omega I) \int_0^{+\infty} \frac{d^{q-1}}{dt^{q-1}} \psi_{n-q, \alpha+q}(t) (e^{-\omega t} T(t)) x dt \\
&= (W - \omega I)^q \int_0^{+\infty} \psi_{n-q, \alpha+q}(t) (e^{-\omega t} T(t)) x dt \\
&= \int_0^{+\infty} \psi_{n-q, \alpha+q}(t) (e^{-\omega t} T(t)) (W - \omega I)^q x dt.
\end{aligned}$$

So

$$\begin{aligned}
\| (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} Cx \| &= \left\| \int_0^{+\infty} \psi_{n, \alpha}(t) (e^{-\omega t} T(t)) x dt \right\| \\
&= \left\| \int_0^{+\infty} (\psi_{n-q, \alpha+q}(t) (e^{-\omega t} T(t)) (W - \omega I)^q x) dt \right\| \\
&\leq \int_0^{+\infty} | \psi_{n-q, \alpha+q}(t) (e^{-\omega t} T(t)) x | dt \| (W - \omega I)^q x \| \\
&\leq \| (W - \omega I)^q x \| M \| \psi_{n-q, \alpha+q} \|_1 \\
&\leq \| (W - \omega I)^q x \| \frac{C_{\alpha+q}}{(n-q)^{\frac{\alpha+q}{2}}} \text{ (according to (5)).}
\end{aligned}$$

On the other hand, we are looking to increase the quantity

$$e^{\omega t} \left\| \sum_{m=0}^{+\infty} (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} x \phi_{m, \alpha}(t) - \sum_{m=0}^n (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} x \phi_{m, \alpha}(t) \right\| \quad (11)$$

We have

$$\begin{aligned}
(11) &\leq \sum_{m=n+1}^{+\infty} \| e^{\omega t} (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \phi_{m, \alpha}(t) \| \\
&\leq e^{\omega t} \sum_{m=n+1}^{+\infty} \| (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \| | \phi_{m, \alpha}(t) | \\
&\leq e^{\omega t} \sum_{m=n+1}^{+\infty} \| (W - \omega I)^q x \| \frac{C_{\alpha+q}}{(m-q)^{\frac{\alpha+q}{2}}} c_t m^{\frac{\alpha}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \| (W - \omega I)^q x \| e^{\omega t} c_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \frac{m^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha}{2}}} \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{(m-q)^{\frac{\alpha}{2}}}{(m-q)^{\frac{\alpha+q}{2}}} \left(\frac{m}{m-q}\right)^{\frac{\alpha}{2}} \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \sum_{m=n+1}^{+\infty} \frac{1}{(m-q)^{\frac{q}{2}}} \times 2 \\
&\leq \| (W - \omega I)^q x \| c'_t c_{\alpha+q} \frac{1}{\left(\frac{q}{2} - 1\right)} \frac{1}{n^{\frac{q}{2}-1}}
\end{aligned}$$

where

$$\| T(t)x - \sum_{m=0}^n (\omega I - W)^m ((\omega + 1)I - W)^{-m-\alpha-1} Cx \varphi_{m,\alpha}(t) \| \leq \frac{c_{t,\alpha,q} \| (W - \omega I)^q x \|}{n^{\frac{q}{2}-1}}.$$

Theorem 2.2 Let $(T(t))_{t \in \mathbb{R}^+}$ be an exponentially bounded C -semigroup in Banach space E with generator $(W, \mathcal{D}(W))$, $\alpha \in \mathbb{R}$ such that $\alpha > 1$.

1. For $x \in E$ and $n \in \mathbb{N}$, we have

$$\int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I) x dt = \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds.$$

2. For $x \in \mathcal{D}(W)$ we have

$$R_C(t, W - \omega I) x = \sum_{n=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s) x ds \right) \phi_{n,\alpha}(t), \text{ for all } t > 0.$$

Proof.

1. For all $t > 0$, $t + \omega \in \rho_C(W)$, then the function $t \mapsto R_C(t, W - \omega I) = R_C(t + \omega, W)$ is analytic in \mathbb{R}_+^* .

