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L_∞ NORM MINIMIZATION FOR NOWHERE-ZERO INTEGER EIGENVECTORS OF THE BLOCK GRAPHS OF STEINER TRIPLE SYSTEMS AND JOHNSON GRAPHS

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ABSTRACT. We study nowhere-zero integer eigenvectors of the block graphs of Steiner triple systems and the Johnson graphs. For the first eigenvalue we obtain the minimums of L_∞-norm for several infinite series of Johnson graphs, including $J(n,3)$ as well as general upper and lower bounds. The minimization of L_∞-norm for nowhere-zero integer eigenvectors with the second eigenvalue of the block graph of a Steiner triple system S is equivalent to finding the minimum nowhere-zero flow for Steiner triple system S. For the all Assmuss-Mattson Steiner triple systems of order at least 99 we prove that this minimum is bounded above by 4.

Keywords: Steiner triple system, flow, strongly regular graph, Johnson graph, Grassmann graph, eigenvalue

1. Introduction

A vector is called *nowhere-zero integer* (shortly NZI vector) if all of its elements are nonzero integers. The infinity norm $\|v\|_\infty$ of a vector v is defined as the maximum of the absolute values of its elements.

Let W_S be the point-block incidence matrix of Steiner triple system S. A nowhere-zero integer vector u: $W_S u = 0$ is called a nowhere-zero $(\|u\|_\infty + 1)$-*flow* for Steiner triple system S [1]. It is not hard to see that the right nullspace of the incidence matrix W_S coincides with the second eigenspace of the block graph of S (see Proposition 4 below).

The minimum of L_∞ norm for flows of Steiner triple systems were considered in works [1], [2], [4]. Akbari, Burgess, Danziger and Mendelsohn [3] showed that the norm of a flow in Steiner triple system of order n is upper bounded by $O(n^2)$.

On the other hand, studies show that for particular families of Steiner triple systems, the actual minimum of norm of flow is much smaller than $O(n^2)$ and this

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fact finds similarities in a conjecture of Tutte on existence of a 5-flow for the graphs [24]. Speaking more precisely, all Steiner triple systems of order 15 have 3-flows [1]. Furthermore, it was proven that some well-known recursive classes of Steiner triple systems, such as direct product construction, $2v + 7$-construction admit a 3-flow, given a 3-flow in the original Steiner triple system [4]. As for the $2v+1$-construction, the resulting Steiner triple systems has 3, 4 or 5-flow [3]. In Section 5 we establish that any Assmuss-Mattson [6] Steiner triple system obtained from a Steiner triple system S of order at least 49 has a 5-flow, regardless of the flow in the original system S.

One might consider a more general definition of a flow for any given natural matrix W, which is, for example, the inclusion matrix of subsets [1] or subspaces [23]. A flow in these cases is the sum of eigenvectors of a Johnson or Grassmann graph with specific eigenvalues. For example, let W be the inclusion matrix of 2-subsets and k-subsets of n-element set. In [1] it was shown that for $k = 3$, there is a nowhere-zero integer vector v: $Wv = 0$, $\|v\|_\infty = 2$, i.e. a generalized 3-flow. Note that any such vector is an eigenvector of the Johnson graph $J(n, 3)$ with third eigenvalue. Relying on the properties of higher order inclusion matrices, the authors of [1] extended the result and showed that a 3-flow exists for the inclusion matrix of 2-subsets and k-subsets for any $k \geq 3$.

The perspective of the continuing studies of flows in Steiner triple systems and some other structures implies the following natural question.

Problem 1. Given a distance-regular graph Γ of diameter d and its rational eigenvalue $\theta_i(\Gamma)$, $0 \leq i \leq d$, find

$$min\{\|u\|_\infty + 1 : u \text{ is a nowhere-zero integer } \theta_i(\Gamma)\text{-eigenvector of } \Gamma\},$$

which we denote as $m(i, \Gamma)$ in below.

From results of Akbari et al. there is always a solution for Problem 1.

Theorem 1. [1], [2] *Let Γ be a distance-regular graph with a rational eigenvalue $\theta_i(\Gamma)$, $0 \leq i \leq d$. Then there is a NZI $\theta_i(\Gamma)$-eigenvector of Γ.*

Proof. By [2, Theorem 3] there is a nowhere-zero real eigenvector for every eigenvalue of any distance-regular graph. By [1, Lemma 3.3] the existence of a nowhere-zero real vector, belonging to the null space of a rational matrix implies the existence of nowhere-zero integer vector in the null space. By taking the matrix to be $A_\Gamma - \theta_i I$, where A_Γ is the adjacency matrix of Γ, we obtain the required. □

