

LOCAL AND 2-LOCAL AUTOMORPHISMS OF FINITE-DIMENSIONAL NILPOTENT ASSOCIATIVE ALGEBRAS

F.N.ARZIKULOV, I.A.KARIMJANOV, AND S.M.UMRZAQOV

ABSTRACT. In the present paper automorphisms, local and 2-local automorphisms of n -dimensional null-filiform and filiform associative algebras are studied. Namely, a common form of the matrix of automorphisms and local automorphisms of these algebras is clarified. It turns out that the common form of the matrix of an automorphism on these algebras does not coincide with the local automorphism's matrices common form on these algebras. Therefore, these associative algebras have local automorphisms that are not automorphisms. Also, that each 2-local automorphism of these algebras is an automorphism is proved.

INTRODUCTION

The Gleason-Kahane-Żelazko theorem [10, 13], which is a fundamental contribution to the theory of Banach algebras, asserts that every unital linear local homomorphism from an unital complex Banach algebra A into \mathbb{C} is multiplicative. We recall that a linear map T from a Banach algebra A into a Banach algebra B is said to be a local homomorphism if for every a in A there exists a homomorphism $\Phi_a : A \rightarrow B$, depending on a , such that $T(a) = \Phi_a(a)$.

Later, in [12], R. Kadison introduces the concept of local derivations and proves that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation. B. Jonson [11] extends the above result by proving that every local derivation from a C^* -algebra into its Banach bimodule is a derivation. In particular, Johnson gives an automatic continuity result by proving that local derivations of a C^* -algebra A into a Banach A -bimodule X are continuous even if not assumed a priori to be so (cf. [11, Theorem 7.5]). Based on these results, many authors have studied local derivations on operator algebras.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [17] as 2-local automorphisms. He described such maps on the algebra $B(H)$ of all bounded linear operators on an infinite dimensional separable Hilbert space H .

The first results concerning local derivations and automorphisms on finite-dimensional Lie algebras were obtained in [2]. Namely, in [2] the authors have proved that every local derivation on semi-simple Lie algebras is a derivation

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and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. Sh.A.Ayupov, K.K.Kudaybergenov, B.A.Omirov proved similar results concerning local derivations and automorphisms on simple Leibniz algebras in their recent paper [3]. Local automorphisms of certain finite-dimensional simple Lie and Leibniz algebras are investigated in [4]. Concerning local automorphism, T.Becker, J.Escobar, C.Salas, and R.Turdibaev in [7] established that the set of local automorphisms $LAut(sl_2)$ coincides with the group $Aut^\pm(sl_2)$ of all automorphisms and anti-automorphisms. Later in [9] M.Costantini proved that a linear map on a simple Lie algebra is a local automorphism if and only if it is either an automorphism or an anti-automorphism. Similar results concerning local derivations and automorphisms on Lie superalgebras were obtained in [8, 18] and [19].

In the paper [5], local derivations of solvable Lie algebras are studied, and it is proved that in the class of solvable Lie algebras, there exist algebras that admit local derivations which are not derivation. Also, algebras, every local derivation of which is a derivation, are found. Moreover, every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and the maximal dimension of complementary space is a derivation. Sh.A.Ayupov, A.Kh.Khudoyberdiyev, and B.B.Yusupov proved similar results concerning local derivations on solvable Leibniz algebras in their recent papers [6, 20]. F.N.Arzikulov, I.A.Karimjanov, and S.M.Umrzaqov established that every local and 2-local automorphisms on the solvable Leibniz algebras with null-filiform and naturally graded non-Lie filiform nilradicals, whose dimension of complementary space is maximal, is an automorphism [1].

In the paper [15], I.A.Karimjanov, S.M.Umrzaqov, and B.Yusupov describe automorphisms, local and 2-local automorphisms of solvable Leibniz algebras with a model or abelian null-radicals. They show that any local automorphisms on solvable Leibniz algebras with a model nilradical, the dimension of the complementary space of which is maximal, is an automorphism. But solvable Leibniz algebras with an abelian nilradical with a 1-dimensional complementary space admit local automorphisms which are not automorphisms.

In the present paper automorphisms, local and 2-local automorphisms of n -dimensional filiform and null-filiform associative algebras are studied. Namely, a common form of the matrix of automorphisms and local automorphisms of these algebras is clarified. It turns out that the common form of the matrix of an automorphism on these algebras does not coincide with the local automorphism's matrix's common form on these algebras. Therefore, these associative algebras have local automorphisms that are not automorphisms. Also, that each 2-local automorphism of these algebras is an automorphism is proved.

1. PRELIMINARIES

Null-filiform and filiform associative algebras. For an algebra \mathcal{A} of an arbitrary variety, we consider the series

$$\mathcal{A}^1, \quad \mathcal{A}^{i+1} = \sum_{k=1}^i \mathcal{A}^k \mathcal{A}^{i-k+1}, \quad i \geq 1.$$

We say that an algebra \mathcal{A} is nilpotent if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathcal{A}^i = 0$ is called the index of nilpotency or nilindex of \mathcal{A} .

Definition. An n -dimensional algebra \mathcal{A} is called null-filiform $\dim \mathcal{A}^i = (n+1) - i, 1 \leq i \leq n+1$.

Theorem 1.1 ([16]). *An arbitrary n -dimensional null-filiform associative algebra is isomorphic to the following algebra:*

$$\mu_0 : \quad e_i e_j = e_{i+j}, 2 \leq i+j \leq n,$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra \mathcal{A} and the omitted products vanish.

