Quasi-isometric embedding between $*$-algebras

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Abstract. We define a new version of quasi-isometric embedding maps between $*$-algebras and obtain some basic results related to this notion. Similar to quasi-isometric embedding maps on metric spaces, under some conditions, we give a necessary and sufficient condition on a $*$-homomorphism to be a quasi-isometric embedding between $*$-algebras. Moreover, we show that if φ be an injective quasi-isometry between $*$-Banach algebras A and B, then amenability of B implies amenability of A.

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1. Introduction

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is called quasi-isometric embedding if there exist $\alpha \geq 1$ and $\beta \geq 0$ such that

$$\frac{1}{\alpha} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta, \tag{1}$$

for all $x, y \in X$. In the above inequalities if $\beta = 0$ then the quasi-isometric embedding f is called bi-Lipschitz map where for some results we refer to [1, 3, 9, 12, 14]. The concept of quasi-isometric embedding is a very useful tool for investigating Cayley and hyperbolic graphs, for more details and applications, we refer to [2, 6, 7, 13]. Authors in [4] considered the notion of quasi-isometric on finitely generated algebras, where they have obtained many interesting geometric properties. For more details about geometric properties and other works related to these properties we refer to [4, Introduction] and the references there in.

Throughout this paper by a $*$-algebra we mean a Banach algebra with the involution $*$. Let A and B be two $*$-algebras, here for simplifying, we show the involutions on A and B by $*$. A linear mapping $\varphi : A \longrightarrow B$ is called a $*$-map, if $\varphi(a^*) = \varphi(a)^*$, for every $a \in A$. Moreover, a linear map $\varphi : A \longrightarrow B$ is called a homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in B$.

Amenability of Banach algebras was introduced by Johnson in [10] and its relation between homological properties of Banach algebras was introduced by Helemskii in [8]. Let A be a Banach algebra and X is a Banach A-bimodule. A derivation from A into X is a bounded linear map D such that $D(ab) = a \cdot D(b) + D(a) \cdot b$, for all $a, b \in A$. Moreover, if there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a$, for every $a \in A$, then D is called an inner derivation. A Banach algebra A is called amenable if all derivation from A into the first dual of any Banach A-bimodule X are inner. For a locally compact group G, Johnson proved that the group algebra $L^1(G)$ is amenable if and only if G is amenable [10]. Let A and B be two Banach algebras and φ be a continuous dense range homomorphism, if, A is amenable, then B is too [10, Proposition 5.3]. This result shows that amenability of Banach algebras can be transferred under a continuous dense range homomorphism. In this paper, we consider $*$-algebras and quasi-isometric embedding maps on these algebras where our definition is different from the quasi-isometric embedding maps on metric spaces and algebras that was defined before. By this new definition, we will consider the inverse of [10, Proposition 5.3], i.e., we investigate under which condition, if the right hand of a map is amenable, the left hand side (domain) can be amenable?

2. Quasi-isometric Embedding

In this section, we define a new version of quasi-isometric embedding maps between $*$-Banach algebras and we characterize these maps. We commence with the following definitions:

Definition 2.1. *Let A and B two $*$-algebras. We say a continuous map $\varphi : A \longrightarrow B$ is a quasi-isometric embedding of A into B if the following two conditions hold:*

(i) *for every finite subset $F \subset A$ there is a finite subset $F' \subset B$ such that, if, $a_1^* a_2 \in F$, then $\varphi(a_1)^* \varphi(a_2) \in F'$, for all $a_1, a_2 \in A$.*
(ii) *for every finite subset $F' \subset B$ there is a finite subset $F \subset A$ such that, if, $\varphi(a_1)^* \varphi(a_2) \in F'$, then $a_1^* a_2 \in F$, for all $a_1, a_2 \in A$.*

Definition 2.2. *Let A and B two $*$-algebras. A quasi-isometry from A into B is a quasi-isometric embedding $\varphi : A \longrightarrow B$ for which there is a finite dimensional subspaces $K \subset B$ such that $\varphi(A) + K = B$ i.e. for every $b \in B$ there exists $k \in K$ and $a \in A$ such that $\varphi(a) + k = b$. If there is a quasi-isometry from A into B, then we say that A is quasi-isometric to B.*

We continue with the following example that says there is an example of quasi-isometric embedding between $*$-algebras that is not isometry in the classical case.