Definitions, notation and basic theory are in Section 2. The results of Section 3 are presented in a general context: for q-ary Steiner triple systems and Grassmann graphs; the classic Steiner triple systems and Johnson graph are treated as a particular case. Recently, q-ary Steiner triple systems were shown to exist asymptotically [19], however there is only one explicit example [9] of order 13. We consider a description of the eigenspaces of the block graph of q-ary Steiner triple system in terms of the point-block incidence matrix of STS. In particular, for the first eigenvalues, we see that eigenvectors of the Grassmann graph $J_q(n, 3)$ are in a natural one-to-one correspondence with that of the block graph of q-ary Steiner triple system S order n: the restriction of any eigenvector of $J_q(n, 3)$ to the blocks of S is an eigenvector of the block graph of S. This relation between the first eigenspaces of STSs and $J_q(n, k)$ is in spirit of [25], where an extension of eigenvectors of Johnson graphs to that of Hamming graphs was established. Despite this strong connection, in Section 4 we show that the minimums of the L_∞ norm for nowhere-zero eigenvectors for both graphs are different for $q = 1$. We establish lower and upper bounds on the optimum norms of NZI $\theta_1(J(n,k))$-eigenvectors of the Johnson graphs $J(n,k)$ and obtain the exact minimums for infinite series of Johnson graphs $J(n,k)$. In particular, we completely solve the problem for $k = 3$ and all $n > 63$ (see Theorem 5). The bounds on the L_∞ norm of the NZI eigenvectors of the block graphs of STSs with the second eigenvalue (which is equivalent to finding i-flow for STSs for small i) are given in Section 5. We start Section 5.1 with reviewing the existing results for flows in projective and Bose Steiner triple systems which utilize the aspects of cyclicity and resolvability of these designs. In Section 5.2 we show that any Steiner triple system constructed by Assmuss-Mattson approach [6] of order at least 99 has a 5-flow. The results of this section are described in terms of flows of Steiner triple systems rather than eigenvectors of their block graphs.

In Section 6 we discuss completely regular codes in the block graphs of Steiner triple system. These objects are in the scope of current study, as the minimum possible value of L_∞ norm for nowhere-zero integer eigenvectors is attained on a vector arising from a specific completely regular code (see Proposition 2). From perspective of Cameron-Liebler line classes [10], the completely regular codes in the block graphs of Steiner triple system with the covering radius 1 and the first eigenvalue are of interest as they provide one of different variations [13], [17] of such objects. The block graph of projective (Hamming) Steiner triple system of order $2^r - 1$ is isomorphic to the Grassmann graph $J_2(r,2)$ and all such completely regular codes are exactly Cameron-Liebler line classes in $PG(n-1,2)$. We conjecture that these codes comprise only the following classic examples of Cameron-Liebler line classes: a point, hyperplane and nonincident point-hyperplane pair (see Problem 2 in Section 6) and show that there are no other codes for STSs of orders 13 and 15. These codes in the block graphs of Steiner triple systems and affine Steiner triple systems in particular, were considered in [18], where these objects and the above mentioned conjecture were treated from the perspective of small support eigenvectors of the block graphs.

2. Definitions and notations

2.1. Johnson, Grassmann graphs, q-ary Steiner triple systems and their block graphs. A regular graph of diameter d is called *distance-regular* if there is an array of integers

$$\{\beta_0, \dots, \beta_{d-1}; \gamma_1, \dots, \gamma_d\},$$

such that for any vertices x and y at distance i, $i \in \{0, \dots, d\}$ there are exactly β_i neighbors of y at distance $i+1$ from x and γ_i neighbors of y at distance $i-1$ from x. The array of integers $\{\beta_0, \dots, \beta_{d-1}; \gamma_1, \dots, \gamma_d\}$ is called the *intersection array* of the distance-regular graph Γ. We say that a nonzero vector v is $\theta(\Gamma)$-*eigenvector* if $A_\Gamma v = \theta(\Gamma)v$, where A_Γ is the adjacency matrix of Γ. In this case θ is called an *eigenvalue* of Γ. It is well known that any distance-regular graph of diameter d has exactly $d+1$ distinct eigenvalues, which we index in descending order: $\theta_0(\Gamma) > \theta_1(\Gamma) > \ldots > \theta_d(\Gamma)$, note that $\theta_0(\Gamma)$ is the valency of the graph.

The vertices of *the Grassmann graph* $J_q(n,k)$ are k-subspaces of the finite vector space $\mathbb{F}_q^n$ over the field $\mathbb{F}_q$ and the edges are pairs of subspaces meeting in $(k-1)$-subspace. We also include a limit case of $q = 1$ as we define *the Johnson graph*

$J(n,k)$ (also denoted by $J_1(n,k)$) to be the graph with the vertex set being k-subsets of $\{1,\dots,n\}$ and edges being pairs of subsets meeting in $(k-1)$ set. Let $\left[{n \atop k}\right]_q$ be the Gaussian binomial coefficient and $\left[{n \atop k}\right]_1$ be the ordinary binomial coefficient. The $k+1$ eigenvalues of $J_q(n,k)$, $q \geq 1$ are as follows:

$$\theta_i(J_q(n,k)) = q^{k+1}\left[{k-i \atop 1}\right]_q\left[{n-k-i \atop 1}\right]_q - \left[{i \atop 1}\right]_q, 0 \leq i \leq k.$$

By a q-ary *Steiner triple system* (briefly, STS) S we mean a collection of 3-subspaces (called blocks) of $\mathbb{F}_q^n$ such that any 2-subspace of $\mathbb{F}_q^n$ is in exactly one subspace in S. By letting $q=1$ we include Steiner triple system in the traditional sense in the definition above, i.e. a collection of 3-subsets of $\{1,\dots,n\}$ (called blocks or triples) such that any 2-subset is in exactly one block. A 1-subspace of $\mathbb{F}_q^n$ (1-subset of $\{1,\dots,n\}$) is called *a point* of S. By *the order* of S, we mean the dimension n of the ambient space F_q^n (the number of points for $q=1$).

