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CRITICAL MULTITYPE BRANCHING PROCESSES ON A GRAPH AND THE MODEL OF THE HIV INFECTION DEVELOPMENT

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Abstract. We consider the Crump-Mode-Jagers branching process on an oriented graph in an application to modeling the development of HIV-1 infection in a human organism. For all particles of the same global type, located at each of the vertexes or arcs of the graph, different types are assigned. Checking the criticality condition and searching for the eigenvectors of an offspring mean matrix in the critical case for the original process are reduced to an offspring mean matrix for some Galton-Watson process. The last has the types of particles corresponding only to the vertexes of the graph.

Keywords: Crump-Mode-Jagers branching process on an oriented graph, Yaglom type limit theorem for critical branching process, eigenvectors for the mean matrix of high dimension, stochastic model of HIV-1 infection.

1. Introduction

Various mathematical models are used to study the development of HIV-1 infection in the human organism (see [1] – [5]). If we consider the process of developing HIV-1 infection within a relatively short period of time after infection of a healthy person (from several days to 3-4 weeks), then the main components of the process are mature viral particles (virions) and productively infected cells. Virions come into contact with target cells, for example, CD4+ T-lymphocytes, enter these cells and start the process of converting target cells into productively infected cells.

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Productively infected cells produce new viral particles, which in turn infect new target cells. Each virion and each productively infected cell can be located in the lymph nodes and move between them, and may also die due to the influence of various factors. The transitions of virions and productively infected cells between two connected lymph nodes are unidirectional, which is due to the specificity of lymph flow in the lymphatic vessels. Traditionally, there are two approaches to the study of the problem. This is its study using a system of differential equations with delay (see [6]) and the method of stochastic simulation (see [7] – [9]).

We will consider a stochastic model of the development of HIV-1 infection in the human organism in the form of a branching process with several types of particles. Suppose that the particles of type A are mature viral particles (virions) and of type B are productively infected cells. We assume that particles of global types A and B can be in any of the n lymph nodes or move between some pairs of lymph nodes in one direction. The system of lymph nodes and connections between them is interpreted as a one-connected directed graph. The evolution of particles of global types A and B in each of the nodes and on the vessels (or in terms of graph theory: at each of the vertexes and on the arcs) are individual, and it is natural to assign different types to them. Since the number of lymph nodes in the human organism is quite large ($n \approx 100$), a very large number of particle types are presented in the model. In addition, the particles ability of moving between nodes is limited due to the specific structure of the human lymphatic system. When conducting research, we assume that all transformations of particles occur independently of the behavior of other particles, but significantly depend on their location.

Note that in the general setting, it is possible to simulate the development of HIV-1 infection in the human organism within the framework of a process that is non-homogeneous in time. Non-homogeneity arises when describing the duration of particle transitions between lymph nodes using some functions that depend on time. Non-homogeneity leads to significant difficulties when using classical analytical methods and requires the use of simulation methods. The proposed analytical results are important for developing efficient algorithms and testing the correctness of computational procedures in a simulation model.

Later we will define the model of moving two type particles through the directed graph in terms of Crump-Mode-Jagers branching process (see [10] or below in section 3). It will be named as $\mathcal{M}$ model. If the graph contains n vertices, then there can be up to $n(n-1)/2$ unidirectional edges. On each of these objects, the branching process develops according to different laws, i.e. we have to define $n(n+1)/2$ two-dimensional branching processes with transitions along directed edges. The result is a Crump-Mode-Jagers process with dimensions up to $(n(n+1)) \times (n(n+1))$, which we will call the $\mathcal{M}$–CMJ process.

A detailed description of the $\mathcal{M}$–CMJ process associated with the lifetime of particles, as will be shown below, is not essential for our research. Our goal is to simplify the calculation of the eigenvalue (Perron root) of the offspring mean matrix (the mean numbers of all types of offspring born to all parent types until the death of the last) for $\mathcal{M}$–CMJ process. Branching processes are traditionally divided into supercritical (the number of all types of particles grows exponentially), subcritical (the number of all types of particles dies exponentially quickly), and critical, where the averages for all types of particles are asymptotically constant and the process decays (degenerates) with probabilities inversely proportional to time (see [11], [12],

[10]). We will concentrate on the algorithm for calculating the eigenvectors of the offspring mean matrix only in critical case. The number of offspring and lifetime of parents are described in terms of some point processes (see Section 3). We assume that all second moments of these point processes are finite. The explicit forms and asymptotic behavior of the first and second moments for critical Crump-Mode-Jagers model are described in [13]. In [13] Yaglom type limit Theorem is also given for the number of particles of all types under the condition that the process is non-degenerate.

