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SOLVABILITY OF A REGULARIZED BOUNDARY-VALUE PROBLEM FOR THE SYSTEM OF EQUATIONS OF DYNAMICS OF MIXTURES OF VISCOUS COMPRESSIBLE HEAT-CONDUCTING FLUIDS

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Abstract. We consider the boundary–value problem for the system of nonlinear partial differential equations which arise in the analysis of stationary motions of mixtures of viscous compressible heat–conducting fluids in a bounded domain of three–dimensional space. We prove the existence of strong solutions to the regularized boundary value problem.

Keywords: existence theorem, stationary boundary value problem, viscous compressible heat–conducting fluid, multi–velocity mixture

1. Introduction

Suppose that the mixture of $N \geqslant 2$ viscous compressible heat-conducting fluids occupies a bounded domain $\Omega \subset \mathbb{R}^3$ of the space of points $\boldsymbol{x} = (x_1, x_2, x_3)$ with the boundary $\partial\Omega$ of the class C^2. The sought physical quantities are described by $N+2$ ($3N+2$, if the dimension of the vectors is taken into account) functions defined in Ω which are as follows: vector fields of the velocities $\boldsymbol{u}_i(\boldsymbol{x}) = (u_{1i}(\boldsymbol{x}), u_{2i}(\boldsymbol{x}), u_{3i}(\boldsymbol{x}))$ for each component of the mixture with the number $i = 1, \dots, N$, the scalar field of the density $\rho(\boldsymbol{x}) \geqslant 0$ and the scalar field for the

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temperature of the mixture $\theta(\boldsymbol{x}) > 0$. To find these quantities it is necessary to solve one continuity equation

$$\operatorname{div}(\rho \boldsymbol{v}) = 0 \text{ in } \Omega, \tag{1.1}$$

N vector-valued (i. e. $3N$ scalar) momentum equations

$$\sum_{j=1}^{N} L_{ij}\boldsymbol{u}_j + \operatorname{div}(\rho_i \boldsymbol{v} \otimes \boldsymbol{u}_i) + \beta_i \nabla p = \rho_i \boldsymbol{f}_i \text{ in } \Omega, \quad i = 1, \dots, N \tag{1.2}$$

and one equation for the total energy of the mixture

$$\operatorname{div}\left(\mathcal{E}\boldsymbol{v} + p\boldsymbol{v} - \sum_{i=1}^{N} \mathbb{S}_i \boldsymbol{u}_i + \boldsymbol{q}\right) = \sum_{i=1}^{N} \rho_i \boldsymbol{f}_i \cdot \boldsymbol{u}_i \text{ in } \Omega. \tag{1.3}$$

The following notations are used in the above equations:

$$\boldsymbol{v} = \sum_{j=1}^{N} \beta_j \boldsymbol{u}_j$$

is the average-in-mass (barycentric) velocity of the mixture, $\beta_i = \text{const} \in (0,1)$, $\sum_{j=1}^{N} \beta_j = 1$; $\rho_i = \alpha_i \rho$ is the density of the i–th component of the mixture, $\alpha_i = \text{const} \in (0,1)$, $\sum_{j=1}^{N} \alpha_j = 1$; $\boldsymbol{f}_i(\boldsymbol{x}) = (f_{1i}(\boldsymbol{x}), f_{2i}(\boldsymbol{x}), f_{3i}(\boldsymbol{x}))$ are the known external body forces; p is the pressure, for which the constitutive equation

$$p = \rho^\gamma + \rho\theta,$$

is supposed, where the constant γ is supposed to be large enough (the exact values are formulated below); $\boldsymbol{q}$ is the vector of the heat flux, which is defined by the Fourier law

$$\boldsymbol{q} = -k(\theta)\nabla\theta,$$

with the coefficient of thermal conductivity, taken in the form

$$k(\theta) = 1 + \theta^m,$$

where the constant m is described below; $\mathcal{E} = \sum_{j=1}^{N} \mathcal{E}_j$ is the total specific energy of the mixture, where $\mathcal{E}_i$ is the total specific energy of the i–th component, defined as the sum of the kinetic and internal energies:

$$\mathcal{E}_i = \frac{\rho_i |\boldsymbol{u}_i|^2}{2} + \rho_i e_i,$$

where the specific internal energy of the i–th component e_i is defined by the constitutive equation

$$e_i = \frac{1}{\gamma - 1}\rho^{\gamma-1} + \theta;$$

$\mathbb{S}_i$ is the viscous part of the stress tensor of the i–th component of the mixture, defined by the constitutive equation

$$\mathbb{S}_i = \sum_{j=1}^{N} \left(\lambda_{ij}\,(\operatorname{div}\boldsymbol{u}_j)\mathbb{I} + 2\mu_{ij}\mathbb{D}(\boldsymbol{u}_j)\right),$$

where the (constant) viscosity coefficients λ_{ij} and μ_{ij} must satisfy definite restrictions specified below, $\mathbb{I}$ is the identity tensor, and $\mathbb{D}(\boldsymbol{w}) = \dfrac{1}{2}\left((\nabla \otimes \boldsymbol{w}) + (\nabla \otimes \boldsymbol{w})^*\right)$ is the strain tensor of the vector field $\boldsymbol{w}$ ($*$ stands for the transposition); finally, the Lamé operators are denoted as

$$L_{ij} = -(\lambda_{ij} + \mu_{ij})\nabla \mathrm{div} - \mu_{ij}\Delta, \quad i, j = 1, \ldots, N,$$

so that $\operatorname{div} \mathbb{S}_i = -\sum\limits_{j=1}^{N} L_{ij}\boldsymbol{u}_j$, $i = 1, \ldots, N$.

The equations (1.1)–(1.3) have to be supplemented by the boundary conditions for the velocities and temperature, for example as

$$\boldsymbol{u}_i = 0 \text{ at } \partial\Omega, \quad i = 1, \ldots, N \tag{1.4}$$

(i. e. here the boundary $\partial\Omega$ of the domain Ω is supposed to be a unmovable rigid wall),

$$k(\theta)\nabla\theta \cdot \boldsymbol{n} + L(\theta)(\theta - \widehat{\theta}) = 0 \text{ at } \partial\Omega, \quad i = 1, \ldots, N \tag{1.5}$$