In the other hand by equation (9), we know that $\| R_C(t + \omega, W) \| \leq \frac{M}{t}$ for all $t > 0$, so

$$\begin{aligned} \int_0^{+\infty} \| \psi_{n,\alpha}(t) R_C(t, W - \omega I) x \| dt &= \int_0^{+\infty} | \psi_{n,\alpha}(t) | \| (R_C(t, W - \omega I) x \| dt \\ &\leq \int_0^{+\infty} \frac{n!M}{\Gamma(n + \alpha + 1)t} e^{-t} t^\alpha | \phi_{n,\alpha}(t) | dt \\ &\leq \frac{n!M}{\Gamma(n + \alpha + 1)} \int_0^{+\infty} e^{-t} t^{\alpha-1} | \phi_{n,\alpha}(t) | dt \\ &< +\infty. \end{aligned}$$

If we posed

$$H := \Gamma(n + \alpha + 1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I) x dt,$$

then

$$\begin{aligned} H &= \Gamma(n + \alpha + 1) \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I) x dt \\ &= \int_0^{+\infty} \frac{d^n}{dt^n} (e^{-t} t^{n+\alpha}) R_C(t, W - \omega I) x dt \\ &= \underbrace{\left[\frac{d^{n-1}}{dt^{n-1}} (e^{-t} t^{n+\alpha} R_C(t, W - \omega I)) \right]_0^{+\infty}}_{=0} - \int_0^{+\infty} \frac{d^{n-1}}{dt^{n-1}} (e^{-t} t^{n+\alpha}) \frac{d}{dt} (R_C(t, W - \omega I) x) dt \\ &= \left[-\frac{d^{n-2}}{dt^{n-2}} (e^{-t} t^{n+\alpha}) \frac{d}{dt} (R_C(t, W - \omega I) x) \right]_0^{+\infty} \\ &+ (-1)^2 \int_0^{+\infty} \frac{d^{n-2}}{dt^{n-2}} (e^{-t} t^{n+\alpha}) \frac{d^2}{dt^2} (R_C(t, W - \omega I) x) dt \\ &= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} \frac{d^n}{dt^n} (R_C(t, W - \omega I) x) dt \quad (\text{integration by parts}) \\ &= (-1)^n \int_0^{+\infty} e^{-t} t^{n+\alpha} (-1)^n n! (R_C(t, W - \omega I))^{-n-1} x dt \\ &= \int_0^{+\infty} e^{-t} t^{n+\alpha} (n!) \frac{1}{n!} \int_0^{+\infty} s^n e^{-ts} e^{-\omega s} T(s) x ds dt \\ &= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left(\int_0^{+\infty} t^{n+\alpha} e^{-t} e^{-ts} dt \right) ds \quad (\text{Fubini's Theorem}) \end{aligned}$$

$$\begin{aligned}
&= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left(\int_0^{+\infty} t^{n+\alpha} e^{-(s+1)t} dt \right) ds \\
&= \int_0^{+\infty} s^n e^{-\omega s} T(s) x \left(\int_0^{+\infty} \frac{u^{n+\alpha}}{(s+1)^{n+\alpha}} e^{-ut} \frac{du}{s+1} \right) ds \quad (u = (s+1)t) \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x \left(\int_0^{+\infty} u^{n+\alpha} e^{-ut} du \right) ds \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x \Gamma(n+\alpha+1) ds \\
&= \Gamma(n+\alpha+1) \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds,
\end{aligned}$$

hence the result.

2. Let $x \in \mathcal{D}(W)$.

The function $t \mapsto R_C(t, W - \omega I)x = R_C(t + \omega, W)x$ is differentiable in \mathbb{R}_+^* (because it's analytic in \mathbb{R}_+^*), and

$$\begin{aligned}
\int_0^{+\infty} t^\alpha e^{-t} \|R_C(t, W - \omega I)x\|^2 dt &= \int_0^{+\infty} t^\alpha e^{-t} \|((t + \omega)I - W)^{-1}Cx\|^2 dt \\
&\leq \int_0^{+\infty} t^\alpha e^{-t} \frac{M^2}{t^2} \|x\|^2 dt \quad (\text{equation (9)}) \\
&\leq M^2 \|x\|^2 \int_0^{+\infty} t^{\alpha-2} e^{-t} dt \\
&\leq M^2 \|x\|^2 \Gamma(\alpha - 1) \quad (\text{because } \alpha > 1) \\
&< +\infty.
\end{aligned}$$

Thus, the series $\sum_{n \in \mathbb{N}} c_n(R_C(\cdot, W - \omega I)x) \phi_{n,\alpha}$ converges pointwise to $R_C(\cdot, W - \omega I)x$ on \mathbb{R}_+^* , where

$$\begin{aligned}
c_n(R_C(\cdot, W - \omega I)x) &= \int_0^{+\infty} \psi_{n,\alpha}(t) R_C(t, W - \omega I)x dt \\
&= \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-\omega s} T(s) x ds \quad (\text{according to (a) of theorem 2.2}).
\end{aligned}$$

Therefore

$$(\forall t > 0) \quad R_C(t, W - \omega I)x = ((t + \omega)I - W)^{-1}Cx = \sum_{n=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n e^{-\omega s}}{(s+1)^{n+\alpha+1}} T(s) x ds \right\} \phi_{n,\alpha}(t).$$