These asymptotic results are expressed in terms of the eigenvectors of the mean matrix and the functionals associated with the age of the parent at the time of generation of the offspring. The specificity of the model makes it possible to transfer transformations on edges to vertices, find eigenvectors for auxiliary processes of small dimension, and then return to the original problem.

More precisely, in the critical case for $\mathcal{M}$–CMJ process, a simple algorithm of calculating the eigenvectors for the matrix $\mathbf{A}$ of dimension up to $(n(n+1)) \times (n(n+1))$ is proposed in terms of the eigenvectors for some matrix $\mathbf{A}_1$ of dimension $(2n) \times (2n)$.

The structure of the paper is as follows. In section 2, we provide a formal description of the development of HIV-1 infection in terms of moving of some particles on a graph and state the main result. In section 3, the classical multitype models of branching processes and offspring mean matrix are reminded. In section 4, an auxiliary Bellman-Harris process of a lower dimension is constructed on the probabilistic space of the $\mathcal{M}$–CMJ process. Section 5 proves the main result. In section 6, some generalization of the $\mathcal{M}$–CMJ model and empirical results are discussed.

2. Evolution of particles on a graph

Let Γ be a directed simply connected graph without loops with weighted arcs. The vertexes of the graph Γ are denoted by N_i, and the arc going from N_i to N_j – by $N_{i,j}$, $1 \le i, j \le n$, where n is the number of vertexes, respectively.

The graph structure is determined by the stochastic weight matrix $\mathbf{G} = \big(g_{i,j}\big)_{i,j=1}^{n}$, in particular, there are no arcs if their weight is zero. The weights $g_{i,j}$ of the arcs $N_{i,j}$ in the graph Γ are satisfying the conditions

$$0 \le g_{i,j} \le 1, \quad g_{i,j} g_{j,i} = 0, \quad \sum_{j=1}^{n} g_{i,j} = 1, \quad 1 \le i, j \le n.$$

The stochastic matrix $\mathbf{G}$ can be interpreted as the transitions probabilities from states N_i to N_j, where $j \ne i$, for the Markov chain. We assume that all states of this chain are communicating.

Define two sets of pairs

$$\mathcal{S} = \big\{(i,j) : 1 \le i, j \le n,\ g_{i,j} > 0\big\}, \quad \mathcal{S}_0 = \mathcal{S} \cup \big\{(i,i), 1 \le i \le n\big\}.$$

Assume that a lexico-graphic order has been introduced for them. We denote by $\big|S\big|$ the cardinality (number of elements) of an arbitrary finite set S.

Two kinds of particles $A_{i,j}$ and $B_{i,j}$, $(i,j) \in \mathcal{S}_0$, can be located at the vertex N_i of the graph Γ for $j = i$ or on arcs $N_{i,j}$ for $(i,j) \in \mathcal{S}$. In our conditions on the graph Γ we have $2n + 2\big|\mathcal{S}\big|$ types of particles. Obviously, $4n \le 2n + 2\big|\mathcal{S}\big| \le (n+1)n$.

From the moment of birth, the evolution of any particle and its offspring does not depend on the further behavior of the particles and their offspring that were present at the moment.

Define the model $\mathcal{M}$. The evolution of $A_{i,i}$ and $B_{i,i}$ in vertexes N_i, $i = 1, \cdots, n$ are defined through traditional two-dimensional Crump-Mode-Jagers branching process and are kept as $A_{i,j}$ and $B_{i,j}$ in the arcs $N_{i,j}$, $1 \le i, j \le n$. The main property in $\mathcal{M}$ for vertexes N_i is that sometime the death of the particle is interpreted as emigration to one of arcs. Set the probabilities of emigration for $A_{i,i}$ and $B_{i,i}$ are equal to $p(A_{i,i}) > 0$ and $p(B_{i,i}) > 0$, correspondingly. The transition of the particles $A_{i,i}$ and $B_{i,i}$ to the arcs $N_{i,j}$, $(i, j) \in \mathcal{S}$, occurs with probabilities $g_{i,j}$. It means that the particle $A_{i,i}$ (or $B_{i,i}$) at the death time generate exactly one particle of type $A_{i,j}$ (or $B_{i,j}$) with probability $g_{i,j}p(A_{i,i})$ (or $g_{i,j}p(B_{i,i})$).