(i. e. the heat exchange occurs with an external environment with a known temperature distribution $\widehat{\theta} > 0$), and also an additional condition for the density, which is taken standardly in the form

$$\int\limits_{\Omega} \rho \, d\boldsymbol{x} = M, \tag{1.6}$$

where the positive constant M stands for the total mass of the mixture and is supposed to be given. The symbol $\boldsymbol{n}$ in the condition (1.5) denotes the unit external normal vector of $\partial\Omega$, and the coefficient of boundary heat transfer is taken in the form

$$L(\theta) = 1 + \theta^{m-1}.$$

As can be seen, the considered model of the dynamics of mixtures is not a trivial generalization of the system of Navier—Stokes—Fourier equations, since in the equations of momentum (1.2) and energy (1.3) there are higher order terms responsible for the interaction between the components of the mixture, namely, the terms responsible for the viscous friction between the components of the mixture. The viscosity coefficients form two matrices $\mathbf{\Lambda} = \{\lambda_{ij}\}_{i,j=1}^{N}$ and $\mathbf{M} = \{\mu_{ij}\}_{i,j=1}^{N}$ (bulk and shear viscosities), the off-diagonal components of which are responsible for the indicated interaction. The matrix of total viscosities $\mathbf{N} = \mathbf{\Lambda} + 2\mathbf{M}$ (with the components $\nu_{ij} = \lambda_{ij} + 2\mu_{ij}$, $i, j = 1, \ldots, N$) also plays an important role. If these matrices are diagonal, interaction does not occur, and the corresponding problem does not present new significant mathematical difficulties in comparison with one-component movement. We consider the case of non-diagonal viscosity matrices in which there is no possibility of direct transfer of methods developed in the Navier—Stokes—Fourier theory of single-component heat-conducting fluids [1]–[4].

From thermodynamic considerations it follows that viscosity matrices cannot be arbitrary, but must satisfy certain requirements of positive or non-negative definiteness (see the details in [5], [6]). Here we impose the following (close to minimal) requirements:

$$\mathbf{M} > 0, \quad 3\mathbf{\Lambda} + 2\mathbf{M} \geqslant 0, \tag{1.7}$$

which,together with (1.4), provide the following inequalities

$$\sum_{i=1}^{N} \mathbb{S}_i : \mathbb{D}(\boldsymbol{u}_i) \geqslant 0 \tag{1.8}$$

(which corresponds to the non-negativity of the entropy production) and

$$\sum_{i,j=1}^{N} \int\limits_{\Omega} L_{ij}\boldsymbol{u}_j \cdot \boldsymbol{u}_i \, d\boldsymbol{x} \geqslant B_0 \sum_{i=1}^{N} \int\limits_{\Omega} |\nabla \otimes \boldsymbol{u}_i|^2 \, d\boldsymbol{x} \tag{1.9}$$

(which provides the ellipticity that is important from the mathematical point of view), where $B_0 = B_0(\mathbf{\Lambda}, \mathbf{M})$ is a positive constant. Let us note that (1.7) leads to $\mathbf{N} > 0$.

2. Statement of the problem and formulation of the main result

The problem (1.1)–(1.6) is referred to as Problem $\mathcal{H}S$ below. The main object of the paper is the regularization of Problem $\mathcal{H}S$. This regularized problem referred to as Problem $\mathcal{H}S^\varepsilon$ below is formulated as follows. It is required to find functions ρ^ε, θ^ε and $\boldsymbol{u}_i^\varepsilon$, $i = 1, \ldots, N$ (here and below the upper index ε is not an exponent), which satisfy the following equations, boundary and additional conditions, containing the parameter $\varepsilon \in (0, 1]$:

$$-\varepsilon\Delta\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon \boldsymbol{v}^\varepsilon) + \varepsilon\rho^\varepsilon = \varepsilon\frac{M}{|\Omega|} \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\begin{gathered} \sum_{j=1}^{N} L_{ij}\boldsymbol{u}_j^\varepsilon + \frac{\varepsilon}{2}\rho_i^\varepsilon \boldsymbol{u}_i^\varepsilon + \frac{\varepsilon}{2}\frac{M}{|\Omega|}\alpha_i \boldsymbol{u}_i^\varepsilon + \frac{1}{2}\rho_i^\varepsilon(\boldsymbol{v}^\varepsilon \cdot \nabla)\boldsymbol{u}_i^\varepsilon + \\ + \frac{1}{2}\operatorname{div}(\rho_i^\varepsilon \boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}_i^\varepsilon) + \beta_i \nabla p^\varepsilon = \rho_i^\varepsilon \boldsymbol{f}_i \quad \text{in} \quad \Omega, \quad i = 1, \ldots, N, \end{gathered} \tag{2.2}$$

$$\begin{gathered} -\operatorname{div}\left(k(\theta^\varepsilon)\frac{\varepsilon + \theta^\varepsilon}{\theta^\varepsilon}\nabla\theta^\varepsilon\right) + \operatorname{div}(\rho^\varepsilon\theta^\varepsilon\boldsymbol{v}^\varepsilon) + \rho^\varepsilon\theta^\varepsilon \operatorname{div}\boldsymbol{v}^\varepsilon = \\ = \sum_{i=1}^{N} \mathbb{S}_i^\varepsilon : (\nabla \otimes \boldsymbol{u}_i^\varepsilon) + \varepsilon\gamma(\rho^\varepsilon)^{\gamma-2}|\nabla\rho^\varepsilon|^2 \quad \text{in} \quad \Omega, \end{gathered} \tag{2.3}$$

$$\boldsymbol{u}_i^\varepsilon = 0, \quad i = 1, \ldots, N, \quad \nabla\rho^\varepsilon \cdot \boldsymbol{n} = 0 \quad \text{at} \quad \partial\Omega, \tag{2.4}$$

$$k(\theta^\varepsilon)\frac{\varepsilon + \theta^\varepsilon}{\theta^\varepsilon}\nabla\theta^\varepsilon \cdot \boldsymbol{n} + \varepsilon \ln\theta^\varepsilon + L(\theta^\varepsilon)(\theta^\varepsilon - \widehat{\theta}) = 0 \quad \text{at} \quad \partial\Omega, \tag{2.5}$$

$$\int\limits_{\Omega} \rho^\varepsilon \, d\boldsymbol{x} = M, \tag{2.6}$$

where

$$\boldsymbol{v}^\varepsilon = \sum_{j=1}^{N} \beta_j \boldsymbol{u}_j^\varepsilon, \quad \rho_i^\varepsilon = \alpha_i\rho^\varepsilon, \quad p^\varepsilon = (\rho^\varepsilon)^\gamma + \rho^\varepsilon\theta^\varepsilon,$$
$$\mathbb{S}_i^\varepsilon = \lambda_{ij}\operatorname{div}\boldsymbol{u}_j^\varepsilon\mathbb{I} + 2\mu_{ij}\mathbb{D}(\boldsymbol{u}_j^\varepsilon), \quad i = 1, \ldots, N,$$

and $|\Omega|$ stands for the Lebesgue measure of Ω. It is clear that Problem $\mathcal{H}S^\varepsilon$ is nothing else than a uniformly elliptic regularization of the problem $\mathcal{H}S$ plus additional terms and boundary conditions designed to preserve the useful properties of the original problem, which play an important role in the theory of viscous gas, for example, the integral orthogonality of convective terms to the velocities. The conditions (2.6) follow from (2.1) and (2.4), but they are included in the statement of Problem $\mathcal{H}S^\varepsilon$ for consistency with original Problem $\mathcal{H}S$.

