

# Finslerian Hypersurfaces of a Randers type exponential form of a special Finsler $(\alpha, \beta)$ - Metric

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## Abstract

In the present paper we have studied the Finslerian hypersurfaces of a Finsler space which is equipped with Randers type exponential form metric given by  $L(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{\alpha + \beta}}$ . We also examined the hypersurfaces of this special Randers type exponential metric as a hyperplane of first, second and third kinds. Further we obtained the condition under which this hypersurfaces will be semi-C-reducible, C-reducible, C2-like Finsler Space and the value of main scalar  $I$  in two dimensional case to be obtained.

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## 1 Introduction

The n-dimensional Finsler space  $F^n = (M^n, L)$ , is a combination of an n-dimensional differential manifold  $M^n$  and a fundamental function  $L(x, y)$  which is defined by:

**Definition 1.1.** A Finsler space  $F^n$  of dimension  $n$  is a differentiable manifold such that the length  $s$  of the curve  $x^i(t)$  of  $F^n$  is defined by  $S = \int L(x, y)dt$ , where  $L(x, y) = L(x^1, \dots, x^n, y^1, \dots, y^n)$  is a fundamental function supposed to be differentiable for  $y \neq (0)$  and satisfies the following condition:-  
(i) Positively homogeneous:  $L(x, py) = pL(x, y)$ ,  $p > 0$ .  
(ii) Positive:  $L(x, y) > 0$ ,  $y \neq 0$ .  
(iii) Positive definite metric  $g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} > 0$ .

If the fundamental function  $L(x, y)$  is the function of the Riemannian metric  $\alpha$  and one form metric  $\beta$  then this metric is known as  $(\alpha, \beta)$ - metric and it is defined by

**Definition 1.2.** A Finsler metric  $L(\alpha, \beta)$  in a differentiable manifold  $M^n$  is called an  $(\alpha, \beta)$ -metric, if  $L$  is a positively homogeneous function of degree one of a Riemannian metric  $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$  and a one-form  $\beta = b_i(x)y^i$  on  $M^n$ .

The interesting and important examples of an  $(\alpha, \beta)$ - metric are Randers metric  $\alpha + \beta$ , Kropina metric  $\frac{\alpha^2}{\beta}$  and Matsumoto metric [9]  $\frac{\alpha^2}{(\alpha - \beta)}$ . The notion of an  $(\alpha, \beta)$ -metric was introduced by M. Matsumoto [7] and has been studied by many authors.

A Hypersurface is a generalization of the concept of hyperplane. The definition of hypersurface is given by

**Definition 1.3.** Suppose an enveloping manifold  $M$  of  $n$  dimension, then any submanifold of  $M$  of  $(n - 1)$  dimension is a hypersurface. The co-dimension of hypersurface is one. If  $m > n$ , then  $V_n$  is called a subspace of  $V_m$ .  $V_m$  is also enveloping space of  $V_n$  if  $m > n$ . In particular if  $m = n + 1$ , then  $V_n$  is called hypersurface of the enveloping space  $V_{n+1}$ .

A hypersurface  $M^{n-1}$  of the  $M^n$  may be represented parametrically by the equation  $x^i = x^i(u^\alpha)$ ,  $\{\alpha = 1, 2, \dots, (n-1)\}$ , where  $u$  are Gaussian coordinates on  $M^{n-1}$ . If the supporting element  $y^i$  at a point  $(u^\alpha)$  of  $M^{n-1}$  is assumed to be tangential to  $M^{n-1}$ , we may be write

$$y^i = B_\alpha^i(u)v^\alpha$$

so that  $v^\alpha$  thought of as the supporting element of  $M^{n-1}$  at a point  $(u^\alpha)$ . Since the function  $\underline{L}(u, v) := L(x(u), y(u, v))$  gives rise to a Finsler metric of  $M^{n-1}$ , we get an  $(n - 1)$ -dimensional Finsler space  $F^{(n-1)} = (M^{n-1}, \underline{L}(u, v))$ .

The concept of Finslerian hypersurface is first introduced by Matsumoto in the year 1985 and further he defined three types of hypersurfaces that

were called a hyperplane of the first, second and third kinds. Further many authors studied these hyperplanes in different changes of the Finsler metric [2, 3, 4, 5, 6, 10, 11, 12] and obtained different results.

In the present paper we consider Randers type exponential form of a special Finsler  $(\alpha, \beta)$ -metric and determine Induced Riemannian metric, second fundamental v tensor also obtain to the conditions for hypersurfaces of this special metric as a hyperplane of first, second and third kinds. Further we obtained the condition under which this hypersurfaces will be semi-C-reducible, C-reducible, C2-like Finsler Space and the value of main scalar  $I$  in two dimensional case to be obtained.