Example 2.3. (i) *Let A,B two $*$-algebras and $\varphi : A \longrightarrow B$ be an injective $*$-homomorphism. We claim that φ is a quasi-isometric embedding of A into B. We must show that φ satisfies in the conditions of Definition*

2.1. *Assume that $F \subset A$ is a finite subset and set $\varphi(F) = F'$. Clearly, $F' \subset B$ is finite. Then for all $a_1, a_2 \in A$ such that $a_1^* a_2 \in F$, we have*

$$\varphi(a_1)^* \varphi(a_2) = \varphi(a_1^* a_2) \in F'. \tag{2}$$

This implies that the condition (i) holds. For (ii), suppose that F' is a finite subset of B and set $F = \varphi^{-1}(F')$. Then F is a finite subset of A, because of that φ is injective. For all $a_1, a_2 \in A$ satisfying $\varphi(a_1)^* \varphi(a_2) \in F'$, we have $\varphi(a_1^* a_2) \in F'$. This implies that $a_1^* a_2 \in F$.

(ii) *Let G be a locally compact group and H be a subgroup of H. Consider the group algebras $L^1(G)$ and $L^1(H)$ with the convolution product and the involution $f^*(x) = \Delta_G\left(x^{-1}\right)\overline{f\left(x^{-1}\right)}$, where Δ_G is the modular function of G, similarly the involution on H can defined by the modular function Δ_H. Now, consider the inclusion function $\imath : L^1(H) \longrightarrow L^1(G)$ by $\imath(f) = f$, for all $f \in L^1(H)$. Then, by (i), $\imath$ is a quasi-isometric embedding.*

(iii) *Let A be a $*$-Banach algebra and I be a closed ideal of A. Then similar to the above example, $\imath : I \longrightarrow A$ is a quasi-isometric embedding.*

Let A be a $*$-algebra and $F_1, F_2 \subset A$ be arbitrary sets, by $\bigoplus_{f \in F_2}(F_1 + f)$ we means disjoint union of $F_1 + f$'s. In the following, we give necessary and sufficient conditions on defined map in the Example 2.3 that becomes a quasi-isometry.

Proposition 2.4. *Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be a quasi-isometric embedding with the closed range. Then φ is a quasi-isometry if and only if $\varphi(A)$ is a subspace of B with a finite codimension.*

Proof. Assume that φ is a quasi-isometry. Hence, there is a finite dimensional subspace K of B such that $\varphi(A) + K = B$. Then

$$\dim \frac{B}{\varphi(A)} \leq \dim K < \infty. \tag{3}$$

Conversely, suppose that $\varphi(A)$ has a finite codimension in B. Let K be the set of representatives for the right cosets of $\varphi(A)$ in B. Thus, K is a finite dimensional subspace of B and

$$B = \bigoplus_{k \in K} (\varphi(A) + k) = \varphi(A) + K. \tag{4}$$

This implies that φ is a quasi-isometry. □

Corollary 2.5. *Let A be a commutative $*$-Banach algebra and I be a maximal ideal of A. Then $\varphi : I \longrightarrow A$ is a quasi-isometry.*

Note that in the above result, commutativity of A is essential, because if we set $A = \mathcal{B}(E)$, the Banach algebra contains all linear bounded operators on the infinite dimensional Banach space E. Then $M = \{T \in \mathcal{B}(E) : Tx = 0, \text{ for all } x \in E\}$ is a maximal ideal of $A = \mathcal{B}(E)$ which has infinite codimension.

Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be a $*$-map i.e. $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. Then φ is called unitary preserving $*$-map if for every unitary element $a \in A$, $\varphi(a)$ is a unitary element in B. Assume that e_A and e_B are the unit elements of A and B, respectively. We denote the set of all unitary elements of a $*$-algebra A by $U(A)$ and by $|U(A)|$ we mean the cardinal number of $U(A)$. If φ is a quasi-isometric embedding, then we could not say that $\varphi(e_A) = e_B$, in general. By the following, we are seeking a quasi-isometric embedding that has this property. A Banach algebra A is called without order if, $ab = 0$ implies that $a = 0$ or $b = 0$.

Lemma 2.6. *Let A and B be two unital $*$-algebras with units e_A and e_B, respectively. If there is a quasi-isometric embedding from A into B that is a unitary preserving $*$-map, then there is a quasi-isometric embedding ϕ from A into B such that $\phi(e_A) = e_B$. Moreover, if the exited map is a quasi-isometry and B is without order, then ϕ is a quasi-isometry.*

Proof. Suppose that $\varphi : A \longrightarrow B$ is a quasi-isometric embedding that satisfies the stated conditions. Define $\phi : A \longrightarrow B$ by $\phi(a) = \varphi(e_A)^*\varphi(a)$ for all $a \in A$. Then

$$\phi(a_1)^*\phi(a_2) = \varphi(a_1)^*\varphi(e_A)\varphi(e_A)^*\varphi(a_2) = \varphi(a_1)^*\varphi(a_2), \tag{5}$$

for all $a_1, a_2 \in A$. Thus, ϕ is a quasi-isometric embedding. Also, according to definition of ϕ, we have $\phi(e_A) = e_B$.

Now, let φ be a a quasi-isometry. Thus, there is a finite dimensional subspace K of B such that $\varphi(A) + K = B$. Set $K' = \varphi(e_A)^*K$. Clearly, K' is a finite dimensional subspace of B. Then

$$\phi(A) + K' = \varphi(e_A)^*\varphi(A) + \varphi(e_A)^*K = \varphi(a)^*B \cong B. \tag{6}$$

In (5), $\varphi(a)^*B \cong B$ holds because B is without order. This shows that ϕ is a quasi-isometry. □

Proposition 2.7. *Let A and B be two $*$-Banach algebra such that A is finite dimensional and B is unital. If $\varphi : A \longrightarrow B$ is a quasi-isometry and linear map, then B is finite dimensional.*

Proof. Since $\varphi : A \longrightarrow B$ is a quasi-isometry, there is a finite dimensional subspace K_B of B such that $B = \varphi(A) + K_B$. Let e_A, e_B be the units of A and B, respectively, $\dim A = n$ and let $F = \{e_A, a_1, \ldots, a_m\} \subset A$ such that generates A and $m \geq n$. Since φ is a quasi-isometric embedding, there is a finite subset F' of B such that $\varphi(x)^*\varphi(y) \in F'$, whenever $x^*y \in F$. So, if we take $x = e_A$, then by Lemma 2.6, $\varphi(y) \in F'$, whenever, $y \in F$. This means that $\varphi(a_i) \in F'$, for all $a_i \in F$, $1 \leq i \leq m$. We show that $\overline{F} = F' \cup K_B$ generates B. Let $b \in B$, then there exist $b' \in \varphi(A)$ and $k \in K_B$ such that $b = b' + k$. Then there exists $a \in A$ such that $\varphi(a) = b'$, so we have $b = \varphi(a) + k$. Moreover, there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $a = \sum_{i=1}^m \alpha_i a_i$. Then,

$$b = b' + k = \varphi\left(\sum_{i=1}^m \alpha_i a_i\right) + k = \sum_{i=1}^m \alpha_i \varphi(a_i) + k.$$

This shows that $\overline{F}$ generates B and hence B is finite dimensional. □

By the following result we show that the composition of the quasi-isometric embedding maps is a quasi-isometric embedding map.