During the analysis of Problem $\mathcal{H}S^\varepsilon$, it is occasionally convenient to use the function

$$s^\varepsilon = \ln \theta^\varepsilon, \tag{2.7}$$

instead of the temperature θ^ε, so that the relations (2.3) and (2.5) take the form

$$\begin{aligned} -\mathrm{div}((1+e^{ms^\varepsilon})(\varepsilon + e^{s^\varepsilon})\nabla s^\varepsilon) = -\mathrm{div}(\rho^\varepsilon e^{s^\varepsilon}\boldsymbol{v}^\varepsilon) - \rho^\varepsilon e^{s^\varepsilon}\mathrm{div}\boldsymbol{v}^\varepsilon + \\ + \sum_{i=1}^N \mathbb{S}_i^\varepsilon : (\nabla \otimes \boldsymbol{u}_i^\varepsilon) + \varepsilon\gamma(\rho^\varepsilon)^{\gamma-2}|\nabla\rho^\varepsilon|^2 \quad \text{in} \quad \Omega, \end{aligned} \tag{2.8}$$

$$(1+e^{ms^\varepsilon})(\varepsilon + e^{s^\varepsilon})\nabla s^\varepsilon \cdot \boldsymbol{n} + \varepsilon s^\varepsilon + L(e^{s^\varepsilon})(e^{s^\varepsilon} - \widehat{\theta}) = 0 \quad \text{at} \quad \partial\Omega \tag{2.9}$$

respectively, and the modified problem (2.1), (2.2), (2.8), (2.4), (2.9), (2.6) for the unknowns ρ^ε, s^ε and $\boldsymbol{u}_i^\varepsilon$, $i = 1, \ldots, N$, is referred to as Problem $\widetilde{\mathcal{H}S}^\varepsilon$ below.

Definition 2.1. *By a strong solution to Problem $\mathcal{H}S^\varepsilon$ we mean nonnegative function $\rho^\varepsilon \in W_p^2(\Omega)$, where $p > 3$, positive function $\theta^\varepsilon \in W_p^2(\Omega)$ and vector fields $\boldsymbol{u}_i^\varepsilon \in W_p^2(\Omega)$, $i = 1, \ldots, N$, which satisfy* (2.6), *the equations* (2.1)–(2.3) *a. e. in Ω, and the conditions* (2.4), (2.5) *a. e. at $\partial\Omega$.*

Definition 2.2. *By a strong solution to Problem $\widetilde{\mathcal{H}S}^\varepsilon$ we mean nonnegative function $\rho^\varepsilon \in W_p^2(\Omega)$, where $p > 3$, function $s^\varepsilon \in W_p^2(\Omega)$ and vector fields $\boldsymbol{u}_i^\varepsilon \in W_p^2(\Omega)$, $i = 1, \ldots, N$, which satisfy* (2.6), *the equations* (2.1), (2.2), (2.8) *a. e. in Ω, and the conditions* (2.4), (2.9) *a. e. at $\partial\Omega$.*

It is obvious that Definitions 2.1 and 2.2 are equivalent via the substitution (2.7). The main result of the paper is formulated as the following Theorem.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\partial\Omega \in C^2$, the viscosity matrices satisfy the conditions* (1.7), *exponents $\gamma > 3$ and $m > 2$, and the rest numeric parameters of the model $\varepsilon \in (0, 1]$, $\beta_i, \alpha_i \in (0, 1)$, $i = 1, \ldots, N$, $\sum_{j=1}^N \beta_j = 1$, $\sum_{j=1}^N \alpha_j = 1$, $M > 0$. Then for arbitrary input data of the class $\boldsymbol{f}_i \in C(\overline{\Omega})$, $i = 1, \ldots, N$, $\widehat{\theta} \in C^1(\partial\Omega)$, $\widehat{\theta} > 0$ Problem $\mathcal{H}S^\varepsilon$ has at least one strong solution.*

3. The proof of the solvability of the regularized problem

In view of the observations made at the end of the previous section, it suffices to prove the existence of a strong solution to Problem $\widetilde{\mathcal{H}S}^\varepsilon$. This solution will be constructed as a fixed point of the operator $\boldsymbol{\Psi}$ formed below.

Let us define the operator $\mathcal{R}$ which acts as $\mathcal{R}:\ \boldsymbol{w}\mapsto r$, where

$$\boldsymbol{w}\in B_p(\Omega)=\{\,\boldsymbol{v}\in W_p^2(\Omega):\ \boldsymbol{v}|_{\partial\Omega}=0\,\},$$

and r is the solution to the problem

$$-\varepsilon\Delta r+\operatorname{div}(r\boldsymbol{w})+\varepsilon r=\varepsilon\frac{M}{|\Omega|},\quad \nabla r\cdot\boldsymbol{n}|_{\partial\Omega}=0.$$

Then $r=\mathcal{R}(\boldsymbol{w})\geqslant 0$ [7] and satisfies (2.6). In view of the standard properties of elliptic boundary value problems (see, for example, [8], [9]) the operator

$$\mathcal{R}:\ B_p(\Omega)\to W_p^2(\Omega),$$

moreover, it is continuous, since

$$\|\mathcal{R}(\boldsymbol{v})-\mathcal{R}(\boldsymbol{w})\|_{W_p^2(\Omega)}\leqslant B_1(p,\varepsilon,\|\boldsymbol{v}\|_{C^1(\overline{\Omega})},\|\boldsymbol{w}\|_{C^1(\overline{\Omega})},\Omega,M)\|\boldsymbol{v}-\boldsymbol{w}\|_{W_p^2(\Omega)}.$$

The second auxiliary operator is $\mathcal{U}:\ \mathbf{g}\mapsto\mathbf{h}$, where $\mathbf{g}=(\boldsymbol{g}_1,\dots,\boldsymbol{g}_N)$ with the components $\boldsymbol{g}_i\in L_p(\Omega)$, $i=1,\dots,N$, and $\mathbf{h}=(\boldsymbol{h}_1,\dots,\boldsymbol{h}_N)$ is the solution to the problem

$$\sum_{j=1}^N L_{ij}\boldsymbol{h}_j=\boldsymbol{g}_i,\quad \boldsymbol{h}_i|_{\partial\Omega}=0,\quad i=1,\dots,N.$$

It is obvious that

$$\mathcal{U}:\ L_p(\Omega)\to B_p(\Omega)$$

continuously.