In a number of applied publications the evolution of particles $A_{i,j}$ (or $B_{i,j}$), $(i, j) \in \mathcal{S}$, located on arcs $N_{i,j}$ depends on their type and the birth times of these particles. We suppose that our process is homogeneous and the evolution of particles $A_{i,j}$ and $B_{i,j}$ is independent of their birth times. In the model of HIV-1 infection the arcs $N_{i,j}$ correspond to lymphatic vessels of various lengths with lymphatic fluid moving along them. The initial particle $A_{i,j}$ (or $B_{i,j}$) that appears in $N_{i,j}$ from N_i and all of its descendants simultaneously leave the vessel. In this case, we cannot control the offspring on $N_{i,j}$ within the branching process on all graph. The problem of particles evolution on arcs $N_{i,j}$ will be investigated separately. As a result we define that all particles $A_{i,j}$ and $B_{i,j}$ are kept on the arcs $N_{i,j}$ (do not produced offspring and no die) and only at the moment of moving of the particles $A_{i,j}$ and $B_{i,j}$ to N_j they die and, depending on age, spawn a random number of offspring $A_{j,j}$ and $B_{j,j}$. This determines the evolution of $A_{i,j}$ and $B_{i,j}$ on the arcs $N_{i,j}$ in terms of the Crump-Mode-Jagers processes in the model $\mathcal{M}$.

Denote $\mathcal{M}$–CMJ branching process by $\mathbf{Z}(t)$. Define the offspring mean matrices of the average number of offspring $A_{s,k}$ and $B_{s,k}$ from the particles $A_{i,j}$ throughout their life in the first row and from the particles $B_{i,j}$ throughout their life in the second line through $\mathbf{N}_{(i,j)(s,k)}$. In the general case, we will use the notation

$$\mathbf{N}_{(i,j)(s,k)} = \begin{pmatrix} m(A_{i,j}; A_{s,k}) & m(A_{i,j}; B_{s,k}) \\ m(B_{i,j}; A_{s,k}) & m(B_{i,j}; B_{s,k}) \end{pmatrix}.$$

But many of these matrices can be written explicitly for the model $\mathcal{M}$.

Write the offspring mean matrix $\mathbf{A}$ for $\mathcal{M}$–CMJ process in terms of block matrices $\mathbf{N}_{(i,j)(s,k)}$ of dimension 2×2

$$\mathbf{A} = \left(\mathbf{N}_{(i,j)(s,k)}\right)_{(i,j),(s,k) \in \mathcal{S}_0}. \tag{1}$$

The previously introduced definitions and conditions lead to the relations

$$\mathbf{N}_{(i,i)(i,i)} = \begin{pmatrix} m(A_{i,i}; A_{i,i}) & m(A_{i,i}; B_{i,i}) \\ m(B_{i,i}; A_{i,i}) & m(B_{i,i}; B_{i,i}) \end{pmatrix}, \tag{2}$$

$$\mathbf{N}_{(i,i)(i,j)} = g_{i,j} \begin{pmatrix} p(A_{i,i}) & 0 \\ 0 & p(B_{i,i}) \end{pmatrix}, \text{ for } (i,j) \in \mathcal{S}, \tag{3}$$

$$\mathbf{N}_{(i,j)(j,j)} = \begin{pmatrix} m(A_{i,j}; A_{j,j}) & m(A_{i,j}; B_{j,j}) \\ m(B_{i,j}; A_{j,j}) & m(B_{i,j}; B_{j,j}) \end{pmatrix}, \text{ for } (i,j) \in \mathcal{S}, \tag{4}$$

$$\mathbf{N}_{(i,j)(s,k)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \forall (i,j), (s,k) \in \mathcal{S}, \text{ or } \{(i,j), (s,k) \in \mathcal{S}_0\} \& \{j \neq s\}. \tag{5}$$

Suppose that

$$m(A_{i,i};A_{i,i})m(B_{i,i};B_{i,i}) > 0,\ m(A_{i,i};B_{i,i}) + m(B_{i,i};A_{i,i}) > 0 \\ m(A_{i,j};A_{j,j})m(B_{i,j};B_{j,j}) > 0 \tag{6}$$

Note that in a lot of applications, the $A_{i,j}$ and $B_{i,j}$ types of particles can only perish on the arcs $N_{i,j}$ or traverse the entire arc $N_{i,j}$. In such a case the evolution of particles is described in terms of Crump-Mode-Jagers process and we are dealing with real moments of particles $A_{i,j}$ and $B_{i,j}$ death. For such a model, the transformations of the matrices of means used by us are preserved after replacing the matrices $\mathbf{N}_{(i,j)(j,j)}$ from (4) with diagonal ones with the probabilities of passing the corresponding arcs on the diagonal.