## 2 Preliminaries

In the present paper we consider an  $n$ -dimensional Finsler space  $F^n = \{M^n, L(\alpha, \beta)\}$ , that is, a pair consisting of an  $n$ - dimensional differentiable manifold  $M^n$  equipped with a Fundamental function  $L$  as a Randers type of exponential form of a special Finsler Space with the metric

$$(2.1) \quad L(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{\alpha+\beta}}$$

Differentiating equation (2.1) partially with respect to  $\alpha$  and  $\beta$  are given by

$$\begin{aligned} L_\alpha &= \frac{\alpha}{(\alpha+\beta)}e^{\frac{\beta}{\alpha+\beta}}, & L_\beta &= \frac{(2\alpha+\beta)}{(\alpha+\beta)}e^{\frac{\beta}{\alpha+\beta}} \\ L_{\alpha\alpha} &= \frac{\beta^2}{(\alpha+\beta)^3}e^{\frac{\beta}{\alpha+\beta}} & L_{\beta\beta} &= \frac{\alpha^2}{(\alpha+\beta)^3}e^{\frac{\beta}{\alpha+\beta}} \\ L_{\alpha\beta} &= -\frac{\alpha\beta}{(\alpha+\beta)^3}e^{\frac{\beta}{\alpha+\beta}} \end{aligned}$$

where  $L_\alpha = \frac{\partial L}{\partial \alpha}$ ,  $L_\beta = \frac{\partial L}{\partial \beta}$ ,  $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$ ,  $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$ ,  $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$ .

In Finsler space  $F^n = \{M^n, L(\alpha, \beta)\}$  the normalized element of support  $l_i = \partial_i L$  and angular metric tensor  $h_{ij}$  are given by [8]:

$$\begin{aligned} l_i &= \alpha^{-1}L_\alpha Y_i + L_\beta b_i \\ h_{ij} &= pa_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j \end{aligned}$$

where  $Y_i = a_{ij}y^j$ . For the fundamental metric function (2.1) above constants are

$$(2.2) \quad \begin{aligned} p &= e^{\frac{2\beta}{\alpha+\beta}}, \\ q_0 &= \frac{\alpha^2}{(\alpha+\beta)^2} e^{\frac{2\beta}{\alpha+\beta}}, \\ q_{-1} &= -\frac{\beta}{(\alpha+\beta)^2} e^{\frac{2\beta}{\alpha+\beta}}, \\ q_{-2} &= -\frac{(\alpha+2\beta)}{\alpha(\alpha+\beta)^2} e^{\frac{2\beta}{\alpha+\beta}} \end{aligned}$$

Fundamental metric tensor  $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$  and its reciprocal tensor  $g^{ij}$  for  $L = L(\alpha, \beta)$  are given by [8]

$$(2.3) \quad g_{ij} = pa_{ij} + p_0b_ib_j + p_{-1}(b_iY_j + b_jY_i) + p_{-2}Y_iY_j$$

where

$$(2.4) \quad \begin{aligned} p_0 &= \frac{\{\alpha^2 + (2\alpha + \beta)^2\}}{(\alpha + \beta)^2} e^{\frac{2\beta}{\alpha+\beta}}, \\ p_{-1} &= \frac{2\alpha}{(\alpha + \beta)^2} e^{\frac{2\beta}{\alpha+\beta}}, \\ p_{-2} &= -\frac{2\beta}{\alpha(\alpha + \beta)^2} e^{\frac{2\beta}{\alpha+\beta}} \end{aligned}$$

The reciprocal tensor  $g^{ij}$  of  $g_{ij}$  is given by

$$(2.5) \quad g^{ij} = p^{-1}a^{ij} - s_0b^ib^j - s_{-1}(b^iy^j + b^jy^i) - s_{-2}y^iy^j$$

where  $b^i = a^{ij}b_j$  and  $b^2 = a_{ij}b^ib^j$

$$(2.6) \quad \begin{aligned} s_0 &= \frac{1}{\tau p} \{pp_0 + (p_0p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} &= \frac{1}{\tau p} \{pp_{-1} + (p_0p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} &= \frac{1}{\tau p} \{pp_{-2} + (p_0p_{-2} - p_{-1}^2)b^2\}, \\ \tau &= p(p + p_0b^2 + p_{-1}\beta) + (p_0p_{-2} - p_{-1}^2)(\alpha^2b^2 - \beta^2) \end{aligned}$$

The hv-torsion tensor  $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$  is given by [11]

$$(2.7) \quad 2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1m_im_jm_k$$

where,

$$(2.8) \quad \gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \quad m_i = b_i - \alpha^{-2}\beta Y_i$$

Here  $m_i$  is a non-vanishing covariant vector orthogonal to the element of support  $y^i$ .