The third auxiliary operator is $\mathcal{S}:\ (d,b,t)\mapsto z$, where

$$(d,b,t)\in L_p(\Omega)\times C^1(\overline{\Omega})\times W_p^{1-\frac{1}{p}}(\partial\Omega),$$

with $b>0$, and z is the solution to the problem

$$-\operatorname{div}(b\nabla z)=d,\quad (b\nabla z\cdot\boldsymbol{n}+\varepsilon z)|_{\partial\Omega}=t.$$

Again, the general theory provides the inequality

$$\|\mathcal{S}(d,b,t)\|_{W_p^2(\Omega)}\leqslant B_2(\|b\|_{C^1(\overline{\Omega})},\min_{\overline{\Omega}}b,p,\varepsilon,\Omega)\left(\|d\|_{L_p(\Omega)}+\|t\|_{W_p^{1-\frac{1}{p}}(\partial\Omega)}\right),\tag{3.1}$$

i. e.

$$\mathcal{S}:\ L_p(\Omega)\times C^1(\overline{\Omega})\times W_p^{1-\frac{1}{p}}(\partial\Omega)\to W_p^2(\Omega),$$

and after the application of the same estimate to the difference of two problems, i. e. the analysis of $z_k=\mathcal{S}(d_k,b_k,t_k)$, $k=1,2$, and noting that

$$(z_2-z_1)=\mathcal{S}\left((d_2-d_1)+\operatorname{div}((b_2-b_1)\nabla z_1),b_2,(t_2-t_1)-(b_2-b_1)\frac{\partial z_1}{\partial n}\right),$$

from (3.1) we obtain the estimate

$$\|\mathcal{S}(d_2,b_2,t_2)-\mathcal{S}(d_1,b_1,t_1)\|_{W_p^2(\Omega)}\leqslant$$

$$\leqslant B_3\left(\|d_2-d_1\|_{L_p(\Omega)}+\|t_2-t_1\|_{W_p^{1-\frac{1}{p}}(\partial\Omega)}+\|b_2-b_1\|_{C^1(\overline{\Omega})}\right),$$

where

$$B_3=B_3\left(\|b_1\|_{C^1(\overline{\Omega})},\|b_2\|_{C^1(\overline{\Omega})},\min_{\overline{\Omega}}b_1,\min_{\overline{\Omega}}b_2,p,\varepsilon,\Omega,\|d_1\|_{L_p(\Omega)},\|t_1\|_{W_p^{1-\frac{1}{p}}(\partial\Omega)}\right),$$

i. e. the operator $\mathcal{S}$ is continuous in the same spaces.

Finally, the fourth set of operators $\mathcal{G}_i$, $i = 1, \ldots, N$, $\mathcal{D}$, $\mathcal{B}$, $\mathcal{T}$ is defined as follows:

$$\mathcal{G}_i(\mathbf{w}, y) = -\frac{\varepsilon}{2}\alpha_i r \boldsymbol{w}_i - \frac{\varepsilon}{2}\frac{M}{|\Omega|}\alpha_i \boldsymbol{w}_i - \frac{1}{2}\alpha_i r \left(\left(\sum_{j=1}^{N} \beta_j \boldsymbol{w}_j \right) \cdot \nabla \right) \boldsymbol{w}_i -$$

$$-\frac{1}{2}\mathrm{div}\left(\alpha_i r \left(\sum_{j=1}^{N} \beta_j \boldsymbol{w}_j \right) \otimes \boldsymbol{w}_i \right) - \beta_i \nabla r^{\gamma} - \beta_i \nabla (r e^{y}) + \alpha_i r \boldsymbol{f}_i, \quad i = 1, \ldots, N,$$

$$\mathcal{D}(\mathbf{w}, y) = -\mathrm{div}\left(r e^{y} \left(\sum_{j=1}^{N} \beta_j \boldsymbol{w}_j \right) \right) - r e^{y} \mathrm{div}\left(\sum_{j=1}^{N} \beta_j \boldsymbol{w}_j \right) +$$

$$+\sum_{i=1}^{N} \left(\sum_{j=1}^{N} (\lambda_{ij} (\mathrm{div} \boldsymbol{w}_j) \mathbb{I} + 2\mu_{ij} \mathbb{D}(\boldsymbol{w}_j)) \right) : (\nabla \otimes \boldsymbol{w}_i) + \varepsilon \gamma r^{\gamma - 2} |\nabla r|^2,$$

$$\mathcal{B}(y) = (1 + e^{my})(\varepsilon + e^{y}),$$

$$\mathcal{T}(y) = -(1 + e^{(m-1)y})(e^{y} - \widehat{\theta})|_{\partial\Omega},$$

where $\mathbf{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N) \in B_p(\Omega)$, $r = \mathcal{R}\left(\sum\limits_{j=1}^{N} \beta_j \boldsymbol{w}_j \right) \in W_p^2(\Omega)$, $y \in W_p^2(\Omega)$. It is clear that

$$\mathcal{G}_i : \ B_p(\Omega) \times W_p^2(\Omega) \to C(\overline{\Omega}), \quad i = 1, \ldots, N,$$

$$\mathcal{D} : \ B_p(\Omega) \times W_p^2(\Omega) \to C(\overline{\Omega}), \quad \mathcal{B} : \ W_p^2(\Omega) \to C^1(\overline{\Omega}), \quad \mathcal{T} : \ W_p^2(\Omega) \to C^1(\partial\Omega).$$

Moreover, it is easy to check the following properties: $\mathcal{G}_i$ $(i = 1, \ldots, N)$ and $\mathcal{D}$ are well defined, bounded and continuous as operators from $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ into $C(\overline{\Omega})$, and hence compact (completely continuous) as operators from $B_p(\Omega) \times W_p^2(\Omega)$ into $L_p(\Omega)$; $\mathcal{B}$ is well defined, bounded and continuous as an operator from $C^1(\overline{\Omega})$ into $C^1(\overline{\Omega})$, and hence compact (completely continuous) as an operator from $W_p^2(\Omega)$ into $C^1(\overline{\Omega})$; $\mathcal{T}$ is well defined, bounded and continuous as an operator from $C^1(\partial\Omega)$ into $C^1(\partial\Omega)$, and hence compact (completely continuous) as an operator from $W_p^2(\Omega)$ into $W_p^{1-\frac{1}{p}}(\partial\Omega)$, moreover, the corresponding estimates depend only on $\{\|\boldsymbol{f}_i\|_{C(\overline{\Omega})}\}$, $\|\widehat{\theta}\|_{C^1(\partial\Omega)}$, $\mathbf{\Lambda}$, $\mathbf{M}$, ε, γ, m, p, M and Ω.