Define accompanying mean matrix $\mathbf{A}_1 = (\mathbf{N}_{i,j})_{i,j=1}^n$ of dimension $2n \times 2n$ in block form, where

$$\mathbf{N}_{i,i} = \begin{pmatrix} m(A_{i,i};A_{i,i}) & m(A_{i,i};B_{i,i}) \\ m(B_{i,i};A_{i,i}) & m(B_{i,i};B_{i,i}) \end{pmatrix} = \mathbf{N}_{(i,i)(i,i)} \tag{7}$$

and $\mathbf{N}_{i,j} = \mathbf{N}_{(i,i)(i,j)}\mathbf{N}_{(i,j)(j,j)}$, for $i \neq j$, or

$$\mathbf{N}_{i,j} = g_{i,j} \begin{pmatrix} p(A_{i,i})m(A_{i,j};A_{j,j}) & p(A_{i,i})m(A_{i,j};B_{j,j}) \\ p(B_{i,i})m(B_{i,j};A_{j,j}) & p(B_{i,i})m(B_{i,j};B_{j,j}) \end{pmatrix}. \tag{8}$$

The selection of these matrices will be explained when proving the main result.

Define positive $2|\mathcal{S}_0|$ – dimensional left and right eigenvectors $\mathbf{v}$ and $\mathbf{u}$ for the matrix $\mathbf{A}$

$$\mathbf{vA} = \mathbf{v}, \quad \mathbf{Au}^\top = \mathbf{u}^\top. \tag{9}$$

where

$$\mathbf{v} = \left(v_{(i,j)}\right)_{(i,j)\in\mathcal{S}_0}, \quad \mathbf{u} = \left(u_{(i,j)}\right)_{(i,j)\in\mathcal{S}_0},$$

and $v_{(i,j)} = \left(v_{(i,j),1}, v_{(i,j),2}\right)$, $u_{(i,j)} = \left(u_{(i,j),1}, u_{(i,j),2}\right)$.

Hereinafter symbol $^\top$ means transposing of a matrix (vector).

In the same way define positive $2n$ – dimensional left and right eigenvectors $\mathbf{v}_1$ and $\mathbf{u}_1$ for the matrix $\mathbf{A}_1$

$$\mathbf{v}_1\mathbf{A}_1 = \mathbf{v}_1, \quad \mathbf{A}_1\mathbf{u}_1^\top = \mathbf{u}_1^\top. \tag{10}$$

where

$$\mathbf{v}_1 = (v_i)_{1\le i\le n}, \quad \mathbf{u}_1 = (u_i)_{1\le i\le n},$$

and $v_i = (v_{i,1}, v_{i,2})$, $u_i = (u_{i,1}, u_{i,2})$.

An essential condition for applying of asymptotic results for critical branching processes of all types is that the offspring mean matrix $\mathbf{A}$ is indecomposable. This means the simultaneous positiveness of all elements of $\mathbf{A}^{n_0}$ for some $n_0 \in \mathbb{N}$.

Theorem 1. *Fix $\mathcal{M}$–CMJ branching process $\mathbf{Z}(t)$ with the offspring mean matrix $\mathbf{A}$, defined in (1) and (2)–(5). Let lifetime distribution in terms of point processes with finite second moments for this branching process, be also fixed.*

Then, under the conditions (6) the matrix $\mathbf{A}$ and the accompanying matrix $\mathbf{A}_1$, defined in (7) and (8) are indecomposable and have equal Perron roots and in critical case the eigenvectors of the matrices $\mathbf{A}$ and $\mathbf{A}_1$ are linked by formulas

$$v_{(i,i)} = v_i, \quad v_{(i,k)} = v_i\mathbf{N}_{(i,i)(i,k)}, \quad 1 \le i,k \le n, \tag{11}$$

$$u_{(i,i)} = u_i, \quad u_{(i,j)} = u_j\mathbf{N}^\top_{(i,j)(j,j)}, \quad 1 \le i,j \le n. \tag{12}$$

3. Branching process models and an accompanying Galton-Watson process

Let us give brief definitions of abstract models of branching processes with N types of particles. All processes $\mathbf{Z}(t) \in \mathbb{N}_0^N$ will be interpreted as a number of particles of all types at time t, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In all cases will be defined $\mathbf{Z}(0)$ and stochastic evolution of each of the particles. From the moment of particle birth its evolution does not depend on the behavior of the other particles (branching condition).