Proposition 2.8. *Let A, B and C be three $*$-Banach algebras. Assume that $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ are quasi-isometric embedding maps, then $\psi \circ \varphi : A \longrightarrow C$ is a quasi-isometric embedding map.*

Proof. Let F be a finite subset of A. Then one can find a finite subset F' of B such that if $a_1^* a_2 \in F$, then $\varphi(a_1)^*\varphi(a_2) \in F'$. Moreover, since ψ is a quasi-isometric embedding, there is finite subset F'' of C such that $\psi\left(\varphi(a_1)\right)^* \psi\left(\varphi(a_2)\right) \in F''$. Similarly, for any finite subset $\overline{F}$ of C one can find a finite subset F of A such that $\psi\left(\varphi(a_1)\right)^* \psi\left(\varphi(a_2)\right) \in \overline{F}$ implies that $a_1^* a_2 \in F$. □

Corollary 2.9. *Let A and B be $*$-Banach algebras, I be a $*$-subalgebra of A and $\varphi : A \longrightarrow B$ be a quasi-isometric embedding. Then the restriction map $\varphi|_I : I \longrightarrow B$ is a quasi-isometric embedding.*

Proof. Consider the inclusion map $\imath : I \longrightarrow A$. Since $\imath$ is an injective $*$-homomorphism, by Example 2.3, $\imath$ is a quasi-isometric embedding. Then by Proposition 2.8, $\varphi|_I = \varphi \circ \imath$ is a quasi-isometric embedding. □

Proposition 2.10. *Let A, B, C and D be $*$-Banach algebras. Assume that $\varphi : A \longrightarrow B$ and $\psi : C \longrightarrow D$ are quasi-isometry, then*

(i) *$\varphi \times \psi : A \times C \longrightarrow B \times D$ is a quasi-isometry.*
(ii) *$\varphi \otimes \psi : A \otimes C \longrightarrow B \otimes D$ is not a quasi-isometry.*

Proof. There are finite dimensional subspace K_B and K_D of B and D such that $B = \varphi(A) + K_B$ and $D = \psi(C) + K_D$. Then clearly (i) holds. For the case (ii), $\varphi \otimes \psi$ becomes a quasi-isometric embedding but is not a quasi-isometry, because

$$B \otimes D = (\varphi(A) \otimes \psi(C)) + (\varphi(A) \otimes K_D) + (K_B \otimes \psi(C)) + (K_B \otimes K_D)\,.$$

□

Let A and B be two $*$-algebras, let $\varphi : A \longrightarrow B$ be a map and $b \in B$. Define the sets

$$S_A = \{a^* a' : \ \|a - a'\| = 1, \ \text{for all } a, a' \in A\},$$
$$S_B = \{\varphi(a)^* \varphi(a') : \ \|\varphi(a) - \varphi(a')\| = 1, \ \text{for all } a, a' \in A\},$$

and

$$C_\varphi(b) = \{\varphi(a) : \ \varphi(a) = b \text{ for every } a \in A\}.$$

For a finite subset F of A by $\mathrm{Diam}(F)$ we mean the diameter of F i.e.

$$\mathrm{Diam}(F) = \max_{a, a' \in F} \|a - a'\|.$$

Now, we are ready to prove the one of the main results of this paper as follows:

Theorem 2.11. *Let φ be a quasi-isometric embedding between finite dimensional $*$-algebras A and B, then there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\frac{1}{\alpha}\|a-a'\| - \beta \leq \|\varphi(a)-\varphi(a')\| \leq \alpha\|a-a'\| + \beta, \tag{7}$$

for all $a, a' \in A$.

Proof. Let $F \subset A$ be a finite subset such that $a_1^* a_2 \in F$ whenever $\varphi(a_1)^*\varphi(a_2) \in S_B$ for all $a_1, a_2 \in A$. Similarly, assume that $F' \subset B$ is a finite subset such that $\varphi(a_1)^*\varphi(a_2) \in F'$ whenever $a_1^* a_2 \in S_A$ for all $a_1, a_2 \in A$. Also, let $F_1 \subset A$ be a finite subset and $b \in B$ such that $a_1^* a_2 \in F_1$ whenever $\varphi(a_1), \varphi(a_2) \in C_\varphi(b)$ for all $a_1, a_2 \in A$. Now; set

$$\alpha = \max\{\mathrm{Diam}(F), \mathrm{Diam}(F')\} \quad \text{and} \quad \beta = \frac{1}{\alpha}\mathrm{Diam}(F_1). \tag{8}$$