As a result, we set $\mathbf{\Psi} = (\mathcal{U} \circ (\mathcal{G}_1, \ldots, \mathcal{G}_N), \mathcal{S} \circ (\mathcal{D}, \mathcal{B}, \mathcal{T}))$, i. e. for all $(\mathbf{u}, s) = ((\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N), s) \in B_p(\Omega) \times W_p^2(\Omega)$ we set

$$\mathbf{\Psi}(\mathbf{u}, s) = \{\mathcal{U}(\mathcal{G}_1(\mathbf{u}, s), \ldots, \mathcal{G}_N(\mathbf{u}, s)), \mathcal{S}(\mathcal{D}(\mathbf{u}, s), \mathcal{B}(s), \mathcal{T}(s))\}.$$

By construction, the operator $\mathbf{\Psi} : B_p(\Omega) \times W_p^2(\Omega) \to B_p(\Omega) \times W_p^2(\Omega)$ is well defined, continuous compact (i. e. completely continuous), and the desired strong solution to Problem $\widetilde{\mathcal{HS}}^{\varepsilon}$ has the form

$$\left(\mathcal{R}\left(\sum_{j=1}^{N} \beta_j \boldsymbol{u}_j \right), s, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_N \right),$$

where $(\mathbf{u}, s)$ is a fixed point of $\mathbf{\Psi}$.

To apply the Leray—Schauder principle [10] it remains to obtain a priori estimate of solutions to the operator equation $\lambda \mathbf{\Psi}(\mathbf{u}, s) = (\mathbf{u}, s)$ in the space $W_p^2(\Omega)$, uniformly in the parameter $\lambda \in (0, 1]$, i. e. to estimate (in this space) a hypothetical solution

$(\rho, s, \boldsymbol{u}_1, \dots, \boldsymbol{u}_N)$ of the boundary value problem $\widetilde{\mathcal{HS}}^{\varepsilon,\lambda}$ uniformly on $\lambda \in (0,1]$, where Problem $\widetilde{\mathcal{HS}}^{\varepsilon,\lambda}$ consists in the relations (indices ε and λ at the values depending on ε and λ are omitted)

$$\sum_{j=1}^{N} L_{ij}\boldsymbol{u}_j + \frac{\lambda\varepsilon}{2}\rho_i\boldsymbol{u}_i + \frac{\lambda\varepsilon}{2}\frac{M}{|\Omega|}\alpha_i\boldsymbol{u}_i + \frac{\lambda}{2}\rho_i(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_i + \frac{\lambda}{2}\mathrm{div}(\rho_i\boldsymbol{v}\otimes\boldsymbol{u}_i) + \tag{3.2}$$

$$+\lambda\beta_i\nabla\rho^\gamma + \lambda\beta_i\nabla(\rho e^s) = \lambda\rho_i\boldsymbol{f}_i \quad \text{in} \quad \Omega, \quad i = 1, \dots, N,$$

$$-\mathrm{div}\left((1+e^{ms})(\varepsilon + e^s)\nabla s\right) = \Pi \quad \text{in} \quad \Omega, \tag{3.3}$$

$$(1+e^{ms})(\varepsilon+e^s)\nabla s\cdot\boldsymbol{n} = \widehat{\Pi} \quad \text{at} \quad \partial\Omega \tag{3.4}$$

together with (2.1), (2.4) and (2.6). Here we use the notations

$$\Pi = -\lambda\mathrm{div}(\rho e^s\boldsymbol{v}) - \lambda\rho e^s\mathrm{div}\boldsymbol{v} + \lambda\sum_{i=1}^{N}\mathbb{S}_i : (\nabla\otimes\boldsymbol{u}_i) + \lambda\varepsilon\gamma\rho^{\gamma-2}|\nabla\rho|^2 \tag{3.5}$$

and

$$\widehat{\Pi} = -\varepsilon s - \lambda L(e^s)(e^s - \widehat{\theta}). \tag{3.6}$$

Let us form the scalar product of the both sides of the equations (3.2) and $\boldsymbol{u}_i$, integrate over Ω, sum via $i = 1, \dots, N$ and divide by λ. This leads to the inequality

$$\sum_{i=1}^{N}\int_\Omega \mathbb{S}_i : (\nabla\otimes\boldsymbol{u}_i)\,d\boldsymbol{x} + \frac{\varepsilon}{2}\sum_{i=1}^{N}\int_\Omega \rho_i|\boldsymbol{u}_i|^2\,d\boldsymbol{x} + \frac{\varepsilon M}{2|\Omega|}\sum_{i=1}^{N}\alpha_i\int_\Omega|\boldsymbol{u}_i|^2\,d\boldsymbol{x} +$$

$$+\varepsilon\frac{\gamma}{\gamma-1}\int_\Omega\rho^\gamma\,d\boldsymbol{x} + \varepsilon\gamma\int_\Omega\rho^{\gamma-2}|\nabla\rho|^2\,d\boldsymbol{x} \leqslant \frac{\varepsilon M}{|\Omega|}\frac{\gamma}{\gamma-1}\int_\Omega\rho^{\gamma-1}\,d\boldsymbol{x} + \tag{3.7}$$

$$+\int_\Omega\rho\theta\mathrm{div}\boldsymbol{v}\,d\boldsymbol{x} + \sum_{i=1}^{N}\int_\Omega\rho_i\boldsymbol{f}_i\cdot\boldsymbol{u}_i\,d\boldsymbol{x}.$$

Let us multiply (3.3) by $\dfrac{1}{\lambda}\left(1-\dfrac{1}{\theta}\right)$, integrate over Ω and sum with (3.7), then we obtain the inequality

$$\sum_{i=1}^{N}\int_\Omega\frac{\mathbb{S}_i : (\nabla\otimes\boldsymbol{u}_i)}{\theta}\,d\boldsymbol{x} + \int_\Omega k(\theta)\frac{\varepsilon+\theta}{\theta}|\nabla\ln\theta|^2\,d\boldsymbol{x} + \int_{\partial\Omega}L(\theta)\frac{\widehat{\theta}}{\theta}d\sigma +$$

$$+\int_{\partial\Omega}L(\theta)\theta d\sigma + \frac{\varepsilon}{2}\sum_{i=1}^{N}\int_\Omega\rho_i|\boldsymbol{u}_i|^2\,d\boldsymbol{x} + \frac{\varepsilon M}{2|\Omega|}\sum_{i=1}^{N}\alpha_i\int_\Omega|\boldsymbol{u}_i|^2\,d\boldsymbol{x} +$$

$$+\varepsilon\frac{\gamma}{\gamma-1}\int_\Omega\rho^\gamma\,d\boldsymbol{x} + \varepsilon\gamma\int_\Omega\frac{\rho^{\gamma-2}}{\theta}|\nabla\rho|^2\,d\boldsymbol{x} + \varepsilon\int_{\partial\Omega}(s^-e^{s^-} + s^+)d\sigma \leqslant \tag{3.8}$$