In the Galton-Watson processes $\mathbf{Z}(t)$ (see [11]) the lifetime of all particles is equal to 1. Particles of the i-th type at death produce a random number of offspring with the generating function $F^{(i)}(\mathbf{s})$, where $\mathbf{s} = (s_1, \cdots, s_N) \in [0,1]^N$.

In the Bellman-Harris processes $\mathbf{Z}(t)$ (see [11]) the lifetime of particles of the i-th type have the distribution $G_i(t)$ and at time of death (upon death) it produces a random number of offspring with the generating function $F^{(i)}(\mathbf{s})$, where $\mathbf{s} = (s_1, \cdots, s_N) \in [0,1]^N$.

The asymptotic properties of critical Bellman-Harris processes under the additional condition of indecomposability of the matrix of means are described in terms of $\mathbf{Z}_a(t)$ – critical Galton-Watson processes (we call it the accompanying Galton-Watson process for $\mathbf{Z}(t)$), with generating functions for the number of offspring $F^{(i)}(\mathbf{s})$ for the particles of the i-th type and some coefficients expressed in terms of moments for $G_i(t)$. Formally, the accompanying Galton-Watson process is obtained by replacing all particles lifetimes with a single one without changing the number of offspring. See Theorem 1 [14, Ch. V, sec.5] for exact asymptotic formulas.

The offspring mean matrix for the Bellman-Harris process and the same for the accompanying Galton-Watson process (in general form without reference to the original problem) is written as

$$\mathbf{A} = \left(a_{i,j}\right)_{i,j=1}^N = \left(\frac{\partial F^{(i)}(\mathbf{s})}{\partial s_j}\bigg|_{\mathbf{s}=\mathbf{1}} \right)_{i,j=1}^N .$$

Sevast'yanov processes $\mathbf{Z}(t)$ with N types of particles (see [12, Ch. VIII]) differ from Bellman-Harris processes in that the number of offspring of a particle may depend on the age of its death. In other words, the generating function $F^{(i)}(\mathbf{s}, u)$ of the number of offspring of a particle of the i-th type that perishes at the age of u has the first moments

$$a_{i,j}(u) = \frac{\partial F^{(i)}(\mathbf{s}, u)}{\partial s_j}\bigg|_{\mathbf{s}=\mathbf{1}} .$$

Consequently the accompanying Galton-Watson process $\mathbf{Z}_a(t)$ for $\mathbf{Z}(t)$ is given by the generating functions

$$F^{(i)}(\mathbf{s}) = \int_0^\infty F^{(i)}(\mathbf{s}, u) dG_i(u)$$

and the offspring mean matrix $\mathbf{A}$ consists of elements

$$a_{i,j} = \int_0^\infty a_{i,j}(u) dG_i(u).$$

The Crump-Mode-Jagers processes $\mathbf{Z}(t)$ are a generalization of Sevast'yanov processes and differ from them in that particles can repeatedly generate random numbers of offspring throughout their life, while the number of groups of offspring, the moments of their generation and the lifetime of the particles themselves may depend on each other. Formally, the offspring generation process is specified using a multidimensional counting process. We will not go into details, but the asymptotic properties of such processes are described in terms of the offspring mean matrix $\mathbf{A}$ for $\mathbf{Z}(t)$ (or $\mathbf{Z}_a(t)$ – its the accompanying Galton-Watson) and some functionals such as moments from counting processes (from the evolution of particles), see [10] and [13].

The asymptotic properties of multidimensional critical Galton-Watson processes are described in [12, Ch. VI, § 3]. In [12], the criticality condition are expressed in terms of the properties of the offspring mean matrix $\mathbf{A}$. Its Perron root will be equal 1. Exact asymptotic formulas for the nonextinction probability of the process and conditional distributions for the normalized number of particles are related to the second moments (provided that they are finite) and the right and left eigenvectors $u > 0$ and $v > 0$ are defined by the relations $\mathbf{A}u^\top = u^\top$, $v\mathbf{A} = v$, $(u, v) = 1$.

4. $\mathcal{M}$–CMJ process and its accompanying process

For multitype Crump-Mode-Jagers model in [13] are presented integral equations for the first two moments in an explicit form and asymptotic representations and conditional Yaglom type limit Theorems for the number of particles of all types. The generating vector-functions for the number of offspring produced by particle of the type i throughout their life was denoted by $F^{(i)}(\mathbf{s})$. These generating functions define multitype accompanying Galton-Watson process $\mathbf{Z}_a(t)$. The critical type for both processes is the same.