Definition. An n -dimensional algebra \mathcal{A} is called filiform if $\dim(\mathcal{A}^i) = n - i, 2 \leq i \leq n$.

Theorem 1.2 ([14]). *For $n > 3$ every n -dimensional filiform associative algebra over an algebraically closed field \mathbb{F} of characteristic zero is isomorphic to one of the following pairwise non-isomorphic algebras with a basis $\{e_1, e_2, \dots, e_n\}$:*

$$\mu_{1,1} : \quad e_i e_j = e_{i+j},$$

$$\mu_{1,2} : \quad e_i e_j = e_{i+j}, e_n e_n = e_{n-1},$$

$$\mu_{1,3} : \quad e_i e_j = e_{i+j}, e_1 e_n = e_{n-1},$$

$$\mu_{1,4} : \quad e_i e_j = e_{i+j}, e_1 e_n = e_n e_n = e_{n-1}$$

where $2 \leq i+j \leq n-1$.

 2. DESCRIPTION OF AUTOMORPHISMS OF FINITE-DIMENSIONAL
 NULL-FILIFORM AND FILIFORM ASSOCIATIVE ALGEBRAS

Here we give a description of automorphisms of the associative algebras from the theorems 1.1 and 1.2.

Theorem 2.1. *A linear map $\varphi : \mu_0 \rightarrow \mu_0$ is an automorphism of the algebra μ_0 if and only if the map φ has the following form:*

$$\varphi(e_1) = \sum_{i=1}^n a_i e_i,$$

$$\varphi(e_i) = \sum_{j=i}^n \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j, \quad 2 \leq i \leq n, \quad (1)$$

where $a_1 \neq 0$.

Proof. Let

$$\varphi(e_1) = \sum_{i=1}^n a_i e_i.$$

Then

$$\begin{aligned} \varphi(e_2) &= \varphi(e_1 e_1) = \varphi(e_1) \varphi(e_1) \\ &= \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n a_i e_i \right) = \left(\sum_{i=1}^n a_i e_i \right)^2 \\ &= \sum_{j=2}^n \left(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2} \right) e_j. \end{aligned}$$

Also we have

$$\begin{aligned} \varphi(e_3) &= \varphi(e_1 e_2) = \varphi(e_1) \varphi(e_2) = \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n a_i e_i \right)^2 \\ &= \left(\sum_{i=1}^n a_i e_i \right)^3 = \sum_{j=3}^n \left(\sum_{k_1+k_2+k_3=j} a_{k_1} \cdot a_{k_2} \cdot a_{k_3} \right) e_j. \end{aligned}$$

Similarly, for any $i = 2, 3, \dots, n$, we have

$$\begin{aligned} \varphi(e_i) &= \varphi(e_1 e_{i-1}) = \varphi(e_1) \varphi(e_{i-1}) = \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n a_i e_i \right)^{i-1} \\ &= \left(\sum_{i=1}^n a_i e_i \right)^i = \sum_{j=i}^n \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j. \end{aligned}$$

The proof is complete. \square

Theorem 2.2. *A linear map $\varphi : \mu_{1,1} \rightarrow \mu_{1,1}$ is an automorphism if and only if the map φ has the following form:*

$$\begin{aligned} \varphi(e_1) &= \sum_{i=1}^n a_i e_i, \\ \varphi(e_i) &= \sum_{j=i}^{n-1} \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j, \quad 2 \leq i \leq n-1 \\ \varphi(e_n) &= b_{n-1} e_{n-1} + b_n e_n \end{aligned}$$

where $a_1 \neq 0$.

Proof. Let

$$\begin{aligned} \varphi(e_1) &= \sum_{i=1}^n a_i e_i, \\ \varphi(e_n) &= \sum_{i=1}^n b_i e_i. \end{aligned}$$

Then, by the table of multiplication of the algebra $\mu_{1,1}$ and equalities

$$\varphi(e_i) = \varphi(e_1 e_{i-1}), \quad 2 \leq i \leq n-1,$$

we have

$$\begin{aligned} \varphi(e_2) &= \varphi(e_1 e_1) = \varphi(e_1) \varphi(e_1) = \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n a_i e_i \right) = \left(\sum_{i=1}^n a_i e_i \right)^2 = \\ &= \sum_{j=2}^{n-1} \left(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2} \right) e_j, \end{aligned}$$

$$\begin{aligned} \varphi(e_3) &= \varphi(e_1 e_2) = \varphi(e_1) \varphi(e_2) = \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n a_i e_i \right)^2 = \\ &= \left(\sum_{i=1}^n a_i e_i \right)^3 = \sum_{j=3}^{n-1} \left(\sum_{k_1+k_2+k_3=j} a_{k_1} \cdot a_{k_2} \cdot a_{k_3} \right) e_j, \end{aligned}$$

and so on

$$\begin{aligned} \varphi(e_i) &= \varphi(e_1 e_{i-1}) = \varphi(e_1) \varphi(e_{i-1}) = \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n a_i e_i \right)^{i-1} = \\ &= \left(\sum_{i=1}^n a_i e_i \right)^i = \sum_{j=i}^{n-1} \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j, \quad 2 \leq i \leq n-1. \end{aligned}$$

We also have

$$\begin{aligned} 0 &= \varphi(e_1 e_n) = \varphi(e_1) \varphi(e_n) = \left(\sum_{i=1}^n a_i e_i \right) \left(\sum_{i=1}^n b_i e_i \right) = \\ &= \sum_{j=1}^{n-2} \sum_{i=j+1}^{n-1} a_j b_{i-1} e_i. \end{aligned}$$

From this it follows that

$$b_i = 0, \quad 1 \leq i \leq n-2, \quad b_{n-1} \neq 0, b_n \neq 0.$$

So,

$$\varphi(e_n) = b_{n-1} e_{n-1} + b_n e_n.$$

The proof is complete. \square

We can similarly prove the following theorems.