Clearly, if $a = a'$ then (7) holds. Suppose that $\|a - a'\| = 1$, then $a^* a' \in S_A$ and consequently, $\varphi(a)^*\varphi(a') \in F'$. Since, φ is a quasi-isometric embedding, we have $\|\varphi(a) - \varphi(a')\| \leq \alpha$. Now, assume that $1 \leq \|a - a'\| = \gamma \leq n$, where $\gamma \in \mathbb{R}^+$ and assume that $a_0, a_1, \ldots, a_n \in A$ such that $a_0 = a$, $a_n = a'$ and $\|a_i - a_{i+1}\| = 1$ for $0 \leq i \leq n-1$. Then

$$\begin{aligned} \|\varphi(a)-\varphi(a')\| &\leq \sum_{i=0}^{n-1}\|\varphi(a_i)-\varphi(a_{i+1})\| \\ &\leq n\alpha \\ &= \alpha\|a-a'\|. \end{aligned} \tag{9}$$

Hence,

$$\|\varphi(a)-\varphi(a')\| \leq \alpha\|a-a'\|, \tag{10}$$

for all $a, a' \in A$. Let $\|\varphi(a) - \varphi(a')\| = 1$, then $\varphi(a)^*\varphi(a') \in S_B$ and consequently, $a^* a' \in F$. Since, φ is a quasi-isometric embedding, we have $\|a - a'\| \leq \alpha$. Similar to the above statements, if assume that $\|\varphi(a) - \varphi(a')\| = n \geq 1$ one can show that

$$\|a-a'\| \leq \alpha\|\varphi(a)-\varphi(a')\|, \tag{11}$$

for all $a, a' \in A$. If $\varphi(a_1), \varphi(a_2) \in C_\varphi(b)$, then by (8), we have $a^* a' \in F_1$. Thus

$$\frac{1}{\alpha}\|a-a'\| \leq \beta. \tag{12}$$

This implies that

$$\frac{1}{\alpha}\|a-a'\| - \beta \leq 0 = \|\varphi(a)-\varphi(a')\|. \tag{13}$$

Thus, the inequalities (11) and (13) imply the inequality (7). □

As a special case of the converse of the above obtained result, we have the following:

Theorem 2.12. *Let φ be a $*$-homomorphism between finite dimensional $*$-algebras A and B that satisfies* (7)*. Then φ is a quasi-isometric embedding.*

Proof. Let $F \subset A$ be a finite subset. Pick $a \in F$ and define

$$F' = \{\varphi(a') : \ a' \in F \text{ and } \|\varphi(a) - \varphi(a')\| < \delta\}, \tag{14}$$

where $\delta = \alpha \max_{a' \in F} \|a - a'\| + \beta$. Clearly, $F' \subset B$ is a finite subset. For all $a_1, a_2 \in A$ such that $a_1^* a_2 \in F$, the relation (7) implies that

$$\begin{aligned} \|\varphi(a) - \varphi(a_1)^* \varphi(a_2)\| &= \|\varphi(a) - \varphi(a_1^* a_2)\| \leq \alpha \|a - a_1^* a_2\| + \beta \\ &< \delta. \end{aligned} \tag{15}$$

Hence, $\varphi(a_1)^* \varphi(a_2) \in F'$. Thus, the condition (i) of Definition 2.1 holds. For the condition (ii), suppose that F' is a finite subset of B. Let $f \in F'$ and assume that a is an element of A such that $\varphi^{-1}(f) = a$ (inverse map). Define

$$F = \{a' : \text{ there is a } f' \in F' \text{ such that } \varphi^{-1}(f') = a' \text{ and } \|a - a'\| < \delta\}, \tag{16}$$

where $\delta = \alpha \max_{f' \in F} \|f - f'\| + \alpha\beta$. Note that in F, we just choose one of elements that $\varphi^{-1}(f')$ contains. Clearly, $F' \subset B$ is a finite subset. For all $a_1, a_2 \in A$ such that $\varphi(a_1)^* \varphi(a_2) \in F'$, then (7) implies that

$$\begin{aligned} \|a - a_1^* a_2\| &\leq \alpha \|\varphi(a) - \varphi(a_1^* a_2)\| + \alpha\beta \\ &= \alpha \|\varphi(a) - \varphi(a_1)^* \varphi(a_2)\| + \alpha\beta \\ &< \delta. \end{aligned} \tag{17}$$