In our $\mathcal{M}$–CMJ process we have $2n+2|\mathcal{S}|$ particle types $A_{i,j}$ and $B_{i,j}$, $(i,j) \in \mathcal{S}_0$ with offspring mean matrix $\mathbf{A}$ defined in (1) and (2)–(5). If any of the means $m(A_{i,j}; A_{s,k})$, $m(A_{i,j}; B_{s,k})$, $m(B_{i,j}; A_{s,k})$, $m(B_{i,j}; B_{s,k})$ is equal to zero, then the particles of the first type from the argument do not produce particles of the second type. In $\mathbf{Z}_a(t)$ the particle of the $A_{i,i}$ (or $B_{i,i}$) type produce a random number of offspring $A_{i,i}$ and $B_{i,i}$ and in some step with probability $g_{i,j}p(A_{i,i})$ (or $g_{i,j}p(B_{i,i})$) gives rise to exactly one offspring of the $A_{i,j}$ (or $B_{i,j}$) type. After that, at the next step, the $A_{i,j}$ (or $B_{i,j}$) type particle produces a random number of offspring of the types $A_{j,j}$ and $B_{j,j}$ with the generating function $F_A^{(i)}(s_1, s_2)$ (or $F_B^{(i)}(s_1, s_2)$).

Let us construct a new process $\mathbf{Z}_1(t)$ on the probabilistic space of the Galton-Watson process $\mathbf{Z}_a(t)$. We divide all one-step transformations of $A_{i,j}$ and $B_{i,j}$ in $\mathbf{Z}_a(t)$ from one state to other states into three groups: inside the vertices, from the vertices to the arcs, and from the arcs to the vertices. In the first group, offspring particles can stay at the same vertex for several steps in a row, while in the rest they leave the groups in one step. In terms of process $\mathbf{Z}_a(t)$ implementations, the transformations of the first type are preserved for the new process $\mathbf{Z}_1(t)$, and the transformations of the second and third types are combined into one of duration 2. Instead of two transitions from $A_{i,i}$ (or $B_{i,i}$) to $A_{i,j}$ (or $B_{i,j}$) and to N_j with generating function $F_A^{(i)}(s_1, s_2)$ (or $F_B^{(i)}(s_1, s_2)$) for number of offspring, we define that in two units of time $A_{i,i}$ (or $B_{i,i}$) generates a random number of offspring $A_{j,j}$ and $B_{j,j}$ with the same generating function $F_A^{(i)}(s_1, s_2)$ (or $F_B^{(i)}(s_1, s_2)$).

New process $\mathbf{Z}_1(t)$ is the Sevast'yanov process with $2n$ particles of the types $A_{i,i}$ and $B_{i,i}$, $1 \le i \le n$, with lifetime 1 or 2, with the lifetime dependent of offspring number and with offspring mean matrix $\mathbf{A}_1$ are defined in (7) and (8).

Recall that the condition of subcriticality, criticality and supercriticality branching processes corresponds to exponential decay, convergence to some positive constant or exponential growth of the average number of all components simultaneously with unlimited growth of the time parameter see [11], [12]). So $\mathbf{Z}_a(t)$ and $\mathbf{Z}_1(t)$ have one and the same critical class. Denote by $\mathbf{Z}_{1a}(t)$ the accompanying Galton-Watson process for the $\mathbf{Z}_1(t)$. These two processes share the common offspring mean matrix $\mathbf{A}_1$.

5. Proof of Theorem 1

For our analysis, the initial distributions of $\mathcal{M}$–CMJ process $\mathbf{Z}(t)$ and its accompanying of the second order Galton-Watson process $\mathbf{Z}_{1a}(t)$ are not important. We assume that the distributions of the number of offspring and the lifetime of particles for $\mathbf{Z}(t)$ are given, i.e. the process $\mathbf{Z}(t)$ is formally defined. The critical type of accompanying Galton-Watson process $\mathbf{Z}_{1a}(t)$ coincides with the critical type of the original $\mathcal{M}$–CMJ process $\mathbf{Z}(t)$. This means that the offspring mean matrices $\mathbf{A}$ and $\mathbf{A}_1$ have the same Perron roots.

First, let us prove that the offspring mean matrices $\mathbf{A}$ and $\mathbf{A}_1$ are indecomposable.