$$\leqslant \int_\Omega(\rho\boldsymbol{v}\cdot\nabla s - \boldsymbol{v}\cdot\nabla\rho)\,d\boldsymbol{x} + \int_{\partial\Omega}L(\theta)(1+\widehat{\theta})d\sigma + \frac{\varepsilon M}{|\Omega|}\frac{\gamma}{\gamma-1}\int_\Omega\rho^{\gamma-1}\,d\boldsymbol{x} +$$

$$+\sum_{i=1}^{N}\int\limits_{\Omega}\rho_i \boldsymbol{f}_i\cdot\boldsymbol{u}_i\,d\boldsymbol{x}+\varepsilon\int\limits_{\partial\Omega}(s^+e^{-s^+}+s^-)d\sigma,$$

where we used the notations of the positive $z^+=z\chi(z)$ and negative $z^-=-z\chi(-z)$ parts of an arbitrary value z (where χ is the Heaviside function). Let us mention obvious properties $(z\varphi(z))^+=z^+\varphi(z^+)$ and $(z\varphi(z))^-=z^-\varphi(-z^-)$, valid for any positive function φ. The left hand sides of the both inequalities above contain only non-negative terms due to (1.8).

Let us estimate the terms in the right hand side of the inequality (3.8). To do this, we first present integrals in the first sum in the following form (which is easy to do via (2.1)):

$$\int\limits_{\Omega}(\rho\boldsymbol{v}\cdot\nabla s-\boldsymbol{v}\cdot\nabla\rho)\,d\boldsymbol{x}=\int\limits_{\Omega}\left(\varepsilon\nabla\rho\cdot\nabla s-\varepsilon\frac{|\nabla\rho|^2}{\rho+\sigma_1}\right)d\boldsymbol{x}-$$

$$-\varepsilon\frac{M}{|\Omega|}\int\limits_{\Omega}s\,d\boldsymbol{x}+\int\limits_{\Omega}\left(\varepsilon\rho s-\varepsilon\rho\ln(\rho+\sigma_1)+\varepsilon\frac{M}{|\Omega|}\ln(\rho+\sigma_1)\right)d\boldsymbol{x}+ \tag{3.9}$$

$$+\sigma_1\int\limits_{\Omega}(\mathrm{div}\boldsymbol{v})\ln(\rho+\sigma_1)\,d\boldsymbol{x},$$

where $\sigma_1\in(0,1]$ is an arbitrary parameter. Summing the elementary inequalities

$$\varepsilon\int\limits_{\Omega}\nabla\rho\cdot\nabla s\,d\boldsymbol{x}\leqslant\frac{\varepsilon\gamma}{2}\int\limits_{\Omega}\frac{(\rho+\sigma_1)^{\gamma-2}}{\theta}|\nabla\rho|^2\,d\boldsymbol{x}+$$

$$+\varepsilon\int\limits_{\Omega}\frac{|\nabla\rho|^2}{\rho+\sigma_1}\,d\boldsymbol{x}+\frac{1}{2}\int\limits_{\Omega}\theta^{\frac{1}{\gamma-1}}|\nabla s|^2\,d\boldsymbol{x},$$

$$-\varepsilon\frac{M}{|\Omega|}\int\limits_{\Omega}s\,d\boldsymbol{x}\leqslant\frac{\varepsilon}{4}\int\limits_{\partial\Omega}s^-e^{s^-}\,d\sigma+\frac{1}{4}\|\nabla s\|^2_{L_2(\Omega)}+B_4(M,\Omega),$$

$$\int\limits_{\Omega}\left(\varepsilon\rho\ln\theta-\varepsilon\rho\ln(\rho+\sigma_1)+\varepsilon\frac{M}{|\Omega|}\ln(\rho+\sigma_1)\right)d\boldsymbol{x}\leqslant\|\rho\theta\|_{L_1(\Omega)}+B_5(M,|\Omega|),$$

$$\sigma_1\int\limits_{\Omega}(\mathrm{div}\boldsymbol{v})\ln(\rho+\sigma_1)\,d\boldsymbol{x}\leqslant\int\limits_{\Omega}|\mathrm{div}\boldsymbol{v}|(\rho+1)\,d\boldsymbol{x}\leqslant$$

$$\leqslant\sum_{j=1}^{N}\|\boldsymbol{u}_j\|^2_{W_2^1(\Omega)}+\frac{3N}{4}\|\rho\|^2_{L_2(\Omega)}+B_6(M,N,|\Omega|),$$

we derive an estimate of the right hand side of (3.9), and after the limit as $\sigma_1\to+0$, we obtain

$$\int\limits_{\Omega}(\rho\boldsymbol{v}\cdot\nabla s-\boldsymbol{v}\cdot\nabla\rho)\,d\boldsymbol{x}\leqslant\frac{\varepsilon\gamma}{2}\int\limits_{\Omega}\frac{\rho^{\gamma-2}}{\theta}|\nabla\rho|^2\,d\boldsymbol{x}+\frac{1}{2}\int\limits_{\Omega}\theta^{\frac{1}{\gamma-1}}|\nabla s|^2\,d\boldsymbol{x}+$$

$$+\frac{\varepsilon}{4}\int\limits_{\partial\Omega}s^-e^{s^-}\,d\sigma+\frac{1}{4}\|\nabla s\|^2_{L_2(\Omega)}+\|\rho\theta\|_{L_1(\Omega)}+\sum_{j=1}^{N}\|\boldsymbol{u}_j\|^2_{W_2^1(\Omega)}+\frac{3N}{4}\|\rho\|^2_{L_2(\Omega)}+B_7,$$

where $B_7 = B_7(B_4, B_5, B_6)$. The last integral in the right hand side of (3.8) is estimated as follows:

$$\varepsilon \int\limits_{\partial\Omega} (s^+ e^{-s^+} + s^-) d\sigma \leqslant \frac{\varepsilon}{4} \int\limits_{\partial\Omega} s^- e^{s^-} d\sigma - \varepsilon \int\limits_{\partial\Omega} s^- d\sigma + 2|\partial\Omega|,$$

and after simple estimations of the rest integrals we derive from (3.8) the inequality

$$\begin{aligned}\int\limits_{\Omega} \frac{1+\theta^m}{\theta^2} |\nabla\theta|^2 \, d\boldsymbol{x} + \int\limits_{\Omega} \rho^\gamma \, d\boldsymbol{x} + \int\limits_{\partial\Omega} \left(L(\theta)\theta + \frac{\widehat{\theta}}{\theta} + \varepsilon|s| \right) d\sigma \leqslant \\ \leqslant B_8 \left(\sum_{j=1}^N \|\boldsymbol{u}_j\|^2_{W_2^1(\Omega)} + \|\rho\theta\|_{L_1(\Omega)} + 1 \right),\end{aligned} \tag{3.10}$$

where $B_8 = B_8 \left(B_7, \{\|\boldsymbol{f}_i\|_{C(\overline{\Omega})}\}, \|\widehat{\theta}\|_{C(\partial\Omega)}, M, N, m, \varepsilon, \gamma, \Omega \right)$. Here is contained the estimate of $\theta^{\frac{m}{2}}$ in $W_2^1(\Omega)$, and hence in $L_6(\Omega)$, and after additional elementary transformations we obtain the inequality

$$\|\theta\|^m_{L_{3m}(\Omega)} \leqslant B_9(B_8, m, \gamma, \Omega) \left(\sum_{j=1}^N \|\boldsymbol{u}_j\|^2_{W_2^1(\Omega)} + 1 \right). \tag{3.11}$$