Example 2.3 Let $m : \mathbb{R} \rightarrow \mathbb{R}^-$ be an even measurable function. In the Banach space $L^1(\mathbb{R})$, we consider the family $T := (T(t))_{t \geq 0} \subset \mathfrak{F}(L^1(\mathbb{R}))$ defined by $\forall t \geq 0, T(t) : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}), f \mapsto T(t)(f) : \mathbb{R} \rightarrow \mathbb{R}, s \mapsto T(t)(f)(s) = e^{t.m(s)} f(-s)$. Clearly, $(T(t))_{t \geq 0} \subset B(L^1(\mathbb{R}))$. If we put, $T(0) = C$, then, A family of operators $(T(t))_{t \geq 0}$ is uniformly bounded C -regularized semigroup with generator $(W, \mathcal{D}(W))$ defined by

$$W : \mathcal{D}(W) = \{f \in L^1(\mathbb{R}) / m.f \in L^1(\mathbb{R})\} \rightarrow L^1(\mathbb{R}), f \mapsto W(f) = m.f.$$

The theorem 2.1 give for $f \in \mathcal{D}(W)$ and $s, t \in \mathbb{R}^+$:

$$\begin{aligned} T(t)(f)(s) &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} (f)(s) \phi_{n,\alpha}(t) \\ &= e^{0 \times t} \sum_{n=0}^{+\infty} (0 \times I - W)^n ((0 + 1)I - W)^{-n-\alpha-1} (f)(s) \phi_{n,\alpha}(t) \\ &= \sum_{n=0}^{+\infty} (-m(s))^n (1 - m(s))^{-n-\alpha-1} f(s) \phi_{n,\alpha}(t). \end{aligned}$$

So $T(t)(f) = \sum_{n=0}^{+\infty} \phi_{n,\alpha}(t) (-m)^n ((1 - m))^{-n-\alpha-1} f$.

And Theorem 2.2 gives

$$R_C(t, W)(f)(.) = \sum_{n=0}^{+\infty} \left\{ \int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-sm(.)} ds \right\} C(f)(.) \phi_{n,\alpha}(t).$$

Example 2.4 The space $X = c_0 = \left\{ (x_k)_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \text{ tq } \lim_{k \rightarrow +\infty} x_k = 0 \right\}$, equipped with the norm $\| (x_k)_{k \in \mathbb{N}} \|_{\infty} = \max_{k \in \mathbb{N}} |x_k|$ becomes a Banach space. For each $n \in \mathbb{N}$ let $e_n = (\delta_{n,k})_{k \in \mathbb{N}}$ be element of

X . Like for all $x = (x_k)_{k \in \mathbb{N}} \in X, x = \sum_{k=0}^{+\infty} x_k e_k$ then $X = \text{span}\{e_n / n \in \mathbb{N}\}$. Considering the family of operators $(T(t))_{t \geq 0}$ defined by :

$$\text{for all } t \in \mathbb{R}^+ \text{ for all } x \left(= \sum_{k=0}^{+\infty} x_k e_k \right) \in X, T(t)x = \sum_{k=0}^{+\infty} e^{-k^2 t} x_k e_k$$

$(T(t))_{t \geq 0}$ is a uniformly bounded C_0 -semigroup ($\| T(t) \| \leq 1$) with generator $(W, \mathcal{D}(W))$ such that $\mathcal{D}(W) = \{x = (x_k)_{k \in \mathbb{N}} \in X / (k^2 x_k)_{k \in \mathbb{N}} \in X\}$ and $(\forall x \in \mathcal{D}(W)) Wx = \sum_{k=0}^{+\infty} -k^2 x_k e_k$. The theorem 2.1

give for $x = (x_k)_{k \in \mathbb{N}} \in \mathcal{D}(W)$ and for all $t \in \mathbb{R}^+$:

$$\begin{aligned}
 T(t)(x) &= e^{\omega t} \sum_{n=0}^{+\infty} (\omega I - W)^n ((\omega + 1)I - W)^{-n-\alpha-1} (x) \phi_{n,\alpha}(t) \\
 &= e^{0 \times t} \sum_{n=0}^{+\infty} (0 \times I - W)^n ((0 + 1)I - W)^{-n-\alpha-1} (x) \phi_{n,\alpha}(t) \\
 &= \sum_{n=0}^{+\infty} (k^{2n} (1 + k^2)^{-n-\alpha-1} x_k)_{k \in \mathbb{N}} \phi_{n,\alpha}(t) \\
 &= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} k^{2n} (1 + k^2)^{-n-\alpha-1} x_k e_k \phi_{n,\alpha}(t).
 \end{aligned}$$

And Theorem 2.2 gives

$$R_C(t, W)x = \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \left(\int_0^{+\infty} \frac{s^n}{(s+1)^{n+\alpha+1}} e^{-k^2 s} ds \right) x_k e_k \phi_{n,\alpha}(t).$$

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