The incidence matrix of a q-ary Steiner triple system S is the matrix W_S, whose rows are indexed by the points of S and the columns are indexed by its blocks:

$$(W_S)_{i,T} = \begin{cases} 1, & i \subseteq T \\ 0, & \text{otherwise} \end{cases}.$$

Similarly, we define the point-block incidence matrix W of $J_q(n,k)$ with rows indexed by the points (1-subspaces) and columns are indexed by the k-subspaces of $\mathbb{F}_q^n$. The *block intersection* graph of q-ary, $q \geq 1$, STS S denoted by Γ_S, has the blocks of S as vertices and blocks having a nonempty intersection as edges. It is well-known that this graph is strongly-regular [5] and has the following eigenvalues:

$$\theta_1(\Gamma_S) = \frac{\left[{n \atop 1}\right]_q - 1}{\left[{3 \atop 1}\right]_q - 1} - \left[{3 \atop 1}\right]_q - 1,$$

$$\theta_2(\Gamma_S) = -\left[{3 \atop 1}\right]_q.$$

The blocks of any q-ary STS, $q \geq 1$, of order n could be treated as a set of vertices of $J_q(n,3)$. So the block intersection graph can be viewed as the subgraph induced by the blocks of STS in the distance-2 graph of $J_q(n,3)$. A Steiner triple system is called *resolvable* if its blocks are parted into *parallel classes*, i.e. collections of pairwise nonintersecting blocks with the size of each class equal to $\frac{n}{3}$.

2.2. **Completely regular codes.** Given $C \subseteq V(\Gamma)$, a *distance partition* with respect to C is $C_0 = C, \dots, C_\rho$:

$$C_i = \{x : d(x,C) = i\}.$$

The maximum of all i's is denoted by ρ and is called the *covering radius* of C. A subset $C \subseteq V(\Gamma)$ is called a *completely regular code* if there are numbers $\alpha_0,\dots,\alpha_\rho$, $\beta_0,\dots,\beta_{\rho-1}$, $\gamma_1,\dots,\gamma_\rho$ such that any vertex of C_i is adjacent to exactly α_i, β_i, γ_i vertices of C_i, C_{i+1} and C_{i-1} respectively, for $i = 0,\dots\rho$ and $\gamma_0 = \beta_{\rho+1} = 0$. The array $\{\beta_0,\dots,\beta_{\rho-1};\gamma_1,\dots,\gamma_\rho\}$ is called *the intersection array* of the completely regular code C. For a completely regular code C consider the following tridiagonal

$(\rho+1)\times(\rho+1)$ matrix

$$\begin{pmatrix} \alpha_0 & \beta_0 & 0 & 0 & \dots & 0 \\ \gamma_1 & \alpha_1 & \beta_1 & 0 & \dots & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & 0 & \gamma_{\rho-1} & \alpha_{\rho-1} & \beta_{\rho-1} \\ 0 & . & 0 & 0 & \gamma_\rho & \alpha_\rho \end{pmatrix},$$

which we call the *intersection matrix* of the completely regular code C.

The eigenvalues of this matrix are called *the eigenvalues* of the completely regular code C. It is well-known that the eigenvalues of a completely regular code in a regular graph are necessarily eigenvalues of the graph, which is known as Lloyd's theorem.

Given a set of vertices C in a graph Γ we denote by χ_C its characteristic vector in the vertex set of the graph.

Proposition 1. *(Folklore) Let C be a completely regular code with $\rho = 1$, eigenvalue $\theta_i(\Gamma)$ and intersection array $\{\beta_0;\gamma_1\}$ in a distance-regular graph Γ. Then the vector $\beta_0\chi_C - \gamma_1\chi_{V(\Gamma)\setminus C}$ is a θ_i-eigenvector of Γ. Moreover, any θ_i-eigenvector (up to multiplicity) with only two values can be obtained in this manner.*

The current study of the completely regular codes from the point of view of Problem 1 is inspired by the following statement.

Proposition 2. *Let Γ be a distance-regular graph. Then $m(i,\Gamma)\geq 2$, $m(i,\Gamma)=2$ if and only if there is a completely regular code with $\rho = 1$, intersection array $\{\beta_0;\beta_0\}$ and eigenvalue $\theta_i(\Gamma)$ in Γ.*

Proof. We obviously have $m(i,\Gamma)\geq 2$ and $m(i,\Gamma)$ attains the lower bound 2 if and only if there is an $\theta_i(\Gamma)$-eigenvector with values $+1$ and -1 only. From Proposition 1 we obtain the required. ☐

For the completely regular codes with $\rho = 1$ and the second eigenvalue, we see that they are equivalent to 1-subdesigns of Steiner triple system.

Proposition 3. *Let S be a Steiner triple system of order n. A set $S' \subset S$ is a completely regular code with $\rho = 1$ and eigenvalue $\theta_2(\Gamma_S)$ if and only if S' is a 1-design.*

Proof. Let S' be a subset of blocks of S. The set S' is 1-$(n,3,\lambda)$-design if and only if the vector $W_S((\frac{n-1}{2}-\lambda)\chi_{S'}-\lambda\chi_{S\setminus S'})$ is all-zero, where W_S is the point-block incidence matrix of STS S. This follows from the fact that any point of S belongs to $\frac{n-1}{2}$ and λ blocks of S and S'. In view of Proposition 4.1 below this is equivalent to $(\frac{n-1}{2}-\lambda)\chi_{S'}-\lambda\chi_{S\setminus S'}$ being a $\theta_2(\Gamma_S)$-eigenvector. As any two-valued eigenvector of the graph corresponds to a completely regular code with $\rho = 1$, see Proposition 1, we obtain the required.

☐

3. A description of the eigenspaces of the block graphs of STS and the first eigenspace of $J_q(n,k)$

Firstly, we consider the following auxiliary fact.