Let us construct an auxiliary Markov chain $\mathbf{M}(n)$ for the $\mathcal{M}$–CMJ process with the states $A_{i,j}$, $B_{i,j}$, $(i,j) \in \mathcal{S}_0$, with a matrix of transition probabilities

$$\tilde{\mathbf{A}} = \left(\mathbf{p}_{(i,j)(s,k)}\right)_{(i,j),(s,k)\in\mathcal{S}_0},$$

where the elements of matrices

$$\mathbf{p}_{(i,j)(s,k)} = \begin{pmatrix} p(A_{i,j}; A_{s,k}) & p(A_{i,j}; B_{s,k}) \\ p(B_{i,j}; A_{s,k}) & p(B_{i,j}; B_{s,k}) \end{pmatrix},$$

are satisfying the conditions $p(*;\star) > 0$ if and only if $m(*;\star) > 0$ with the same arguments.

By virtue of the matrix $\mathbf{G}$ properties and conditions (6), the Markov chain has finite number of communicating states and it is not periodic. Therefore, there exists n_0 such that all elements of $\tilde{\mathbf{A}}^{n_0}$ are positive (see Lemma 2 [12, Ch.IV, §5]).

By definition there exists some constant $c > 0$ such that for all combinations of arguments the inequalities

$$m(*;\star) \geq cp(*;\star)$$

are true. In matrix form, the last inequalities are written as

$$\mathbf{A} \geq c\tilde{\mathbf{A}}.$$

This implies that all elements of the matrix $\mathbf{A}^{n_0}$ are positive.

We proved that the offspring mean matrix $\mathbf{A}$ is indecomposable. By analogy, the same can be proved for the offspring mean matrix $\mathbf{A}_1$.

We are interested in a critical case. This means that the maximal root of the equation

$$\det(\mathbf{A}_1 - \lambda I_{2n}) = 0$$

is equal to 1. In this case, there are positive $2n$-dimensional left and right eigenvectors $\mathbf{v}_1$ and $\mathbf{u}_1$ satisfying the conditions (10).

Rewrite the relations (10) in block form

$$\sum_{i=1}^{n} v_i \mathbf{N}_{i,j} = v_j, \quad \sum_{j=1}^{n} \mathbf{N}_{i,j} u_j^\top = u_i^\top. \tag{13}$$

For $2|\mathcal{S}_0|$ - dimensional offspring mean matrix $\mathbf{A}$ left and right eigenvectors $\mathbf{v}$ and $\mathbf{u}$ are defined in (9).

After rewriting the relations (9) in block form, one obtains the relations

$$\sum_{(i,j)\in\mathcal{S}_0} v_{(i,j)}\mathbf{N}_{(i,j)(s,k)} = v_{(s,k)},$$

or in view of (5)

$$v_{(s,k)} = \sum_{(i,j)\in\mathcal{S}_0,\, j=s} v_{(i,j)}\mathbf{N}_{(i,j)(s,k)} = \sum_{\{i:(i,s)\in\mathcal{S}_0\}} v_{(i,s)}\mathbf{N}_{(i,s)(s,k)}. \tag{14}$$

In the case $s = k$ (14) implies $\sum_{\{i:(i,k)\in\mathcal{S}_0\}} v_{(i,k)}\mathbf{N}_{(i,k)(k,k)} = v_{(k,k)}$ or taking into account (7)

$$v_{(k,k)}\mathbf{N}_{k,k} + \sum_{\{i:(i,k)\in\mathcal{S}\}} v_{(i,k)}\mathbf{N}_{(i,k)(k,k)} = v_{(k,k)}. \tag{15}$$

If $k \neq s$, then $s = i$ and for $(i,k)\in\mathcal{S}$

$$v_{(i,i)}\mathbf{N}_{(i,i)(i,k)} = v_{(i,k)}. \tag{16}$$

From (3) and (4) it follows

$$\mathbf{N}_{(i,i)(i,k)}\mathbf{N}_{(i,k)(k,k)} = \mathbf{N}_{i,k}, \tag{17}$$

therefore (15) and (16) imply

$$v_{(k,k)}\mathbf{N}_{k,k} + \sum_{\{i:(i,k)\in\mathcal{S}\}} v_{(i,i)}\mathbf{N}_{i,k} = \sum_{i=1}^{n} v_{(i,i)}\mathbf{N}_{i,k} = v_{(k,k)}. \tag{18}$$

From (16)–(18) it follows that the left eigenvector (without normalization by unity) can be chosen in the form

$$v_{(i,i)} = v_i, \quad v_{(i,k)} = v_{(i,i)}\mathbf{N}_{(i,i)(i,k)}$$

where v_k is an eigenvector from (13). The representation (11) is proved.