Theorem 2.3. *A linear map $\varphi : \mu_{1,2} \rightarrow \mu_{1,2}$ is an automorphism if and only if the map φ has the following form:*

$$\begin{aligned} \varphi(e_1) &= \sum_{i=1}^n a_i e_i, \\ \varphi(e_2) &= \sum_{j=2}^{n-1} \left(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2} \right) e_j + a_n^2 e_{n-1} \end{aligned}$$

$$\begin{aligned}\varphi(e_i) &= \sum_{j=i}^{n-1} \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j, \quad 3 \leq i \leq n-1 \\ \varphi(e_n) &= -a_n \sqrt{a_1^{n-3}} e_{n-2} + b_{n-1} e_{n-1} + \sqrt{a_1^{n-1}} e_n\end{aligned}$$

where $a_1 \neq 0$.

Theorem 2.4. *A linear map $\varphi : \mu_{1,3} \rightarrow \mu_{1,3}$ is an automorphism if and only if the map φ has the following form:*

$$\begin{aligned}\varphi(e_1) &= \sum_{i=1}^n a_i e_i, \\ \varphi(e_2) &= \sum_{j=2}^{n-1} \left(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2} \right) e_j + a_1 a_n e_{n-1} \\ \varphi(e_i) &= \sum_{j=i}^{n-1} \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j, \quad 3 \leq i \leq n-1 \\ \varphi(e_n) &= b_{n-1} e_{n-1} + a_1^{n-2} e_n\end{aligned}$$

where $a_1 \neq 0$.

Theorem 2.5. *A linear map $\varphi : \mu_{1,4} \rightarrow \mu_{1,4}$ is an automorphism if and only if the map φ has the following form:*

$$\begin{aligned}\varphi(e_1) &= \sum_{i=1}^n a_i e_i, \\ \varphi(e_2) &= \sum_{j=2}^{n-1} \left(\sum_{k_1+k_2=j} a_{k_1} \cdot a_{k_2} \right) e_j + (a_1 a_n + a_n^2) e_{n-1}, \\ \varphi(e_i) &= \sum_{j=i}^{n-1} \left(\sum_{k_1+k_2+\dots+k_i=j} a_{k_1} \cdot a_{k_2} \cdot \dots \cdot a_{k_i} \right) e_j, \quad 3 \leq i \leq n-1, \\ \varphi(e_n) &= -a_n e_{n-2} + b_{n-1} e_{n-1} + e_n\end{aligned}$$

where $a_1 = 1$.

3. DESCRIPTION OF LOCAL AUTOMORPHISMS OF FINITE-DIMENSIONAL NULL-FILIFORM AND FILIFORM ASSOCIATIVE ALGEBRAS

Now we give a description of local automorphisms of the associative algebras from the theorems 1.1 and 1.2.

Definition. Let A be an algebra. A linear map $\Phi : A \rightarrow A$ is called a local automorphism, if for any element $x \in A$ there exists an automorphism $\varphi_x : A \rightarrow A$ such that $\Phi(x) = \varphi_x(x)$.

Theorem 3.1. *A linear map Φ is a local automorphism of μ_0 if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 0 \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}$$

Proof. Let Φ be an arbitrary local automorphism on μ_0 . By the definition, for any element $x \in \mu_0$, there exists an automorphism φ_x on μ_0 such that

$$\Phi(x) = \varphi_x(x).$$

By theorem 2.1, the automorphism φ_x has a matrix of the following form:

$$A_x = \begin{pmatrix} a_1^x & 0 & \dots & 0 & 0 \\ a_2^x & (a_1^x)^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^x & \sum_{k_1+k_2=n-1} a_{k_1}^x a_{k_2}^x & \dots & (a_1^x)^{n-1} & 0 \\ a_n^x & \sum_{k_1+k_2=n} a_{k_1}^x a_{k_2}^x & \dots & \sum_{k_1+k_2+\dots+k_{n-1}=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_{n-1}}^x & (a_1^x)^n \end{pmatrix}.$$

Let A be the matrix of Φ and

$$A = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

Then, by choosing subsequently $x = e_1, x = e_2, \dots, x = e_n$ and using $\Phi(x) = \varphi_x(x)$, i.e. $A\bar{x} = A_x\bar{x}$, where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to $x = x_1e_1 + \dots + x_n e_n$, we have $b_{i,j} = 0, i < j$, and $b_{k,k} \neq 0, 1 \leq k \leq n$, which implies

$$A = \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & 0 \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}$$

Now we prove that the linear operator, defined by the matrix A is a local automorphism. If, for each element $x \in \mu_0$, there exists a matrix A_x of the form in theorem 2.1 such that

$$A\bar{x} = A_x\bar{x}, \tag{1}$$

then the linear operator, defined by the matrix A is a local automorphism. In other words, if, for each element $x \in \mu_0$, the system of equations

$$\begin{cases} b_{1,1}x_1 = a_1^x x_1, \\ \sum_{j=1}^i b_{i,j}x_j = a_i^x x_1 + \sum_{j=2}^i \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j, \quad 2 \leq i \leq n, \end{cases} \quad (2)$$

obtained from (1.1), has a solution with respect to the variables

$$a_1^x, a_2^x, \dots, a_n^x,$$

then the linear operator, defined by the matrix A , is a local automorphism.