Theorem 2. [7, Theorem 1] *Let $\overline{\Gamma}$ be a biregular bipartite graph with valencies c and c' and the halved graphs Γ and Γ' and let I denote the incidence matrix of two parts of $\overline{\Gamma}$. Let any pair of vertices of $V(\Gamma)$ at distance 2 in $\overline{\Gamma}$ have exactly m common neighbors and any pair of vertices of $V(\Gamma')$ at distance 2 in $\overline{\Gamma}$ have exactly m' common neighbors. The following holds*

1. Let u be a θ-eigenvector of Γ, $\theta \neq -\frac{c}{m}$. Then the vector Iu is a $\frac{c-c'+m\theta}{m'}$-eigenvector of Γ'.

2. Given a vector u, Iu is all-zero vector if and only if u is $-\frac{c}{m}$-eigenvector of Γ'.

Example 1. Consider the biregular graph whose parts are the blocks and the points of a q-ary STS S with adjacency being point-block inclusion. The halved graphs are the block graph Γ_S (as Γ in Theorem 2) and the complete graph $K_{[{n \atop 1}]_q}$ (as Γ' in Theorem 2) on the points of S, I is the point-block incidence matrix W_S of the design S. It is not hard to see that parameters in Theorem 2 are $m = m' = 1$, $c = [{3 \atop 1}]$, $c' = \frac{[{n \atop 1}]_q - 1}{[{3 \atop 1}]_q - 1}$.

We obtain the following relation between the null-spaces of the incidence matrix of Steiner triple system and the second eigenspace.

Proposition 4. *Let S be a q-ary STS S, $q \geq 1$ with the point-block intersection matrix W_S. Then*

1. A vector u fulfills $W_S u$=0 if and only if u is a $\theta_2(\Gamma_S)$-eigenvector of Γ_S.

2. A vector v is a $(\|v\|_\infty + 1)$-flow for S if and only if v is a nowhere-zero integer $\theta_2(\Gamma_S)$-eigenvector of the block graph Γ_S of the Steiner triple system S.

Proof. We apply Theorem 2.2 for the graph in Example 1. □

3.1. **The first eigenspaces of the block graphs of STSs, Grassmann and Johnson graphs.** In the theorem below by W we denote the incidence matrix of all 1-subspaces of $\mathbb{F}_q^n$ (1-subsets of $\{1, \ldots, n\}$ for $q = 1$) vs k-subspaces $\mathbb{F}_q^n$ (1-subsets of $\{1, \ldots, n\}$ for $q = 1$). A vector is a *restriction* of a vector v to a set S if the vector is obtained deleting all the elements of v having indices outside of S. This operation in coding theory is also known as *puncturing*. Contrary, when we say that a vector is an *extention* of v if it is obtained by appending some extra elements to v.

Theorem 3. *Let U be the set of all real-valued vectors of the vertices of the graph $J_q(n, 1)$, $q \geq 1$ with the sum of values is zero. Then*

1. U is the set of all $\theta_1(J_q(n, 1))$-eigenvectors of $J_q(n, 1)$.

2. [14][20] *$W^T(U)$ is the set of all $\theta_1(J_q(n, k))$-eigenvectors of $J_q(n, k)$.*

3. If S is a q-ary Steiner triple system of order n, then $W_S^T(U)$ is the set of all $\theta_1(\Gamma_S)$-eigenvectors of its block graph Γ_S.

4. The restriction of each $\theta_1(J_q(n, 3))$-eigenvector of $J_q(n, 3)$ to the blocks of S is $\theta_1(\Gamma_S)$-eigenvector of its block graph Γ_S and each $\theta_1(\Gamma_S)$-eigenvectors of its block graph Γ_S is extended to a unique $\theta_1(J_q(n, 3))$-eigenvector of $J_q(n, 3)$.

5. If S is a q-ary Steiner triple system of order n, then $m(1, \Gamma_S) \leq m(1, J_q(n, [{3 \atop 1}]_q))$.

Proof. 1. The graph $J_q(n, 1)$ is the complete graph $K_{[{n \atop 1}]_q}$. Since $\theta_1(J_q(n, 1)) = -1$, we see a (-1)-eigenvector u of $K_{[{n \atop 1}]_q}$ is such that $\sum\limits_{x \in K_{[{n \atop 1}]_q}} u_x = 0$ and vice versa.

2. For the Johnson graphs $J(n, k)$ this property was established by Delsarte [14]. For Grassmann graphs the proof could be found in [20]. We note that the result

follows by consecutively applying Theorem 2 to the pairs of the graphs $J_q(n,i)$ and $J_q(n,i+1)$ for $i=0,\dots,k-1$. The graph are the halved graphs of the bipartite graph, with adjacency being incidence of i-subspaces into $i+1$-subspace of $\mathbb{F}_q^n$ (which is $\overline{\Gamma}$ in Theorem 2).

3. In view of Theorem 2 consider the bipartite graph $\overline{\Gamma}$ where adjacency is the containment relation for the vertices of $J_q(n,1)$ and $J_q(n,k)$ which are subspaces (or subsets for q=1). The graph is the same as in Example 1 but conversely to it, the complete graph $K_{\left[{n \atop 1}\right]_q}$ is denoted by Γ and the block intersection graph of S is denoted by Γ'. The values mentioned in Theorem 2 are $c=\frac{\left[{n \atop 1}\right]_q-1}{\left[{3 \atop 1}\right]_q-1}$, $c'=\left[{3 \atop 1}\right]$, $m=m'=1$. By Theorem 2, we see that any vector of $W_S^T(U)$ is a $c-c'+\theta_1(J_q(n,1))=\frac{\left[{n \atop 1}\right]_q-1}{\left[{3 \atop 1}\right]_q-1}-\left[{3 \atop 1}\right]_q-1=\theta_1(\Gamma_S)$-eigenvector of the block graph Γ_S.