Let  $\{\overset{i}{j}k\}$  be the component of christoffel symbols of the associated Riemannian space  $R^n$  and  $\nabla_k$  be the covariant derivative with respect to  $x^k$  relative to this christoffel symbol. Now we define,

$$(2.9) \quad 2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji}$$

where  $b_{ij} = \nabla_j b_i$ .

Let  $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, \Gamma_{jk}^i)$  be the cartan connection of  $F^n$ . The difference tensor  $D_{jk}^i = \Gamma_{jk}^{*i} - \{\overset{i}{j}k\}$  of the special Finsler space  $F^n$  is given by

$$(2.10) \quad D_{jk}^i = B^i E_{jk} + F_k^i B_j + F_j^i B_k + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} \\ - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i)$$

where

$$(2.11) \quad B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F_i^k = g^{kj} F_{ji} \\ B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B_i^k = g^{kj} B_{ji} \\ A_k^m = B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m \\ \lambda^m = B^m E_{00} + 2B_0 F_0^m, \quad B_0 = B_i y^i$$

where '0' denote contraction with  $y^i$  except for the quantities  $p_0, q_0$  and  $s_0$ .

### 3 Induced Cartan Connection

Let  $F^{n-1}$  be a hypersurface of  $F^n$  given by the equation  $x^i = x^i(u^\alpha)$  where  $\{\alpha = 1, 2, 3, \dots, (n-1)\}$ . The element of support  $y^i$  of  $F^n$  is to be taken tangential to  $F^{n-1}$ , that is [8],

$$(3.1) \quad y^i = B_\alpha^i(u) v^\alpha$$

the metric tensor  $g_{\alpha\beta}$  and hv-tensor  $C_{\alpha\beta\gamma}$  of  $F^{n-1}$  are given by

$$g_{\alpha\beta} = g_{ij}B_{\alpha}^iB_{\beta}^j, \quad C_{\alpha\beta\gamma} = C_{ijk}B_{\alpha}^iB_{\beta}^jB_{\gamma}^k$$

and at each point  $(u^{\alpha})$  of  $F^{n-1}$ , a unit normal vector  $N^i(u, v)$  is defined by

$$g_{ij}\{x(u, v), y(u, v)\}B_{\alpha}^iN^j = 0, \quad g_{ij}\{x(u, v), y(u, v)\}N^iN^j = 1$$

Angular metric tensor  $h_{\alpha\beta}$  of the hypersurface are given by

$$(3.2) \quad h_{\alpha\beta} = h_{ij}B_{\alpha}^iB_{\beta}^j, \quad h_{ij}B_{\alpha}^iN^j = 0, \quad h_{ij}N^iN^j = 1$$

$(B_i^{\alpha}, N_i)$  inverse of  $(B_{\alpha}^i, N^i)$  is given by

$$B_i^{\alpha} = g^{\alpha\beta}g_{ij}B_{\beta}^j, \quad B_{\alpha}^iB_i^{\beta} = \delta_{\alpha}^{\beta}, \quad B_i^{\alpha}N^i = 0, \quad B_{\alpha}^iN_i = 0$$

$$N_i = g_{ij}N^j, \quad B_i^k = g^{kj}B_{ji}, \quad B_{\alpha}^iB_j^{\alpha} + N^iN_j = \delta_j^i$$

The induced connection  $ICT\Gamma = (\Gamma_{\beta\gamma}^{*\alpha}, G_{\beta}^{\alpha}, C_{\beta\gamma}^{\alpha})$  of  $F^{n-1}$  from the Cartan's connection  $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^{*i})$  is given by [8].

$$\Gamma_{\beta\gamma}^{*\alpha} = B_i^{\alpha}(B_{\beta\gamma}^i + \Gamma_{jk}^{*i}B_{\beta}^jB_{\gamma}^k) + M_{\beta}^{\alpha}H_{\gamma}$$

$$G_{\beta}^{\alpha} = B_i^{\alpha}(B_{0\beta}^i + \Gamma_{0j}^{*i}B_{\beta}^j), \quad C_{\beta\gamma}^{\alpha} = B_i^{\alpha}C_{jk}^iB_{\beta}^jB_{\gamma}^k$$

where

$$M_{\beta\gamma} = N_iC_{jk}^iB_{\beta}^jB_{\gamma}^k, \quad M_{\beta}^{\alpha} = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_{\beta} = N_i(B_{0\beta}^i + \Gamma_{0j}^{*i}B_{\beta}^j)$$

and

$$B_{\beta\gamma}^i = \frac{\partial B_{\beta}^i}{\partial u^{\gamma}}, \quad B_{0\beta}^i = B_{\alpha\beta}^i v^{\alpha}$$