Let us consider the following cases

- If $x_1 \neq 0$ then $a_1^x = b_{1,1}, a_2^x = b_{2,1} + \frac{1}{x_1}(b_{2,2} - (a_1^x)^2)x_2$,
 $a_i^x = b_{i,1} + \frac{1}{x_1} \sum_{j=2}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j$, where $3 \leq i \leq n$
 $(a_1^x \neq 0)$.
- If $x_1 = 0$ and $x_2 \neq 0$ then $(a_1^x)^2 = b_{2,2}$,
 $a_{i-1}^x = \frac{1}{2a_1^x} (b_{i,2} - \sum_{k_1 \neq 1, k_2 \neq 1}^{k_1+k_2=i} a_{k_1} a_{k_2} + \frac{1}{x_2} \sum_{j=3}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j)$,
where $3 \leq i \leq n$, $(a_1^x \neq 0)$.
- If $x_1 = 0, x_2 = 0$ and $x_3 \neq 0$ then $(a_1^x)^3 = b_{3,3}$,
 $a_{i-2}^x = \frac{1}{3(a_1^x)^2} (b_{i,3} - \sum_{k_1+k_2+k_3=i}^{k_1+k_2 \neq 2} a_{k_1} a_{k_2} a_{k_3} + \frac{1}{x_3} \sum_{j=4}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j)$, where $4 \leq i \leq n$, $l_1, l_2 \in \{1, 2, 3\}$, $(a_1^x \neq 0)$.
- If $x_1 = x_2 = \dots = x_{m-1} = 0$ and $x_m \neq 0$ then $(a_1^x)^m = b_{m,m}$,
 $a_{i-m+1}^x = \frac{1}{m(a_1^x)^{m-1}} \left(b_{i,k} - \sum_{k_1+k_2+\dots+k_m=i}^{k_1+k_2+\dots+k_{l_1}+\dots+k_{l_{m-1}} \neq m-1} a_{k_1} a_{k_2} \dots a_{k_m} + \right.$
 $\left. + \frac{1}{x_k} \sum_{j=k+1}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x)x_j \right)$, where
 $m \leq i \leq n$, $l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$, $(a_1^x \neq 0)$.

Hence, the system of equation (2) always has a solution. Therefore, the linear operator, defined by the matrix A is a local automorphism. This completes the proof. \square

Theorem 3.2. *A linear map Φ is a local automorphism of $\mu_{1,1}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Proof. Let Φ be an arbitrary local automorphism on $\mu_{1,1}$. By the definition, for any element $x \in \mu_{1,1}$, there exists an automorphism φ_x on $\mu_{1,1}$ such that

$$\Phi(x) = \varphi_x(x).$$

By theorem 2.2, the automorphism φ_x has a matrix of the following form:

$$A_x = \begin{pmatrix} a_1^x & 0 & 0 & \dots & 0 & 0 \\ a_2^x & (a_1^x)^2 & 0 & \dots & 0 & 0 \\ a_3^x & \sum_{k_1+k_2=3} a_{k_1}^x a_{k_2}^x & (a_1^x)^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^x & \sum_{k_1+k_2=n-1} a_{k_1}^x a_{k_2}^x & \sum_{k_1+k_2+k_3=n-1} a_{k_1}^x a_{k_2}^x a_{k_3}^x & \dots & (a_1^x)^{n-1} & b_{n-1}^x \\ a_n^x & 0 & 0 & \dots & 0 & b_n^x \end{pmatrix}.$$

Let A be the matrix of Φ and

$$A = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

Then, by choosing subsequently $x = e_1, x = e_2, \dots, x = e_n$ and using $\Phi(x) = \varphi_x(x)$, i.e. $A\bar{x} = A_x\bar{x}$, where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to $x = x_1e_1 + \dots + x_ne_n$, we have $b_{i,j} = 0, i < j, i \neq n-1$, and $b_{k,k} \neq 0, 1 \leq k \leq n$, which implies

$$A = \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Similar to the proof of Theorem 3.1 we prove that, for each element $x \in \mu_{1,1}$, the system of equations

$$\begin{cases} b_{1,1}x_1 = a_1^x x_1, \\ \sum_{j=1}^i b_{i,j}x_j = a_i^x x_1 + \sum_{j=2}^i \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j, & 2 \leq i \leq n-2, \\ \sum_{j=1}^n b_{n-1,j}x_j = a_{n-1}^x x_1 + \sum_{j=2}^{n-1} \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j + b_{n-1}^x x_n, \\ b_{n,1}x_1 + b_{n,n}x_n = a_n^x x_1 + b_n^x x_n, \end{cases} \quad (3)$$

obtained from the equality $A\bar{x} = A_x\bar{x}$, has a solution with respect to the variables

$$a_1^x, a_2^x, \dots, a_n^x, b_{n-1}^x, b_n^x.$$

Let us consider the following cases

- If $x_1 \neq 0$ then $a_1^x = b_{1,1}$,

$$a_i^x = b_{i,1} + \frac{1}{x_1} \sum_{j=2}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j, \text{ where } 2 \leq i \leq n-2,$$

$$a_{n-1}^x = b_{n-1,n} + \frac{1}{x_1} \sum_{j=2}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + \frac{1}{x_1} (b_{n,n} - b_n^x) x_n,$$

$$a_n^x = b_{n,1} + \frac{1}{x_1} (b_{n,n} - b_n^x) x_n,$$
 where b_{n-1}^x and b_n^x are defined arbitrarily.
- If $x_1 = 0$ and $x_2 \neq 0$ then $(a_1^x)^2 = b_{2,2}$,