Similar calculations can be performed for the right eigenvector, namely:

$$\sum_{(s,k)\in\mathcal{S}_0} \mathbf{N}_{(i,j)(s,k)}u^{\top}_{(s,k)} = u^{\top}_{(i,j)}. \tag{19}$$

Considering the property (5), we rewrite (19) in the form

$$u^{\top}_{(i,j)} = \sum_{\{k:(j,k)\in\mathcal{S}_0\}} \mathbf{N}_{(i,j)(j,k)}u^{\top}_{(j,k)} \tag{20}$$

In case $i = j$ (20) implies $\sum_{\{k:(i,k)\in\mathcal{S}_0\}} \mathbf{N}_{(i,i)(i,k)}u^{\top}_{(i,k)} = u^{\top}_{(i,i)}$ or taking into account (2) and (3) we have

$$\mathbf{N}_{i,i}u^{\top}_{(i,i)} + \sum_{\{k:(i,k)\in\mathcal{S}\}} \mathbf{N}_{(i,i)(i,k)}u^{\top}_{(i,k)} = u^{\top}_{(i,i)}. \tag{21}$$

In case $i \neq j$ (20) implies

$$\mathbf{N}_{(i,j)(j,j)}u^{\top}_{(j,j)} = u^{\top}_{(i,j)}.$$

or taking into account (17)

$$\sum_{\{k:(i,k)\in\mathcal{S}_0\}} \mathbf{N}_{i,k}u^{\top}_{(k,k)} = \sum_{k=1}^{n} \mathbf{N}_{i,k}u^{\top}_{(k,k)} = u^{\top}_{(i,i)}. \tag{22}$$

The representations (21) and (22) imply that as the right eigenvector (without normalization to unity) can take the form

$$u_{(i,i)} = u_i,\ u_{(i,j)} = u_j \mathbf{N}^{\top}_{(i,j)(j,j)}.$$

where u_k is an eigenvector from (13). The representation (12) is proved.

6. Comments on applications and generalization

The problem under consideration in a general case is traditionally solved by methods of statistical simulation. To test the operation of programs, it is convenient to use analytical results. The latter are also useful for creating simulation algorithms. Particular cases of the proposed model were investigated in [7] – [9]. A number of specific models and some results of numerical experiments are presented there. The behavior of the process on arcs in real problems excludes its description in terms of processes simpler than the Sevast'yanov branching processes. Here we used a more general the Crump-Mode-Jagers model.

The value of the Perron root for the the offspring mean matrix significantly affects the global behavior of processes. Supercritical and subcritical processes are well studied and quite simply arranged. With critical processes and processes close to them in terms of the value of the Perron root, there is a special interest. They are boundary when epidemics appear and have nontrivial properties for various parameters. When simulating complex processes, random experiments are repeated many times and conclusions are drawn based on statistical estimates. In the study of critical processes, experimenters encountered strange behavior on a number of trajectories: with a traditionally small number of virions, sharp surges in their numbers occurred. This effect is easily explained by Yaglom's type limit Theorem for critical branching process (large deviation Theorem). If process beginning from one particle of any fixed type and up to time t it is not degenerate (this happens with probability of the order 1/t), then the vector of particles number multiplied to t^{-1} convergence to some random vector, as $t \to \infty$. The precise formula of the limiting random vector distribution essentially depends from the eigenvectors for the initial model. It means that the number of particles at this moment is random, but this number has the order t. This phenomenon must be taken into account in practice for repetitive diseases.

The main results of this article are naturally generalized to a more general model. Instead of two global types of particles A and B, one can take any finite number of types $A^{(l)}$, $1 \le l \le k$, $1 \le i \le k$. At the vertex of the graph, their transformations can be specified by some common branching process for particles $A^{(l)}_{i,i}$, $1 \le l \le k$, with the possibility of particle escape $A^{(l)}_{i,i}$ from vertex N_i to arc $N_{i,j}$, $(i,j) \in \mathcal{S}$, (transformation of $A^{(l)}_{i,i}$ into $A^{(l)}_{i,j}$) with the probabilities $g_{i,j}$ defined above, where without dying they pass the entire arc and at the moment of moving to N_j they die and spawn a random number of offspring $A^{(l)}_{j,j}$, $1 \le l \le k$, as in the Crump-Mode-Jagers process. Further, the matrices $\mathbf{A}$ and $\mathbf{A}_1$ are defined by relations similar to (1)–(8) in terms of new matrices of the size $k \times k$. The statement of the theorem should be written in terms of these matrices of the size $k \times k$ like (11) and (12).

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