The quantities  $M_{\beta\gamma}$  and  $H_{\beta}$  are called the second fundamental v-tensor and normal curvature vector respectively [8]. The second fundamental h-tensor  $H_{\beta\gamma}$  is defined as [8]

$$(3.3) \quad H_{\beta\gamma} = N_i(B_{\beta\gamma}^i + \Gamma_{jk}^{*i}B_{\beta}^jB_{\gamma}^k) + M_{\beta}H_{\gamma}$$

where

$$(3.4) \quad M_{\beta} = N_iC_{jk}^iB_{\beta}^jN^k$$

The relative h and v-covariant derivatives of projection factor  $B_{\alpha}^i$  with respect to  $ICT\Gamma$  are given by

$$B_{\alpha|\beta}^i = H_{\alpha\beta}N^i, \quad B_{\alpha}^i|_{\beta} = M_{\alpha\beta}N^i$$

It is obvious from the equation (3.3) that  $H_{\beta\gamma}$  is generally not symmetric and

$$(3.5) \quad H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta}H_{\gamma} - M_{\gamma}H_{\beta}$$

The above equation yield

$$(3.6) \quad H_{0\gamma} = H_{\gamma}, \quad H_{\gamma 0} = H_{\gamma} + M_{\gamma}H_0$$

We shall use following lemmas which are due to Matsumoto [8] in the coming section

**Lemma 3.1.** *The normal curvature  $H_0 = H_{\beta}v^{\beta}$  vanishes if and only if the normal curvature vector  $H_{\beta}$  vanishes.*

**Lemma 3.2.** *A hypersurface  $F^{(n-1)}$  is a hyperplane of the first kind with respect to connection  $C\Gamma$  if and only if  $H_{\alpha} = 0$ .*

**Lemma 3.3.** *A hypersurface  $F^{(n-1)}$  is a hyperplane of the second kind with respect to connection  $C\Gamma$  if and only if  $H_{\alpha} = 0$  and  $H_{\alpha\beta} = 0$ .*

**Lemma 3.4.** *A hypersurface  $F^{(n-1)}$  is a hyperplane of the third kind with respect to connection  $C\Gamma$  if and only if  $H_{\alpha} = 0$  and  $H_{\alpha\beta} = M_{\alpha\beta} = 0$ .*

## 4 Hypersurface $F^{(n-1)}(c)$ of a Finsler space with Randers type exponential form of $(\alpha, \beta)$ -metric

Let us consider a Finsler space for Randers type exponential form of  $(\alpha, \beta)$ -metric  $L(\alpha, \beta) = (\alpha + \beta)e^{\frac{\beta}{\alpha + \beta}}$ , where,  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a Riemannian metric and vector field  $b_i(x) = \frac{\partial b}{\partial x^i}$  is a gradient of some scalar function  $b(x)$ . Now we consider a hypersurface  $F^{(n-1)}(c)$  given by equation  $b(x) = c$ , a constant [11].

From the parametric equation  $x^i = x^i(u^{\alpha})$  of  $F^{(n-1)}(c)$ , we get

$$\frac{\partial b(x)}{\partial u^{\alpha}} = 0$$

$$\frac{\partial b(x)}{\partial x^i} \frac{\partial x^i}{\partial u^{\alpha}} = 0$$

$$b_i B_{\alpha}^i = 0$$

Above shows that  $b_i(x)$  are covariant component of a normal vector field of hypersurface  $F^{n-1}(c)$ . Further, we have

$$(4.1) \quad b_i B_\alpha^i = 0 \quad \text{and} \quad b_i y^i = 0 \quad \text{i.e.} \quad \beta = 0$$

and induced metric  $L(u, v)$  of  $F^{n-1}(c)$  is given by

$$(4.2) \quad L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B_\alpha^i B_\beta^j$$

which is a Riemannian metric.