$$a_{i-1}^x = \frac{1}{2a_1^x} (b_{i,2} - \sum_{\substack{k_1 \neq 1, k_2 \neq 1 \\ k_1+k_2=i}} a_{k_1}^x a_{k_2}^x + \frac{1}{x_2} \sum_{j=3}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j),$$
 where $3 \leq i \leq n-2$, $(a_1^x \neq 0)$,

$$a_{n-2}^x = \frac{1}{2a_1^x} (b_{n-1,2} - \sum_{\substack{k_1 \neq 1, k_2 \neq 1 \\ k_1+k_2=n-1}} a_{k_1}^x a_{k_2}^x + \frac{1}{x_2} (\sum_{j=3}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-1,n} - b_{n-1}^x) x_n)),$$

$$b_n^x = b_{n,n}$$
 where a_n^x , a_{n-1}^x and b_{n-1}^x are defined arbitrarily.
- If $x_1 = 0, x_2 = 0$ and $x_3 \neq 0$ then $(a_1^x)^3 = b_{3,3}$,

$$a_{i-2}^x = \frac{1}{3(a_1^x)^2} (b_{i,3} - \sum_{\substack{k_{l_1}+k_{l_2} \neq 2 \\ k_1+k_2+k_3=i}} a_{k_1}^x a_{k_2}^x a_{k_3}^x + \frac{1}{x_3} \sum_{j=4}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j), \text{ where } 4 \leq i \leq n-2, l_1, l_2 \in \{1, 2, 3\}, (a_1^x \neq 0),$$

$$a_{n-3}^x = \frac{1}{3(a_1^x)^2} (b_{n-1,3} - \sum_{\substack{k_{l_1}+k_{l_2} \neq 2 \\ k_1+k_2+k_3=n-1}} a_{k_1}^x a_{k_2}^x a_{k_3}^x + \frac{1}{x_3} (\sum_{j=4}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-1,n} - b_{n-1}^x) x_n))$$
 where $l_1, l_2 \in \{1, 2, 3\}$, $(a_1^x \neq 0)$

$$b_n^x = b_{n,n}$$
 where a_n^x , a_{n-1}^x , a_{n-2}^x and b_{n-1}^x are defined arbitrarily.
- If $x_1 = x_2 = \dots = x_{m-1} = 0$ and $x_m \neq 0$ then $(a_1^x)^m = b_{m,m}$,

$$a_{i-m+1}^x = \frac{1}{m(a_1^x)^{m-1}} \left(b_{i,m} - \sum_{\substack{k_{l_1}+k_{l_2}+\dots+k_{l_{m-1}} \neq m-1 \\ k_1+k_2+\dots+k_m=i}} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x + \frac{1}{x_m} \sum_{j=m+1}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j \right), \text{ where } m \leq i \leq n-2,$$

$$l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}, (a_1^x \neq 0),$$

$$a_{n-m}^x = \frac{1}{m(a_1^x)^{m-1}} \left(b_{n-1,k} - \sum_{\substack{k_{l_1}+k_{l_2}+\dots+k_{l_{m-1}} \neq m-1 \\ k_1+k_2+\dots+k_m=n-1}} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x + \frac{1}{x_m} \sum_{j=m+1}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j \right),$$
 where $l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$, $(a_1^x \neq 0)$.

$$b_n^x = b_{n,n}$$
 where $a_n^x, a_{n-1}^x, \dots, a_{n-m+1}^x$ and b_{n-1}^x are defined arbitrarily.

Hence, the system of equation (3) always has a solution. Therefore, the linear operator, defined by the matrix A is a local automorphism. The proof is complete. \square

Theorem 3.3. *A linear map Φ is a local automorphism of $\mu_{1,2}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Proof. Let Φ be an arbitrary local automorphism on $\mu_{1,2}$. By the definition, for any element $x \in \mu_{1,2}$, there exists an automorphism φ_x on $\mu_{1,2}$ such that

$$\Phi(x) = \varphi_x(x).$$

By theorem 2.3, the automorphism φ_x has a matrix of the following form:

$$A_x = \begin{pmatrix} a_1^x & 0 & 0 & \dots & 0 & 0 \\ a_2^x & (a_1^x)^2 & 0 & \dots & 0 & 0 \\ a_3^x & \sum_{k_1+k_2=3} a_{k_1}^x a_{k_2}^x & (a_1^x)^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^x & \sum_{k_1+k_2=n-1} a_{k_1}^x a_{k_2}^x + (a_n^x)^2 & \sum_{k_1+k_2+k_3=n-1} a_{k_1}^x a_{k_2}^x a_{k_3}^x & \dots & (a_1^x)^{n-1} & b_{n-1}^x \\ a_n^x & 0 & 0 & \dots & 0 & \sqrt{(a_1^x)^{n-1}} \end{pmatrix}.$$

Let A be the matrix of Φ and

$$A = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & \dots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