On the other hand, we apply Theorem 2 to $\overline{\Gamma}$ with interchanged the roles of Γ and Γ'. We see that any $\theta_1(\Gamma_S)$-eigenvector v implies that W_Sv is a $\theta_1(J_q(n,1))$-eigenvector of the graph $J_q(n,1)$. Therefore, W_S^T establishes an isomorphism between the first eigenspaces of $J_q(n,1)$ and Γ_S.

4. From the second and third statements of the theorem, we see that W_S^T ($W^T_{J_q(n,3)}$ respectively) settles an isomorphism between the first eigenspaces of the complete graph and the block graph (the graph $J_q(n,k)$ respectively). Each STS S could be treated as vertices of $J_q(n,3)$ and the rows of the block-point incidence matrix W_S^T for q-ary STS S form a subset of the rows of the block-point incidence matrix $W^T_{J_q(n,3)}$ for $J_q(n,3)$, which implies the required.

5. From the fourth statement of the current theorem we see that the restriction of a $\theta_1(J_q(n,3))$-eigenvector v of $J_q(n,3)$ to the blocks of S is an $\theta_1(\Gamma_S)$-eigenvector v' of S. Obviously the norm is not increased upon reduction, i.e.: $\|v\|_\infty \geq \|v'\|_\infty$. □

Remark 1. When q is 1 (i.e. for STS in traditional sense and Johnson graphs) the bound in Theorem 3.5 is not sharp for as we show in the next Section that for STS S of order n $m(1,\Gamma_S)\leq 5$ whereas $m(1,J(n,3))\geq 6$.

4. Minimum of L_∞ norm on nowhere-zero integer eigenvector for the block graphs of STSs and Johnson graphs

In the rest of the paper, we set $q=1$ and consider Steiner triple systems in the classical sense.

4.1. **Minimum of L_∞ norm of nowhere-zero integer eigenvectors with the first eigenvalue for Johnson graphs.** In this subsection we denote by (n,k) the greatest common divisor of n and k.

Lemma 1. *Any $\theta_1(J(n,k))$-eigenvector of $J(n,k)$ is equal to W^Tu, where u is a real-valued eigenvector of $J(n,1)$, $\sum\limits_{i=1,\dots,n} u_i=0$. If W^Tu is an integer $\theta_1(n,k)$-eigenvector of $J(n,k)$, then u is such that u_i and u_j have the same fractional parts for all $1\leq i,j\leq n$. Moreover, the fractional parts equal $\frac{r}{s}$, where r and s are some non-negative integers such that $0\leq r<s$, $(r,s)=1$ and s is a divisor of (n,k).*

Proof. By Theorem 3 we see that any $\theta_1(J(n,k))$-eigenvector of $J(n,k)$ is equal to W^Tu, where u is such that $\sum\limits_{i=1,\dots,n} u_i=0$. Let us prove that u_i and u_j have the same fractional parts for any $i\neq j$. Consider some pairwise distinct positions

$i_1 = i, i_2 \ldots, i_k$, which are different from j. By hypothesis of the theorem we have $u_i + u_{i_2} + \ldots + u_{i_k}$ and $u_j + u_{i_2} + \ldots + u_{i_k}$ are integers. Hence, $u_i - u_j$ is an integer and u_i and u_j have the same fractional parts. Denote this fractional part by α. Consider some k elements of u: $u_{i_1} \ldots, u_{i_k}$. We have that the sum $u_{i_1} + \ldots + u_{i_k}$ is integer (since $W^T u$ is an integer vector). On the other hand, this sum has the same fractional part as $k\alpha$. Hence, α is rational and can be represented as $\frac{r}{s}$, where r and s are non-negative integers, $0 \leq r < s$. Also we have $\frac{kr}{s}$ is integer, hence, s is a divisor of k. Since $\sum\limits_{i=1}^{n} u_i = 0$, $\frac{nr}{s}$ is integer and, hence, s is a divis or of n. Therefore, s is a divisor of (n, k). □

Proposition 5. *If $n \geq 2k$ then we have $m(1, J(n, k)) \leq \frac{n-k}{(n,k)} + 1$.*

Proof. For this statement we take the vector

$$u^T = (\frac{1}{(n,k)}, \ldots, \frac{1}{(n,k)}, -\frac{n-1}{(n,k)}),$$

of length n, where $\frac{1}{(n,k)}$ is repeated $n-1$ times. The vector $W^T u$ has two different values $\frac{k}{(n,k)}$ and $-\frac{n-k}{(n,k)}$. Since $n \geq 2k$, we have $\|W^T u\|_\infty = \frac{n-k}{(n,k)}$. □

Note that in Proposition 5 the bound tends to infinity when n is growing as a function of k. However, for a "small" n, for example $n = 2k$, it can be sharp. In the following statements we provide further upper bounds for odd and even cases of k.

Proposition 6. *Let k be odd. Then*

1. If n is even then $m(1, J(n, k)) \leq k + 1$.

2. If n is odd then $m(1, J(n, k)) \leq 2k + 1$.

Proof. The vector $W^T u$ is an $\theta_1(J(n,k))$-eigenvector of $J(n,k)$ if $\sum\limits_{i=1,\ldots,n} u_i = 0$.

1) For the first statement we take the vector

$$u^T = (1, \ldots, 1, -1, \ldots, -1),$$

of length n with $\frac{n}{2}$ positions with value 1 and $\frac{n}{2}$ positions with value -1.