This means that $a_1^* a_2 \in F$ and this completes the proof. ☐

Theorems 2.11 and 2.12 follow the following result:

Corollary 2.13. *Let φ be a $*$-homomorphism between finite dimensional $*$-algebras A and B. Then φ is a quasi-isometric embedding if and only if there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\frac{1}{\alpha} \|a - a'\| - \beta \leq \|\varphi(a) - \varphi(a')\| \leq \alpha \|a - a'\| + \beta, \tag{18}$$

for all $a, a' \in A$.

Example 2.14. *Let A be a finite dimensional $*$-algebra. Consider $A \times A$ with the coordinate-wise product. It becomes a $*$-algebra with respect to the norm $\|(a, b)\| = \|a\| + \|b\|$ and the involution $(a, b)^* = (a^*, b^*)$ for all $a, b \in A$. Define $\varphi : A \longrightarrow A \times A$ by $\varphi(a) = (a, a)$ for every $a \in A$. Clearly, φ is a $*$-homomorphism and*

$$\|\varphi(a) - \varphi(a')\| = \|(a - a', a - a')\| = 2\|a - a'\|,$$

for all $a, a' \in A$. Obviously, φ satisfies (18) for $\alpha \geq 2$ and $\beta \geq 0$. Thus, it is a quasi-isometric embedding.

Let A and B two C^*-algebras. Suppose that $\varphi : A \longrightarrow B$ is a $*$-homomorphism, then φ is norm-decreasing [5, Corollary 3.2.4]. This fact follows the following result.

Corollary 2.15. *Every $*$-homomorphism between finite dimensional C^*-algebras is a quasi-isometric embedding.*

3. Connection with Amenability

A Banach algebra A is called has property $(\mathbf{G})$ if there exists an amenable locally compact group G and a continuous homomorphism $\varphi : L^1(G) \longrightarrow A$ with dense range, this concept is introduced by Kepert [11] that he used the above mentioned result in [10]. The following result is a special case of the inverse of [10, Proposition 5.3].

Proposition 3.1. *Let A,B be two $*$-algebras and $\varphi : A \longrightarrow B$ be an injective quasi-isometry. If B is amenable, then A is amenable.*

Proof. Assume that B is amenable and K be a finite dimensional subspace of B such that $\varphi(A) + K = B$. This implies that $\varphi(A)$ is amenable. Since $A \cong \varphi(A)$, A is amenable. □

Let A be an amenable Banach algebra and I be a closed two-sided ideal of A. If I possess a bounded approximate identity, then I is amenable [10, Proposition 5.1]. By the following result, for commutative $*$-Banach algebras, we omit the existence of a bounded approximate identity condition for maximal ideals of these algebras. For a Banach algebra A, by $\sigma(A)$ we mean the character space of A.

Corollary 3.2. *Let A be an amenable commutative $*$-Banach algebra and $\varphi \in \sigma(A)$. Then $\ker\varphi$ is amenable.*

Proof. The $\ker\varphi$ is a maximal ideal of A and indeed all maximal ideals of A are the kernels of members of $\sigma(A)$. Now, consider the inclusion map $\imath : \ker\varphi \longrightarrow A$. Then by Corollary 2.5, $\imath$ is an injective quasi-isometry and Proposition 3.1 implies that $\ker\varphi$ is amenable. □

Example 3.3. *Let G be an Abelian locally compact group and $\varphi \in \sigma(L^1(G))$. Then by Corollary 3.2, $\ker\varphi$ is amenable.*

The result in Proposition 3.1 is a special case of the inverse of Property $(\mathbf{G})$. So, similar to property $(\mathbf{G})$, we say a $*$-Banach algebra A has property $(\mathbf{G}')$, if there exist an amenable locally compact group G and an injective quasi-isometry map $\varphi : A \longrightarrow L^1(G)$. Clearly, for every amenable locally compact group G, $L^1(G)$ has property $(\mathbf{G}')$. Moreover, form Example 3.3, $\ker\varphi$ has property $(\mathbf{G}')$.