Writing  $\beta = 0$  in the equations (2.2), (2.3) and (2.5) we get

$$(4.3) \quad \begin{aligned} p &= 1, & q_0 &= 1, & q_{-1} &= 0 & q_{-2} &= -\alpha^{-2} \\ p_0 &= 5 & p_{-1} &= 2\alpha^{-1} & p_{-2} &= 0 & \tau &= 1 + b^2, \\ s_0 &= \frac{1}{1+b^2} & s_{-1} &= \frac{2}{\alpha(1+b^2)} & s_{-2} &= -\frac{4b^2}{\alpha^2(1+b^2)} \end{aligned}$$

from (2.4) we get,

$$(4.4) \quad g^{ij} = a^{ij} - \frac{1}{1+b^2} b^i b_j - \frac{2}{\alpha(1+b^2)} (b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2(1+b^2)} y^i y^j$$

thus along the Finslerian Hypersurface  $F^{n-1}(c)$  with equations (4.4) and (4.1) leads to

$$g^{ij} b_i b_j = \frac{b^2}{(1+b^2)}$$

So we get

$$(4.5) \quad b_i(x(u)) = \sqrt{\frac{b^2}{(1+b^2)}} N_i, \quad b^2 = a^{ij} b_i b_j$$

where  $b$  is the length of the vector  $b^i$

Again from equations (4.4) and (4.5), we get

$$(4.6) \quad b^i = a^{ij} b_j = b \sqrt{1+b^2} N^i + \frac{2b^2 y^i}{\alpha}$$

thus we have,

**Theorem 4.1.** *The Induced Riemannian metric in a Finsler hypersurface  $F^{(n-1)}(c)$  of a Finsler space  $F^{(n)}(c)$  equipped with Randers type exponential form metric defined in equation (2.1) is given by (4.2) and the scalar function  $b(x)$  is given by (4.5) and (4.6).*

Now the angular metric tensor  $h_{ij}$  and the fundamental metric tensor  $g_{ij}$  of  $F^n$  are given by

$$(4.7) \quad h_{ij} = a_{ij} + b_i b_j - \frac{1}{\alpha^2} Y_i Y_j \quad \text{and} \quad g_{ij} = a_{ij} + 5b_i b_j + \frac{2}{\alpha} (b_i Y_j + b_j Y_i)$$

From equations (4.1), (4.7) and (3.2) it follows that if  $h_{\alpha\beta}^{(a)}$  denote the angular metric tensor of the Riemannian  $a_{ij}(x)$  then we have along  $F_{(c)}^{n-1}$ ,  $h_{\alpha\beta} = h_{\alpha\beta}^{(a)}$ .

$$\text{thus along } F_{(c)}^{n-1}, \quad \frac{\partial p_0}{\partial \beta} = \frac{4}{\alpha}$$

from equation (2.8) we get

$$\gamma_1 = -\frac{2}{\alpha}, \quad m_i = b_i$$

then hv-torsion tensor becomes

$$(4.8) \quad C_{ijk} = \frac{1}{\alpha} (h_{ij} b_k + h_{jk} b_i + h_{ki} b_j) - \frac{1}{\alpha} b_i b_j b_k$$

in the Randers type exponential form of  $(\alpha, \beta)$ -metric of a Finsler hypersurface  $F_{(c)}^{(n-1)}$ . Due to fact from equations (3.2), (3.3), (3.5), (4.1) and (4.8) we have

$$(4.9) \quad M_{\alpha\beta} = \frac{1}{\alpha} \sqrt{\frac{b^2}{(1+b^2)}} h_{\alpha\beta} \quad \text{and} \quad M_\alpha = 0$$

Therefore from equation (3.6) it follows that  $H_{\alpha\beta}$  is symmetric. Thus we have

**Theorem 4.2.** *The second fundamental v-tensor for a Finsler hypersurface  $F_{(c)}^{(n-1)}$  of a Finsler space equipped with Randers type exponential form metric defined in equation (2.1) is given by (4.9) and the second fundamental h-tensor  $H_{\alpha\beta}$  is symmetric.*

Now from equation (4.1) we have  $b_i B_\alpha^i = 0$ . Then we have

$$b_{i|\beta} B_\alpha^i + b_i B_{\alpha|\beta}^i = 0$$

Therefore, from (3.5) and using  $b_{i|\beta} = b_{i|j} B_\beta^j + b_i |_{j} N^j H_\beta$ , we have

$$(4.10) \quad b_{i|j} B_\alpha^i B_\beta^j + b_{i|j} B_\alpha^i N^j H_\beta + b_i H_{\alpha\beta} N^i = 0$$

since  $b_i |_{j} = -b_h C_{ij}^h$ , we get

$$b_{i|j}B_\alpha^iN^j = 0$$

Therefore from equation (4.10) we have,

$$(4.11) \quad \sqrt{\frac{b^2}{(1+b^2)}}H_{\alpha\beta} + b_{i|j}B_\alpha^iB_\beta^j = 0$$