2) For the second statement we take the vector

$$u^T = (k+1, -1, \ldots, -1, 1, \ldots, 1),$$

of length n with $\frac{n-k-2}{2}$ positions with value 1, $\frac{n+k}{2}$ positions with value -1, where $u_1 = k + 1$. □

Proposition 7. *Let k be even and γ be the smallest positive integer number that does not divide k. We have the following.*

1. If n is divisible by γ then $m(1, J(n, k)) \leq (\lfloor \frac{\gamma}{2} \rfloor + 1)k + 1$.

2. If n is not divisible by γ then $m(1, J(n, k)) \leq (\lfloor \frac{\gamma}{2} \rfloor + 1)(2k + \beta - 1) + 1$, where β is the remainder of division of $n - (k+1)$ by γ.

Proof. By the hypothesis of the proposition, $n - (\beta + k + 1) = q\gamma$ for some positive integers q and β. Let us describe a vector u, which is an eigenvector of the graph $J(n, 1)$. Divide the first $n - (\beta + k + 1)$ positions into γ blocks. If γ is odd then each block consists from $\lfloor \frac{\gamma}{2} \rfloor$ positions with value $\lfloor \frac{\gamma}{2} \rfloor + 1$ and $\lfloor \frac{\gamma}{2} \rfloor + 1$ positions with value $-\lfloor \frac{\gamma}{2} \rfloor$. If γ is even then each block consists from $\lfloor \frac{\gamma}{2} \rfloor - 1$ positions with value $\lfloor\frac{\gamma}{2}\rfloor + 1$ and $\lfloor\frac{\gamma}{2}\rfloor + 1$ positions with value $-\lfloor\frac{\gamma}{2}\rfloor + 1$. Note that we describe all γ positions in each block and the sum of the values in each block equals 0. We define the values for the remaining $k + \beta + 1$ positions of u. We set one element to be equal to $-(k + \beta)(\lfloor\frac{\gamma}{2}\rfloor + 1)$ and the other $(k + \beta)$ elements to be $(\lfloor\frac{\gamma}{2}\rfloor + 1)$. The sum of values in all positions equals 0 and sum of values in any k positions is integer.

Let us prove that the sum of values in any k positions is not 0. If one of the elements in these positions is $-(\beta+k)(\lfloor\frac{\gamma}{2}\rfloor+1)$ then the sum is less than 0 because the absolute values of all other elements is not greater than $\lfloor\frac{\gamma}{2}\rfloor + 1$.

Consider the case γ is even i.e. $\gamma = 2l$. Take x, $x \in \{0, 1, \ldots, k\}$ positions with values $l + 1$ and $k - x$ positions with values $-l + 1$. The sum of the values in these positions equals $lx + x - kl + k + lx - x = 2lx - k(l - 1)$ which is 0 if and only if

$$2lx = k(l - 1).$$

Let l be odd. From the equality above we see that k is divisible by odd l and by 2 from the condition of the Proposition, so k is divisible by $\gamma = 2l$, which contradicts the definition of γ. If l is even, then $l - 1$ is odd. Hence, again from the equality above we see that k should be divisible by $\gamma = 2l$, a contradiction.

Consider the case when γ is odd, i.e. $\gamma = 2l+1$. Take x positions, $x \in \{0, 1, \ldots, k\}$ with value $l + 1$ and $k - x$ positions with value $-l$. The sum of the values in these positions equals $lx + x - kl + lx$. If this sum equals 0 then $x(2l + 1) = kl$. Since k is not divisible by $\gamma = 2l + 1$, then this sum is equal to 0.

Therefore, $W^T u$ is NZI. If n is divisible by γ, then $\|W^T u\|_\infty = (\lfloor\frac{\gamma}{2}\rfloor + 1)k$. If n is not divisible by γ, then $\|W^T u\|_\infty = (\lfloor\frac{\gamma}{2}\rfloor + 1)(2k + \beta - 1)$. □

In the following Lemmas we study the structural properties of NZI vector as we are working towards lower bounds on $m(1, J(n, k))$.

Lemma 2. *Let u be a real-valued vector indexed by the vertices of the graph $J(n, 1)$ and $\sum\limits_{i=1,\ldots,n} u_i = 0$ that has at least k positions with value that equals $a + \frac{r}{s}$ and at least k positions with value that equals $-b + \frac{r}{s}$ for some non-negative integers a, b, r and s, where $b \neq 0, 0 \leq r < s, (n, s) = 1$ and s is divisor of (n, k),. If $W^T u$ is a NZI $\theta_1(J(n, k))$-eigenvector of $J(n, k)$ then the number $\frac{k(bs-r)}{s(a+b)}$ is not integer.*

Proof. Take x, $0 \leq x \leq k$, positions of u with value $a + \frac{r}{s}$ and $k - x$ positions with the value $-b + \frac{r}{s}$. The sum of these values equals $x(a+b) + \frac{kr}{s} - bk$. This sum equals 0 if and only if $x = \frac{(bs-r)k}{s(a+b)}$ is an integer. □

Lemma 3. *Let u be a real-valued vector indexed by the vertices of the graph $J(n, 1)$, $\sum\limits_{i=1,\ldots,n} u_i = 0$ such that $W^T u$ is NZI and $\|W^T u\|_\infty + 1 = m(1, J(n, k))$. Let $n > j^2 + 2kj + 3k - j - \frac{j^2}{k}$, where $j = 2k$ if k is odd and $j = (\lfloor\frac{\gamma}{2}\rfloor + 1)(2k + \gamma)$ if k is even, γ be the smallest positive integer number that is not a divisor of k. Then u has at least k positions with the same positive value and at least k positions with the same negative value.*

Proof. If vector u has k positions with value 0 then an element of $W^T u$ is zero. Hence, at least $n - k + 1$ positions of u has nonzero values. By Propositions 6 and 7 we have that $\|W^T u\|_\infty \leq j$, where $j = 2k$ if k is odd and $j = (\lfloor\frac{\gamma}{2}\rfloor + 1)(2k + \gamma)$ if k is even. Here γ is the smallest positive integer number that does not divide k.