Then, by choosing subsequently $x = e_1, x = e_2, \dots, x = e_n$ and using $\Phi(x) = \varphi_x(x)$, i.e. $A\bar{x} = A_x\bar{x}$, where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to $x = x_1e_1 + \dots + x_n e_n$, we have $b_{i,j} = 0, i < j, i \neq n-1$, and $b_{k,k} \neq 0, 1 \leq k \leq n$, which implies

$$A = \begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Similar to the proof of Theorem 3.1 we prove that, for each element $x \in \mu_{1,2}$, the system of equations

$$\left\{ \begin{array}{l} b_{1,1}x_1 = a_1^x x_1, \\ \sum_{j=1}^i b_{i,j}x_j = a_i^x x_1 + \sum_{j=2}^i \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j, \quad 2 \leq i \leq n-3, \\ \sum_{j=1}^{n-2} b_{n-2,j}x_j + b_{n-2,n}x_n = \\ = a_{n-2}^x x_1 + \sum_{j=2}^{n-2} \sum_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j - a_n^x \sqrt{(a_1^x)^{n-3}} x_n, \\ \sum_{j=1}^n b_{n-1,j}x_j = a_{n-1}^x x_1 + \sum_{j=2}^{n-1} \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x x_j + (a_n^x)^2 x_2 + b_{n-1}^x x_n, \\ b_{n,1}x_1 + b_{n,n}x_n = a_n^x x_1 + \sqrt{(a_1^x)^{n-1}} x_n, \end{array} \right. \quad (4)$$

obtained from the equality $A\bar{x} = A_x\bar{x}$, has a solution with respect to the variables

$$a_1^x, a_2^x, \dots, a_n^x, b_{n-1}^x.$$

Let us consider the following cases

- If $x_1 \neq 0$ then $a_1^x = b_{1,1}$,

$$a_i^x = b_{i,1} + \frac{1}{x_1} \sum_{j=2}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j, \text{ where } 2 \leq i \leq n-3,$$

$$a_{n-2}^x = b_{n-2,1} + \frac{1}{x_1} \sum_{j=2}^{n-2} (b_{n-2,j} - \sum_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + \frac{1}{x_1} (b_{n-2,n} - a_n^x \sqrt{(a_1^x)^{n-3}}) x_n, \text{ where } a_n^x \text{ is defined arbitrarily,}$$

$$a_{n-1}^x = b_{n-1,1} + \frac{1}{x_1} \sum_{j=2}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + \frac{1}{x_1} ((b_{n-1,n} - b_{n-1}^x) x_n - (a_n^x)^2 x_2), \text{ where } a_n^x, b_{n-1}^x \text{ are defined arbitrarily.}$$

$$a_n^x = b_{n,1} + \frac{1}{x_1} (b_{n,n} - \sqrt{(a_1^x)^{n-1}}) x_n.$$

- If $x_1 = 0$ and $x_2 \neq 0$ then $(a_1^x)^2 = b_{2,2}$,

$$a_{i-1}^x = \frac{1}{2a_1^x} (b_{i,2} - \sum_{k_1+k_2=i}^{k_1 \neq 1, k_2 \neq 1} a_{k_1}^x a_{k_2}^x + \frac{1}{x_2} \sum_{j=3}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j),$$

where $3 \leq i \leq n-3$, $(a_1^x \neq 0)$,

$$a_{n-3}^x = \frac{1}{2a_1^x} (b_{n-2,2} - \sum_{k_1+k_2=n-2}^{k_1 \neq 1, k_2 \neq 1} a_{k_1}^x a_{k_2}^x + \frac{1}{x_2} (\sum_{j=3}^{n-2} (b_{n-2,j} - \sum_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-2,n} + a_n^x \sqrt{(a_1^x)^{n-3}}) x_n)),$$

$$a_{n-2}^x = \frac{1}{2a_1^x} (b_{n-1,2} - \sum_{k_1+k_2=n-1}^{k_1 \neq 1, k_2 \neq 1} a_{k_1}^x a_{k_2}^x - a_n^x + \frac{1}{x_2} (\sum_{j=3}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-1,n} - b_{n-1}^x) x_n)),$$

$$a_1^x = (b_{n,n})^{\frac{2}{n-1}}.$$

- If $x_1 = 0, x_2 = 0$ and $x_3 \neq 0$ then $(a_1^x)^3 = b_{3,3}$,

$$a_{i-2}^x = \frac{1}{3(a_1^x)^2} (b_{i,3} - \sum_{\substack{k_1+k_2 \neq 2 \\ k_1+k_2+k_3=i}} a_{k_1}^x a_{k_2}^x a_{k_3}^x + \frac{1}{x_3} \sum_{j=4}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j),$$
 where $4 \leq i \leq n-3$, $l_1, l_2 \in \{1, 2, 3\}$,
 $(a_1^x \neq 0)$,

$$a_{n-4}^x = \frac{1}{3(a_1^x)^2} (b_{n-2,3} - \sum_{k_1+k_2+k_3=n-2}^{k_1+k_2 \neq 2} a_{k_1}^x a_{k_2}^x a_{k_3}^x + \frac{1}{x_3} (\sum_{j=4}^{n-2} (b_{n-2,j} - \sum_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-2,n} + a_n^x \sqrt{(a_1^x)^{n-3}} x_n))$$
 where $l_1, l_2 \in \{1, 2, 3\}$, $(a_1^x \neq 0)$

$$a_{n-3}^x = \frac{1}{3(a_1^x)^2} (b_{n-1,3} - \sum_{k_1+k_2+k_3=n-1}^{k_1+k_2 \neq 2} a_{k_1}^x a_{k_2}^x a_{k_3}^x + \frac{1}{x_3} (\sum_{j=4}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-1,n} - b_{n-1}^x) x_n))$$
 where $l_1, l_2 \in \{1, 2, 3\}$, $(a_1^x \neq 0)$
 $b_n^x = b_{n,n}$ where $a_n^x, a_{n-1}^x, a_{n-2}^x$ and b_{n-1}^x are defined arbitrarily.
- If $x_1 = x_2 = \dots = x_{m-1} = 0$ and $x_m \neq 0$ then $(a_1^x)^m = b_{m,m}$,