Now, this question arises that: is there any amenable $*$-Banach algebra without property $(\mathbf{G}')$? We answer to this question by the following example.

Example 3.4. *Let A and B be two $*$-Banach algebras that have property $(\mathbf{G}')$. Thus, there are amenable locally compact groups G, H and injective quasi-isometries $\varphi : A \longrightarrow L^1(G)$ and $\psi : B \longrightarrow L^1(H)$. By Proposition 3.1, A and B are amenable. Hence, $A\widehat{\otimes}B$ is amenable [5, Corollary 2.9.62]. Now, consider $\varphi\otimes\psi : A\widehat{\otimes}B \longrightarrow L^1(G)\widehat{\otimes}L^1(H)$ define by $\varphi\otimes\psi(a\otimes b) = \varphi(a)\otimes\psi(b)$, for all $a\otimes b \in A\widehat{\otimes}B$. Then by Proposition 2.10, $\varphi\otimes\psi$ is not a quasi-isometry. Hence, $A\widehat{\otimes}B$ has no property $(\mathbf{G}')$.*

4. Some Problems

Here, we some problems that maybe interesting regarding properties of quasi-isometries:

Availability of data and materials

The data is available inside the paper.

Declarations Competing interests

The authors declare that they have no competing interests

References

[1] I. Benjamini and O. Schramm, *Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant*, Geom. Funct. Anal., 7(1997), 403-419.

[2] N.P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximation*, Graduate Studies in Mathematics, Vol. 88, Amer. Math. Soc., 2008.

[3] V. Capraro, *Amenability, locally finite spaces, and bi-lipschitz embeddings*, Expo. Math., 31(4)(2013), 334-349.

[4] T.G. Ceccherini-Silberstein and A. Y. Samet-Vaillant, *Asymptotic invariants of finitely generated algebras. A generalization of Gromov's quasi-isometric viewpoint*, J. Math. Sci., 156(1)(2009), 56-108.

[5] H.G. Dales, *Banach algebras and automatic continuity*, London Math. Society Monographs, Vol. 24, Clarendon Press, Oxford, 2000.

[6] M. Gromov, *Asymptotic invariants ofinlinite groups, in geometric group theory*, Vol. 2, London Math. Soc. Lecture Note Ser., 182. Cambridge Univ. Press, 1993.

[7] U. Hamenstädt, *Geometry of the mapping class groups I: Boundary amenability*, Invent. Math., 175(2009), 545-609.

[8] A.Ya. Helemskii, *The homology of Banach and topological algebras*, Mathematics and Its Applications, 41, Kluwer Academic Publishers, Dordrecht-Boston-London, 1986.

[9] M. Huang and Y. Li, *On bilipschitz extensions in real Banach spaces*, Abstr. App. Anal., 2013 (2013), Article ID 765685, 9 pages.

[10] B. E. Johnson, *Cohomology in Banach algebras*, Memoirs of the American Mathematical Society, 127, Amer. Math. Soc., Providence, Rhode Island, 1972.

[11] A. Kepert, *Amenability in group algebras and Banach algebras*, Math. Scand., 74(1994), 275-292.

[12] J.R. Lee, A. Naor and Y. Peres, *Trees and Markov convexity*, Geom. Funct. Anal., 18(2009), 1609-1659.

[13] R.E. Schwartz, *Quasi-isometric rigidity and Diophantine approximation*, Acta Math., 177(1996), 75-112.

[14] J. Väisälä, *Banach spaces and bilipschitz map*, Studia Math., 103(3)(1992), 291-294.

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