because  $b_{i|j}$  is symmetric. Now contracting (4.11) with  $v^\beta$  and using (3.1) we get

$$(4.12) \quad \sqrt{\frac{b^2}{(1+b^2)}}H_\alpha + b_{i|j}B_\alpha^iy^j = 0$$

Again contracting by  $v^\alpha$  equation (4.12) and using (3.1), we have

$$(4.13) \quad \sqrt{\frac{b^2}{(1+b^2)}}H_0 + b_{i|j}y^iy^j = 0$$

From lemma (3.1) and (3.2), it is clear that Randers type exponential  $(\alpha, \beta)$ -metric with Finsler hypersurface  $F^{(n-1)}(c)$  is a hyperplane of first kind if and only if  $H_0 = 0$ . Thus from (4.13) it is obvious that  $F^{n-1}(c)$  is a hyperplane of first kind if and only if  $b_{i|j}y^iy^j = 0$ . This  $b_{i|j}$  being the covariant derivative with respect to  $C\Gamma$  of  $F^n$  defined on  $y^i$ , but  $b_{ij} = \nabla_j b_i$  is the covariant derivative with respect to Riemannian connection  $\{^i_{jk}\}$  constructed from  $a_{ij}(x)$ . Hence  $b_{ij}$  does not depend on  $y^i$ . We shall consider the difference  $b_{i|j} - b_{ij}$  where  $b_{ij} = \nabla_j b_i$  in the following. The difference tensor  $D_{jk}^i = \Gamma_{jk}^{*i} - \{^i_{jk}\}$  is given by (2.10). Since  $b_i$  is a gradient vector, then from (2.9) we have

$$E_{ij} = b_{ij} \quad F_{ij} = 0 \quad \text{and} \quad F_j^i = 0$$

Thus (2.10) reduces to

$$(4.14) \quad D_{jk}^i = B^i b_{jk} + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i)$$

where

$$(4.15) \quad B_i = 5b_i + 2\alpha^{-1}Y_i, \quad B^i = \frac{5}{(1+b^2)}b^i + \frac{2(1-b^2)}{\alpha(1+b^2)}y^i, \\ B_i B^i = \frac{4+21b^2}{1+b^2}, \quad \lambda^m = B^m b_{00}, \quad B_{ij} = \frac{1}{\alpha}(a_{ij} - \frac{Y_i Y_j}{\alpha^2}) + \frac{2}{\alpha}b_i b_j, \\ B_j^i = \frac{1}{\alpha}\{(\delta_j^i - \alpha^{-2}y^i Y_j) + \frac{2}{(1+b^2)}b^i b_j - \frac{4b^2}{\alpha(1+b^2)}b_j y^i\}, \\ A_k^m = B_k^m b_{00} + B^m b_{k0}$$

In view of (4.3) and (4.4), the relation in (2.11) becomes to by virtue of (4.15) we have  $B_0^i = 0$ ,  $B_{i0} = 0$  which leads  $A_0^m = B^m b_{00}$ .

Now contracting (4.14) by  $y^k$  we get

$$D_{j0}^i = B^i b_{j0} + B_j^i b_{00} - B^m C_{jm}^i b_{00}$$

Again contracting the above equation with respect to  $y^j$  we have

$$D_{00}^i = B^i b_{00} = \left\{ \frac{5}{(1+b^2)} b^i + \frac{2(1-b^2)}{\alpha(1+b^2)} y^i \right\} b_{00}$$

Paying attention to (4.1), along  $F_{(c)}^{(n-1)}$ , we get

$$(4.16) \quad b_i D_{j0}^i = \frac{5b^2}{(1+b^2)} b_{j0} + \frac{1+3b^2}{\alpha(1+b^2)} b_j b_{00} - \frac{5}{1+b^2} b_i b^m C_{jm}^i b_{00}$$

Now we contract (4.16) by  $y^j$  we have

$$(4.17) \quad b_i D_{00}^i = \frac{5b^2}{1+b^2} b_{00}$$

From (3.3), (4.5), (4.6), (4.9) and  $M_\alpha = 0$ , we have

$$b_i b^m C_{jm}^i B_\alpha^j = \frac{b^2}{(1+b^2)} M_\alpha = 0$$

Thus the relation  $b_{i|j} = b_{ij} - b_r D_{ij}^r$  the equation (4.16) and (4.17) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D_{00}^r = \frac{1-4b^2}{1+b^2} b_{00}$$