Let us estimate the integrals in the right hand side of the inequality (3.7). Using (1.9) and elementary inequalities, we easily obtain the estimate

$$\sum_{j=1}^N \|\boldsymbol{u}_j\|^2_{W_2^1(\Omega)} + \|\rho\|^\gamma_{L_\gamma(\Omega)} \leqslant B_{10} \left(\|\rho\theta\|^2_{L_2(\Omega)} + 1 \right), \tag{3.12}$$

where $B_{10} = B_{10} \left(B_0, \{\|\boldsymbol{f}_i\|_{C(\overline{\Omega})}\}, M, N, \varepsilon, \gamma, \Omega \right)$. Using the obvious inequality

$$\|\rho\theta\|^2_{L_2(\Omega)} \leqslant \frac{1}{2B_{10}} \|\rho\|^\gamma_{L_\gamma(\Omega)} + \frac{1}{2B_9 B_{10}} \|\theta\|^m_{L_{3m}(\Omega)} + B_{11}(B_9, B_{10}, m, \gamma, |\Omega|),$$

it is possible to obtain from (3.11) and (3.12) the estimate of the left hand side of (3.12) by the value $B_{12} := 2B_{10}B_{11} + 2B_{10} + 1$, and hence of the left hand side of (3.11) by the value $B_{13} := B_9 + B_9 B_{12}$. As a result, from (3.10)–(3.12) we obtain the following estimate:

$$\begin{aligned}\sum_{j=1}^N \|\boldsymbol{u}_j\|_{W_2^1(\Omega)} + \|\rho\|_{L_\gamma(\Omega)} + \|\theta\|_{L_{3m}(\Omega)} + \|\nabla\theta\|_{L_2(\Omega)} + \int\limits_{\partial\Omega} (e^s + e^{-s}) d\sigma + \\ + \|\nabla s\|_{L_2(\Omega)} + \|\theta\|_{L_{2m}(\partial\Omega)} \leqslant B_{14} \left(B_{8-13}, \min_{\partial\Omega} \widehat{\theta}, m, \gamma, \Omega \right).\end{aligned} \tag{3.13}$$

Let us denote

$$\boldsymbol{\alpha}_i = \frac{1}{2|\Omega|} \int\limits_{\Omega} \rho_i (\boldsymbol{v} \cdot \nabla) \boldsymbol{u}_i \, d\boldsymbol{x}, \quad i = 1, \ldots, N,$$

$$\boldsymbol{H}_i = \lambda \left(-\frac{\varepsilon}{2} \rho_i \boldsymbol{u}_i - \frac{\varepsilon M}{2|\Omega|} \alpha_i \boldsymbol{u}_i + \rho_i \boldsymbol{f}_i - \boldsymbol{\alpha}_i \right), \quad i = 1, \ldots, N,$$

$$\Phi(z) = \int_0^z (1+e^{my})(\varepsilon + e^y)dy$$

and observe that $\operatorname{sgn}\Phi(z) = \operatorname{sgn} z$, $|\Phi(z)| \leqslant 2 + \varepsilon|z| + e^{(m+1)z}\chi(z)$. Let us denote by $\mathbb{V}_i$, $i = 1,\dots,N$, the solutions to the boundary value problems

$$\operatorname{div}\mathbb{V}_i = \frac{1}{2}\rho_i(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}_i - \boldsymbol{\alpha}_i \text{ in } \Omega, \quad \mathbb{V}_i|_{\partial\Omega} = 0, \quad i = 1,\dots,N, \tag{3.14}$$

and set

$$\mathbb{G}_i = \lambda\left(-\beta_i\rho^\gamma\mathbb{I} - \beta_i\rho\theta\mathbb{I} - \frac{1}{2}\rho_i\boldsymbol{v}\otimes\boldsymbol{u}_i - \mathbb{V}_i\right), \quad i = 1,\dots,N.$$

In these notations, the equations (3.2) take the form

$$\sum_{j=1}^N L_{ij}\boldsymbol{u}_j = \boldsymbol{H}_i + \operatorname{div}\mathbb{G}_i, \quad i = 1,\dots,N,$$

and the problem (3.3), (3.4) is written as (let us remind of the notations (3.5), (3.6))

$$-\Delta\Phi(s) = \Pi \text{ in } \Omega, \quad \nabla\Phi(s)\cdot\boldsymbol{n} = \widehat{\Pi} \text{ at } \partial\Omega. \tag{3.15}$$

The way from (3.13) to the desired estimate of the functions $(\rho, s, \boldsymbol{u}_1,\dots,\boldsymbol{u}_N)$ in $W^2_p(\Omega)$ is a chain of estimates (constants B_k) of norms in the following sequence:

$$\|\boldsymbol{u}_i\|_{L_6(\Omega)}, \quad \|\rho\|_{L_\infty(\Omega)}, \quad \|\rho\boldsymbol{u}_i\|_{L_6(\Omega)}, \quad \|\boldsymbol{H}_i\|_{L_6(\Omega)}, \tag{3.16}$$

$$\|\boldsymbol{u}_i\|_{W^1_3(\Omega)}, \quad \|\boldsymbol{u}_i\|_{L_{4p}(\Omega)}, \quad \|\rho\|_{W^1_{4p}(\Omega)}, \quad \|\rho\|_{W^2_3(\Omega)}, \quad \|\boldsymbol{H}_i\|_{L_{4p}(\Omega)}, \tag{3.17}$$

$$\|\boldsymbol{u}_i\|_{W^1_{\min\{2p,3m\}}(\Omega)}, \ \|\boldsymbol{u}_i\|_{C(\overline{\Omega})}, \ \|\rho\|_{W^2_{\min\{2p,3m\}}(\Omega)}, \ \|\nabla\rho\|_{C^{\sigma_2}(\overline{\Omega})} \ (\exists\,\sigma_2 > 0), \tag{3.18}$$