$$a_{i-m+1}^x = \frac{1}{m(a_1^x)^{m-1}} (b_{i,m} - \sum_{\substack{k_1+k_2+\dots+k_{m-1} \neq m-1 \\ k_1+k_2+\dots+k_m=i}} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x + \frac{1}{x_m} \sum_{j=m+1}^i (b_{i,j} - \sum_{k_1+k_2+\dots+k_j=i} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j),$$
 where $m+1 \leq i \leq n-3$, $l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$, $(a_1^x \neq 0)$,

$$a_{n-m-1}^x = \frac{1}{m(a_1^x)^{m-1}} (b_{n-2,m} - \sum_{k_1+k_2+\dots+k_m=n-2}^{k_1+k_2+\dots+k_{m-1} \neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x + \frac{1}{x_m} ((\sum_{j=m+1}^{n-2} (b_{n-2,j} - \sum_{k_1+k_2+\dots+k_j=n-2} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-2,n} + a_n^x \sqrt{(a_1^x)^{n-3}} x_n))$$
 where $l_1, l_2, \dots, l_{m-1} \in \{1, 2, \dots, m\}$, $(a_1^x \neq 0)$.

$$a_{n-m}^x = \frac{1}{m(a_1^x)^{m-1}} (b_{n-1,m} - \sum_{k_1+k_2+\dots+k_m=n-1}^{k_1+k_2+\dots+k_{m-1} \neq m-1} a_{k_1}^x a_{k_2}^x \dots a_{k_m}^x + \frac{1}{x_m} ((\sum_{j=m+1}^{n-1} (b_{n-1,j} - \sum_{k_1+k_2+\dots+k_j=n-1} a_{k_1}^x a_{k_2}^x \dots a_{k_j}^x) x_j + (b_{n-1,n} - b_{n-1}^x) x_n))$$
 where $a_n^x, a_{n-1}^x, \dots, a_{n-m+1}^x$ and b_{n-1}^x are defined arbitrarily.

Hence, the system of equation (4) always has a solution. Therefore, the linear operator, defined by the matrix A is a local automorphism. This ends the proof. \square

We can similarly prove the following theorems using theorems 2.4 and 2.5.

Theorem 3.4. *A linear map Φ is a local automorphism of $\mu_{1,3}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & b_{n,n} \end{pmatrix}$$

Theorem 3.5. *A linear map Φ is a local automorphism of $\mu_{1,4}$ if and only if the matrix of Φ has the following lower triangular form*

$$\begin{pmatrix} b_{1,1} & 0 & 0 & \dots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & \dots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Remark 3.6. Note that the common form of the matrix of a local automorphism on an algebra includes the common form of the matrix of an automorphism on this algebra. The coincidence of these common forms denotes that every local automorphism of the considering algebra is an automorphism. But the common form of the matrix of an automorphism on the associative algebras μ_0 , $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$ and $\mu_{1,4}$ does not coincide with the common form of the matrix of a local automorphism on these algebras by the appropriate theorems and theorems. Therefore, the associative algebras μ_0 , $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$ and $\mu_{1,4}$ have local automorphisms that are not automorphisms.

Also, note that local automorphisms of an arbitrary low-dimension algebra can be similarly described using a common form of the matrix of automorphisms on this algebra. A technique for constructing a local automorphism, which is not an automorphism, developed by us, can be applied to an arbitrary low-dimension algebra, automorphisms of which have a matrix of a common form.

4. DESCRIPTION OF 2-LOCAL AUTOMORPHISMS OF FINITE-DIMENSIONAL NULL-FILIFORM AND FILIFORM ASSOCIATIVE ALGEBRAS

Theorem 4.1. *Each 2-local automorphism of μ_0 is an automorphism.*

Proof. Let ϕ be an arbitrary 2-local automorphism of μ_0 . Then, by the definition, for every element $x \in \mu_0$,

$$x = \sum_{i=1}^n x_i e_i,$$

there exist a matrix A_{x,e_1}

$$A_{x,e_1} = \begin{pmatrix} a_1^{x,e_1} & 0 & \cdots & 0 & 0 \\ a_2^{x,e_1} & (a_1^{x,e_1})^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^{x,e_1} & \sum_{k_1+k_2=n-1} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} & \cdots & (a_1^{x,e_1})^{n-1} & 0 \\ a_n^{x,e_1} & \sum_{k_1+k_2=n} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} & \cdots & \sum_{k_1+k_2+\cdots+k_{n-1}=n-1} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} \cdots a_{k_{n-1}}^{x,e_1} & (a_1^{x,e_1})^n \end{pmatrix},$$

such that $\phi(x) = \widehat{A_{x,e_1} \bar{x}}$, where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to x , $\widehat{}$ is an operation on \bar{x} such that $\widehat{\widehat{x}} = x$, and

$$\phi(e_1) = \widehat{A_{x,e_1} \bar{e}_1} = (a_1^{x,e_1}, a_2^{x,e_1}, \widehat{a_3^{x,e_1}}, \dots, a_n^{x,e_1})^T.$$