Consequently (4.12) and (4.13) may be written as

$$(4.18) \quad \begin{aligned} \sqrt{\frac{b^2}{1+b^2}} H_\alpha + \frac{1-4b^2}{1+b^2} b_{i0} B_\alpha^i &= 0, \\ \sqrt{\frac{b^2}{1+b^2}} H_0 + \frac{1-4b^2}{1+b^2} b_{00} &= 0 \end{aligned}$$

Thus the condition  $H_0 = 0$  is equivalent to  $b_{00} = 0$ . Using the fact  $\beta = b_i y^i = 0$  the condition  $b_{00} = 0$  can be written as  $b_{ij} y^i y^j = b_i y^i b_j y^j$  for some  $c_j(x)$ . Thus we can write,

$$(4.19) \quad 2b_{ij} = b_i c_j + b_j c_i$$

Now from (4.1) and (4.19) we get

$$b_{00} = 0, \quad b_{ij} B_\alpha^i B_\beta^j = 0, \quad b_{ij} B_\alpha^i y^j = 0$$

Hence from (4.18) we get  $H_\alpha = 0$ , again from (4.19) and (4.15) we get  $b_{i0}b^i = \frac{c_0b^2}{2}$ ,  $\lambda^m = 0$ ,  $A_j^iB_\beta^j = 0$  and  $B_{ij}B_\alpha^iB_\beta^j = \frac{1}{\alpha}h_{\alpha\beta}$ .

Now we use equation (3.3), (4.4), (4.5), (4.6), (4.9) and (4.14) then we have

$$(4.20) \quad b_r D_{ij}^r B_\alpha^i B_\beta^j = -\frac{c_0 b^4}{2\alpha(1+b^2)} h_{\alpha\beta}$$

Thus the equation (4.11) reduces to

$$(4.21) \quad H_{\alpha\beta} + \frac{c_0 b^3}{2\alpha(1+b^2)^{3/2}} h_{\alpha\beta} = 0$$

Hence the hypersurface  $F_{(c)}^{n-1}$  is umbilic.

**Theorem 4.3.** *The necessary and sufficient condition for Finsler hypersurface  $F^{(n-1)}(c)$  of a Finsler space  $F^{(n)}(c)$  equipped with a Randers type exponential form metric defined in equation (2.1) will be a hyperplane of first kind is given by (4.19).*

It is also clear from (4.19) that

**Corollary 4.1.** *The second fundamental h-tensor for a Finsler hypersurface  $F^{(n-1)}(c)$  of a Finsler space  $F^{(n)}(c)$  equipped with a Randers type exponential form metric defined in equation (2.1) is directly proportional to its angular metric tensor.*

Now from lemma (3.3),  $F^{(n-1)}(c)$  is a hyperplane of second kind if and only if  $H_\alpha = 0$  and  $H_{\alpha\beta} = 0$ . Thus from (4.20), we get

$$c_0 = c_i(x)g^i = 0$$

Therefore there exist a function  $\psi(x)$  such that

$$c_i(x) = \psi(x)b_i(x)$$

Therefore (4.19) we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x)b_i b_j$$

**Theorem 4.4.** *The necessary and sufficient condition for a Finsler hypersurface  $F^{(n-1)}(c)$  of a Finsler space  $F^{(n)}(c)$  equipped with a Randers type exponential form metric defined in equation (2.1) to be a hyperplane of second kind is given by (4.21).*

Again lemma (4.4), together with (4.9) and  $M_\alpha = 0$  shows that  $F^{n-1}(c)$  does not become a hyperplane of third kind. Thus

**Theorem 4.5.** *The Finsler hypersurface  $F^{(n-1)}(c)$  of a Finsler space  $F^{(n)}(c)$  equipped with a Randers type exponential form metric defined in equation (2.1) is not a hyperplane of the third kind.*

## 5 Some Important Result of Hypersurfaces $F^{(n-1)}(c)$ of a Finsler Space $F^n(c)$ with Randers type exponential form of $(\alpha, \beta)$ -Metric

The hv-torsion tensor  $C_{ijk}$  of  $F^{(n-1)}(c)$  with Randers type exponential form of  $(\alpha, \beta)$  Metric written in equation (4.8) as

$$C_{ijk} = \frac{1}{\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) - \frac{1}{\alpha}b_i b_j b_k$$

Contracting by  $g^{jk}$ , we have

$$C_i = C_{ijk}g^{jk} = \frac{(2+3b^2)}{\alpha(1+b^2)}b_i$$

This implies that

$$(5.1) \quad b_i = \frac{\alpha(1+b^2)}{(2+3b^2)}C_i$$

Thus the equation (4.8) written as

$$(5.2) \quad C_{ijk} = \frac{(1+b^2)}{(2+3b^2)}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) - \frac{\alpha^2(1+b^2)^3}{(2+3b^2)^3}C_i C_j C_k$$