Also denote by x the number of positions with positive values and by y the number of positions with negative values.

Let us denote $m = \frac{j}{k}$. By Lemma 1 the values of the vector u in all positions have the same fractional part. If u does not have k positions with the same value then there are not more than $k-1$ positions with integer part i for any nonnegative i. So there are not more than $(\lfloor m \rfloor + 1)(k-1)$ positions with integer part not more than $\lfloor m \rfloor$. So, if $x > (m+2)(k-1) \geq \lfloor m \rfloor (k-1)$ then there are at least k positions with the same positive value or there are k positions with values that are more than $\lfloor m \rfloor + 1 > m$. In the latter case we have $\|W^T u\|_\infty > mk = j$, a contradiction. Analogously, if $y > (m+2)(k-1)$ then there are at least k positions with the same negative value or we have a contradiction with the minimality of norm. If $n-k+1 > 2(m+2)(k-1)$ then x or y is more than $(m+2)(k-1)$. Without loss of generality, we assume that $x > (m+2)(k-1)$. If y is also more than $(m+2)(k-1)$, then the Lemma holds, so we consider the case

$$y \leq (m+2)(k-1) \tag{1}$$

in more details below.

In view of Lemma 1, we have that the positive values of u are not less than $\frac{1}{k}$. This, combined with inequality (1) implies that the sum of all positive values in u is such that

$$\sum_{i=1,\dots,n, u_i > 0} u_i \geq \frac{x}{k} \geq \frac{n-k+1-y}{k} \geq \frac{n-k+1-(m+2)(k-1)}{k}. \tag{2}$$

If $y < k$ we consider the sum of y positions with negative values and $k-y$ positions with the smallest positive values in the vector u. Since $\sum\limits_{i=1,\dots,n} u_i = 0$ the absolute value of this sum equals the sum of $x-k+y$ positions with the largest positive values. Since each positive value in u is at least $\frac{1}{k}$, this sum is not less than $\frac{x-k+y}{k} \geq \frac{(n-k+1-y)-k+y}{k} = \frac{n-2k+1}{k}$. Hence, because $n > j^2 + 2kj + 3k - j - \frac{j^2}{k} > jk + 2k + 1$, which we have by the hypothesis of the Lemma, we obtain $\|W^T u\|_\infty > j$, which is a contradiction.

If $y \geq k$ we consider the k minimum negative values of u. The sum of these values is not greater than average negative value k times. Because $\sum\limits_{i=1,\dots,k} u_i = 0$, the absolutes of average negative and average positive coincide. So, from (2) the average amoung all negative is less then or equal to $-\frac{(n-k+1-y)}{ky}$ and there are pairwise distinct $i_1, \dots, i_k$:

$$\sum_{l=1,\dots,k} u_{i_l} \leq -\frac{(n-k+1-y)}{y}. \tag{3}$$

We show that we have $n > (j+1)y + k - 1$. By the hypothesis of the Lemma

$$n > j^2 + 2kj + 3k - j - \frac{j^2}{k} = j(j + 2k - 1 - \frac{j}{k}) + 3k = j(mk + 2k - 1 - m) + 3k =$$

$$j(mk + 2k - 2 - m) + mk + 3k > j(mk + 2k - 2 - m) + (mk + 2k - 2 - m) + k =$$

$$(j+1)(m+2)(k-1) + k$$

From (1) we have that

$$(j+1)(m+2)(k-1)+k \geq (j+1)y+k > y(j+1)+k-1,$$

so we have that $n > (j+1)y+k-1$ and therefore from $\sum\limits_{l=1,\ldots,k} u_{i_l} \leq -\frac{(n-k+1-y)}{y} < -j$, which contradicts $\|Wu\|_\infty \leq j$.

$\square$

We introduce extra notations.

$T(k) = j^2+2kj+3k-j-\dfrac{j^2}{k}$, where $j = (\lfloor\dfrac{\gamma}{2}\rfloor+1)(2k+\gamma)$ if γ is even and $j=2k$ if γ is odd , γ is the smallest positive integer number that is not a divisor of k.

$B(n,k) = \{(a,b,r,s) : \dfrac{(bs-r)k}{s(a+b)}$ is not integer , where a,b,r,s are non-negative integers , $b, \neq 0, 0 \leq r < s, (s,r)=1, s$ is a divisor of $(n,k)\}$.

$$N(n,k) = \min\{\max\{k(a+\frac{r}{s}), k(b-\frac{r}{s})\} : (a,b,r,s) \in B(n,k)\}.$$

$$M(n,k) = \{(a,b,r,s) \in B(n,k) : \max\{k(a+\frac{r}{s}), k(b-\frac{r}{s})\} = N(n,k)\}$$