$$\|\Phi(s)\|_{W^1_2(\Omega)}, \quad \|\theta^{m+1}\|_{L_6(\Omega)}, \quad \|\theta^m\nabla\theta\|_{L_2(\Omega)}, \tag{3.19}$$

$$\|\Phi(s)\|_{W^2_2(\Omega)}, \ \|s\|_{C(\overline{\Omega})}, \ \|\theta\|_{C(\overline{\Omega})}, \ \|\nabla s\|_{L_6(\Omega)}, \ \|\nabla\theta\|_{L_6(\Omega)}, \ \|\boldsymbol{H}_i\|_{C(\overline{\Omega})}, \tag{3.20}$$

$$\|\boldsymbol{u}_i\|_{W^2_6(\Omega)}, \quad \|\boldsymbol{u}_i\|_{C^{\frac{3}{2}}(\overline{\Omega})}, \quad \|\rho\|_{C^2(\overline{\Omega})}, \tag{3.21}$$

$$\|\Phi(s)\|_{W^2_6(\Omega)}, \quad \|s\|_{W^2_3(\Omega)}, \quad \|\theta\|_{W^2_3(\Omega)}, \quad \|\nabla s\|_{C(\overline{\Omega})}, \quad \|\nabla\theta\|_{C(\overline{\Omega})}, \tag{3.22}$$

$$\|\boldsymbol{u}_i\|_{W^2_p(\Omega)}, \quad \|\Phi(s)\|_{W^2_p(\Omega)}, \quad \|s\|_{W^2_p(\Omega)}, \quad \|\theta\|_{W^2_p(\Omega)} \tag{3.23}$$

(here $i = 1,\dots,N$ everywhere), where the horizontal transitions (inside each of the given groups) are quite trivial (they follow from the estimates obtained by the corresponding moment, embedding theorems, standard properties of elliptic problems and properties of the function Φ). Let us explain all the transitions between the lines following the order.

The estimates (3.13), (3.16) and the properties of the problem (3.14) (see [11], [7]) provide the estimates for $\|\mathbb{V}_i\|_{W^1_{\frac{3}{2}}(\Omega)}$, $i = 1,\dots,N$, and hence for $\|\mathbb{G}_i\|_{L_3(\Omega)}$, $i = 1,\dots,N$, that leads to the start of (3.17).

Using again the properties (3.14), from (3.17) we obtain the estimates for $\|\mathbb{V}_i\|_{L_{4p}(\Omega)}$, $i = 1,\dots,N$, which give the start of (3.18) without any difficulties.

Directly from (3.13) we have the estimates

$$\|\Phi(s)\|_{L_{\frac{2m}{m+1}}(\partial\Omega)} \leqslant B_{15}, \quad \int\limits_{\partial\Omega} \Phi(s)\widehat{\Pi} d\sigma \leqslant B_{16}, \tag{3.24}$$

where $B_{15} = B_{15}(B_{14}, m, \Omega)$, $B_{16} = B_{16}\Big(B_{14}, B_{15}, \|\widehat{\theta}\|_{L_\infty(\partial\Omega)}, m, |\partial\Omega|\Big)$, and after (3.18) the norm $\|\Pi\|_{L_2(\Omega)}$ is estimated as well. From (3.15) it follows the identity, the right hand side of which can be estimated via $(3.24)_2$:

$$\int\limits_\Omega |\nabla\Phi(s)|^2 \, d\boldsymbol{x} = \int\limits_\Omega \Phi(s)\Pi \, d\boldsymbol{x} + \int\limits_{\partial\Omega} \Phi(s)\widehat{\Pi} d\sigma \leqslant \|\Pi\|_{L_2(\Omega)}\|\Phi(s)\|_{L_2(\Omega)} + B_{16},$$

and now $(3.24)_1$ admits to pass to the start of (3.19).

After (3.19), the values s and θ^{σ_3} for all $\sigma_3 \in [1, m+1]$ are estimated in $W_2^1(\Omega)$, and hence in $W_2^{\frac{1}{2}}(\partial\Omega)$, so that the estimate $\widehat{\Pi}$ in $W_2^{\frac{1}{2}}(\partial\Omega)$ follows, and finally (3.15) gives the start of (3.20).

Passage from (3.20) to (3.21) follows from the estimates $\operatorname{div}\mathbb{G}_i$, $i = 1, \ldots, N$ in $L_6(\Omega)$.

After (3.21), we have the estimates for $\|\Pi\|_{L_6(\Omega)}$, s and θ^{σ_4} (for all $\sigma_4 \geqslant 1$) in $W_6^1(\Omega)$, and hence in $W_6^{\frac{5}{6}}(\partial\Omega)$, so that the estimate for $\|\widehat{\Pi}\|_{W_6^{\frac{5}{6}}(\partial\Omega)}$ follows, and (3.15) provides (3.22).

Passage from (3.22) to (3.23) follows from the estimate for $\|\operatorname{div}\mathbb{G}_i\|_{C(\overline{\Omega})}$, $i = 1, \ldots, N$. Inside (3.23) in the case $p > 6$, it is necessary to analyze the problem (3.15) again.

Theorem 2.3 is proved.

References

[1] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford University Press, Oxford, 2004. Zbl 1080.76001

[2] E. Feireisl, A. Novotny, *Singular limits in thermodynamics of viscous fluids*, Birkhauser, Basel, 2009. Zbl 1176.35126

[3] P.B. Mucha, M. Pokorny, *On the steady compressible Navier–Stokes–Fourier system*, Comm. Math. Phys, **288**:1 (2009), 349–377. Zbl 1172.35467

[4] P.B. Mucha, M. Pokorny, *Weak solutions to equations of steady compressible heat conducting fluids*, Math. Models Methods Appl. Sci., **20**:5 (2010), 785–813. Zbl 1191.35207

[5] A.E. Mamontov, D.A. Prokudin, *Viscous compressible multi-fluids: modeling and multi–D existence*, Methods Appl. Anal., **20**:2 (2013), 179–196. Zbl 1290.35203

[6] A.E. Mamontov, D.A. Prokudin, *Viscous compressible homogeneous multi–fluids with multiple velocities: barotropic existence theory*, Sib. Electron. Math. Izv., **14** (2017), 388–397. Zbl 1379.35248

[7] A. Novotny, I. Straskraba, *Introduction to the mathematical theory of compressible flow*, Oxford University Press, Oxford, 2004. Zbl 1088.35051

[8] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., **12**:4 (1959), 623–727. Zbl 0093.10401

[9] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math., **17**:1 (1964), 35–92. Zbl 0123.28706

[10] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer–Verlag, Berlin, 1983. Zbl 0562.35001

[11] M.E. Bogovskii, *The solution of some problems of vector analysis related to the operators* div *and* grad, Trudy sem. S.L. Soboleva, **1** (1980), 5–40. Zbl 0479.58018

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