**Definition 5.1.** *A Finsler Space  $F^n$  is Called semi-C-reducible, if the (h) hv-torsion tensor  $C_{ijk}$  is written in the form*

$$(5.3) \quad C_{ijk} = \frac{p}{(n+1)}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{q}{C^2}C_i C_j C_k.$$

Now from equations (5.2) and (5.3) we have

$$(5.4) \quad p = \frac{(n+1)(1+b^2)}{(2+3b^2)} \quad \text{and} \quad q = \frac{-\alpha^2 C^2 (1+b^2)^3}{(2+3b^2)}$$

Thus

**Proposition 5.1.** *The Finslerian Hypersurface  $F^{(n-1)}(c)$  of Finsler Space  $F^n$  equipped with a Randers type exponential form metric defined in equation (2.1) is always to be a semi-C-reducible Finsler space if equation (5.4) satisfied.*

Further now contracting equation (4.8) by  $B_\alpha^i$  and using equation (4.1) we have

$$(5.5) \quad C_{ijk}B_\alpha^i = \frac{1}{\alpha}(h_{ij}b_kB_\alpha^i + h_{ki}b_jB_\alpha^i)$$

Contracting equation (5.5) by  $B_\beta^j$  and using equation (4.1) we have

$$C_{ijk}B_\alpha^iB_\beta^j = \frac{1}{\alpha}h_{ij}b_kB_\alpha^iB_\beta^j$$

This implies that

$$(5.6) \quad C_{ijk}B_\alpha^iB_\beta^j = \frac{1}{\alpha}h_{\alpha\beta}b_k$$

Contracting equation (5.6) and using equation (4.1) we have

$$(5.7) \quad C_{ijk}B_\alpha^iB_\beta^jB_\gamma^k = C_{\alpha\beta\gamma} = 0$$

**Proposition 5.2.** *The Finslerian Hypersurface  $F^{(n-1)}(c)$  of Finsler Space  $F^n$  equipped with a Randers type exponential form metric defined in equation (2.1) is a C-reducible Finsler space if equation (5.7) satisfied.*

**Definition 5.2.** *A Finsler space  $F^n$  is called C2-like, if the (h) hv-tortion tensor  $C_{ijk}$  is written in the form*

$$C_{ijk} = \frac{1}{C^2}C_iC_jC_k$$

Thus using above definition and equation (5.9) we have

$$(5.8) \quad C = \frac{I^{\frac{3}{2}}b^3}{L(1+b^2)^{\frac{3}{2}}}$$

**Proposition 5.3.** *The Finslerian hypersurface  $F^{(n-1)}(c)$  in a two dimensional Finsler space  $F^n$  equipped with Randers type exponential form metric defined in equation (2.1) is always to be a C2-like Finsler space if equation (5.8) is satisfied.*

Since the main scalar of two dimensional Finsler space is defined as

$$LC_{ijk} = Im_im_jm_k$$

Since  $m_i = b_i$  then we have

$$LC_{ijk} = Ib_i b_j b_k$$

Contracting  $g^{jk}$  we have

$$(5.9) \quad LC_i = \frac{Ib^2}{(1+b^2)} b_i$$

Now main scalar of two dimensional Finsler space

$$(5.10) \quad C_{ijl} = \frac{L^2(1+b^2)^3}{I^3 b^6} C_i C_j C_k$$

Now from equation (5.1) and (5.9) we have

$$(5.11) \quad I = \frac{L(2+3b^2)}{\alpha b^2}$$

**Proposition 5.4.** *The main scalar  $I$  in a two dimensional case of a Finslerian hypersurface  $F^{(n-1)}(c)$  for the Finsler space  $F^n(c)$  equipped with Randers type exponential form metric defined in equation (2.1) is given by (5.11).*

## 6 Conclusion

In the present paper we introduced a Randers type exponential form metric in equation (2.1). Further we obtained the necessary and sufficient condition for a Finslerian Hypersurface  $F^{(n-1)}(c)$  of a Finsler space  $F^n(c)$  equipped Randers type exponential form metric will be hyperplane of first, second and third kind in the theorem (4.3), (4.4) and (4.5) respectively.

Further as application point of view for the above exponential form of metric we obtain a proposition (5.1), (5.2), (5.3) in which it stated that this hypersurface will be semi-C-reducible, C-reducible Finsler Space, C2 like Finsler Space in certain conditions. Further in proposition (5.4) we also obtained the value of main scalar  $I$  in two- dimensional case.

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