Since $\phi(e_1) = \varphi_{x,e_1}(e_1) = \varphi_{y,e_1}(e_1)$, we have

$$\begin{aligned} \phi(e_1) &= (a_1^{x,e_1}, a_2^{x,e_1}, \widehat{a_3^{x,e_1}}, \dots, a_n^{x,e_1})^T = \\ &= (a_1^{y,e_1}, a_2^{y,e_1}, \widehat{a_3^{y,e_1}}, \dots, a_n^{y,e_1})^T \end{aligned}$$

for each pair, x, y of elements in μ_0 . Hence, $a_k^{x,e_1} = a_k^{y,e_1}$, $k = 1, 2, \dots, n$. Therefore

$$\phi(x) = \widehat{A_{y,e_1} \bar{x}}$$

for any $x \in \mu_0$, and the matrix of $\phi(x)$ does not depend on x . Hence ϕ is a linear operator, and the matrix of φ_{y,e_1} is the matrix of ϕ . Thus, by Proposition 2.1, ϕ is an automorphism. \square

Theorem 4.2. *Each 2-local automorphism of $\mu_{1,1}$ is an automorphism.*

Proof. Let ϕ be an arbitrary 2-local automorphism of $\mu_{1,1}$. Then, by the definition, for every element $x \in \mu_{1,1}$,

$$x = \sum_{i=1}^n x_i e_i,$$

there exist a matrix A_{x,e_1}

$$A_{x,e_1} = \begin{pmatrix} a_1^{x,e_1} & 0 & \cdots & 0 & 0 \\ a_2^{x,e_1} & (a_1^{x,e_1})^2 & \cdots & 0 & 0 \\ a_3^{x,e_1} & \sum_{k_1+k_2=3} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} & \cdots & (a_1^{x,e_1})^3 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^{x,e_1} & \sum_{k_1+k_2=n-1} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} & \cdots & \sum_{k_1+k_2+k_3=n-1} a_{k_1}^{x,e_1} a_{k_2}^{x,e_1} a_{k_3}^{x,e_1} & \cdots & (a_1^{x,e_1})^{n-1} & b_{n-1}^{x,e_1} \\ a_n^{x,e_1} & 0 & \cdots & 0 & \cdots & 0 & b_n^{x,e_1} \end{pmatrix},$$

such that $\phi(x) = \widehat{A_{x,e_1} \bar{x}}$, where $\bar{x} = (x_1, x_2, \dots, x_n)^T$ is the vector corresponding to x , $\widehat{}$ is an operation on \bar{x} such that $\widehat{\widehat{x}} = x$, and

$$\phi(e_1) = \widehat{A_{x,e_1} \bar{e}_1} = (a_1^{x,e_1}, a_2^{x,e_1}, \widehat{a_3^{x,e_1}}, \dots, a_n^{x,e_1})^T.$$

Since $\phi(e_1) = \varphi_{x,e_1}(e_1) = \varphi_{y_1,e_1}(e_1)$, we have

$$\phi(e_1) = (a_1^{x,e_1}, a_2^{x,e_1}, \widehat{a_3^{x,e_1}}, \dots, a_n^{x,e_1})^T =$$

$$= (a_1^{y_1, e_1}, a_2^{y_1, e_1}, \widehat{a_3^{y_1, e_1}}, \dots, a_n^{y_1, e_1})^T$$

for each pair, x, y_1 of elements in $\mu_{1,1}$. Hence, $a_k^{x, e_1} = a_k^{y_1, e_1}$, $k = 1, 2, \dots, n$.

Similarly, from $\varphi_{e_n, x}(e_n) = \varphi_{e_n, y_2}(e_n)$ it follows that

$$b_{n-1}^{e_n, x} = b_{n-1}^{e_n, y_2}, b_n^{e_n, x} = b_n^{e_n, y_2}.$$

Therefore, if we take $y_1 = e_n, y_2 = e_1$, then

$$\phi(x) = \varphi_{x, e_1}(x) = \varphi_{e_n, x}(x) = \varphi_{e_1, e_n}(x)$$

for any $x \in \mu_{1,1}$, and the matrix of $\phi(x)$ does not depend on x . Hence ϕ is a linear operator and the matrix of φ_{e_1, e_n} is the matrix of ϕ . Thus, by Proposition 2.2, ϕ is an automorphism. \square

The following theorem is proved similar to the proof of Theorem 4.2 using theorems 2.3, 2.4 and 2.5.

Theorem 4.3. *Each 2-local automorphism of the algebras $\mu_{1,2}$, $\mu_{1,3}$ and $\mu_{1,4}$ is an automorphism.*

Conflicts of Interest: *On behalf of all authors, the corresponding author states that there is no conflict of interest.*

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(F.N.Arzikulov) INSTITUTE OF MATHEMATICS UZBEKISTAN ACADEMY OF SCIENCES, 4, UNIVERSITY STREET, OLMAZOR, TASHKENT, 100174, UZBEKISTAN

AND

DEPARTMENT OF MATHEMATICS, ANDIJAN STATE UNIVERSITY, 129, UNIVERSITET STREET, ANDIJAN, 170100, UZBEKISTAN

Email address: arzikulovfn@rambler.ru

(Iqboljon Karimjanov) SAINT-PETERSBURG STATE UNIVERSITY, SAINT-PETERSBURG, RUSSIA,

AND

DEPARTMENT OF MATHEMATICS, ANDIJAN STATE UNIVERSITY, 129, UNIVERSITET STREET, ANDIJAN, 170100, UZBEKISTAN,

AND

V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCES, UNIVERSITY STREET, 9, OLMAZOR DISTRICT, TASHKENT, 100174, UZBEKISTAN

Email address: iqboli@gmail.com

(S.M.Umrzaqov) DEPARTMENT OF MATHEMATICS, ANDIJAN STATE UNIVERSITY, 129, UNIVERSITET STREET, ANDIJAN, 170100, UZBEKISTAN

Email address: sardor.umrzaqov1986@gmail.com