Theorem 4. *If $n > T(k)$, then $m(1, J(n,k)) \geq N(n,k)+1$.*

Proof. In view of Theorem 3 consider the vector u, $\sum\limits_{i=1,\ldots,n} u_i = 0$ such that $W^T u$ is a NZI $\theta_1(J(n,k))$-eigenvector of $J(n,k)$ such that $\|W^T u\|_\infty + 1 = m(1, J(n,k))$. By Lemma 3 we have that there are k positions of u with values $a+\frac{r}{s}$ and k positions with values $-b+\frac{r}{s}$ for some $b \neq 0, (r,s)=1, 0 \leq r < s, s$ is divisor of (n,k). By Lemma 2 we have that $\frac{(bs-r)k}{s(a+b)}$ is not integer. Therefore, $(a,b,r,s) \in B(n,k)$. On the other hand, $\|W^T u\|_\infty$ is not less than $\max\{k(a+\frac{r}{s}), k(b-\frac{r}{s})\}$, hence, $\|W^T u\|_\infty \geq N(n,k)$. $\square$

Corollary 1. *Let (a,b,r,s) be in $M(n,k)$. If $n > T(k)$ and $\frac{(bs-r)n}{s(a+b)}$ is integer, then $m(1, J(n,k)) = N(n,k)+1$.*

Proof. Take the vector u such that it has $\frac{(bs-r)n}{s(a+b)}$ positions with value $(a+\frac{r}{s})$ and $\frac{(as+r)n}{s(a+b)}$ positions with value $(b-\frac{r}{s})$. The norm of the vector $W^T u$ equals $\max\{k(a+\frac{r}{s}), k(-b+\frac{r}{s})\} = N(n,k)$. In the other hand, by Theorem 4 we have $m(1, J(n,k)) \geq N(n,k)+1$, hence, $m(1, J(n,k)) = N(n,k)+1$ $\square$

Corollary 2. *Let γ be the smallest positive integer that does not divide k. If $n > T(k)$ and $(n,k)=1$ and n is divisible by γ, then*
1. *If $\gamma = 2$ then $m(1, J(n,k)) = k+1$.*
2. *If $\gamma > 2$ then $m(1, J(n,k)) = (\lfloor\frac{\gamma}{2}\rfloor+1)k+1$.*

Proof. Let u be the real-valued vector of the vertices of the graph $J(n,1)$ with the sum of values is zero, such that $\|W^T u\|_\infty + 1 = m(1, J(n,k))$. By Proposition 7 we have $\|W^T u\|_\infty \leq (\lfloor\frac{\gamma}{2}\rfloor+1)k+1$ (if $\gamma=2$ then $\|W^T u\|_\infty \leq k+1$ by Proposition 6). Since $(n,k)=1$ the fractional part of value in any position of u equals 0. Let $(a,b,0,1) \in M(n,k)$. We have that $\frac{bk}{a+b}$ is not integer and $(a+b)$ is not divisor of k.

So, $a+b \geq \gamma$, and if $a+b=\gamma$, then $(a,b)=1$. On the other hand, if $a+b=\gamma$, then $\max\{a,b\} \geq (\lfloor \frac{\gamma}{2} \rfloor + 1)$ (if $\gamma=2$ then $\max\{a,b\}=1$). If $a+b>\gamma$ then $\max\{a,b\}$ also not less than $\lfloor \frac{\gamma}{2} \rfloor + 1$ (1 in the case $\gamma=2$). Therefore, $N(n,k) \geq (\lfloor \frac{\gamma}{2} \rfloor + 1)k$ (k if $\gamma=2$) and, hence, $m(1,J(n,k)) = (\lfloor \frac{\gamma}{2} \rfloor + 1)k + 1$ ($k+1$ if $\gamma=2$). □

Proposition 8. *If n is even, k is odd and $n > T(k)$, then $m(1,J(n,k)) = k+1$.*

Proof. Let u be the real-valued vector such that $W^T u$ is NZI eigenvector of $J(n,k)$ with norm $\|W^T u\|_\infty + 1 = m(1,J(n,k))$. By Proposition 6 we have $\|W^T u\|_\infty \leq k$. In the other hand, let $(a,b,r,s) \in M(n,k)$. Then $a \geq 1$ or $b \geq 2$ (otherwise $a+b=1$, and (a,b,r,s) does not belong to $B(n,k)$). Therefore, $N(n,k)$ more than ka and more than $k(b-1)$ and, hence, $N(n,k) \geq k$. By Corollary 1 we have $m(1,J(n,k)) \geq N(n,k)+1 = k+1$ and, hence, $m(1,J(n,k)) = k+1$. □

Theorem 5. *If $n > T(3) = 63$, then we have that*

$$m(1,J(n,3)) = \begin{cases} 4, n \text{ is even} \\ 6, n \text{ is odd and } n = 0,6 \bmod 9 \\ 7, \text{otherwise.} \end{cases}$$

Proof. Let u be the real-valued vector such that $W^T u$ is NZI eigenvector of $J(n,k)$ with norm $\|W^T u\|_\infty + 1 = m(1,J(n,k))$.

The case of even n is a special case of Proposition 8.

In what follows n is odd. By Proposition 6 we have $\|W^T u\|_\infty \leq 6$.

1. Suppose the elements of u are integers. By Lemma 3 there are at least 3 elements of u equal to a and at least 3 elements equal to $-b$. If $a \geq 2$ or $b \geq 2$ then $\|W^T u\|_\infty \geq 6$. Otherwise $a=1$ and $b=1$. Since n is odd and $\sum\limits_{i=1}^{n} u_i = 0$, there is an element of u with even value c. This value cannot be $0, 2, -2$, otherwise we have three elements in u with zero sum. Hence, $c \geq 4$ or $c \leq -4$. In both cases, $\|W^T u\|_\infty \geq 6$.

2. Suppose the elements of u have nonzero fractional parts. By Lemma 1 this fractional part is $\frac{1}{3}$ or $\frac{2}{3}$ and we necessarily have that n is divisible by 3. Without loss of generality, we can assume that this fractional part equals $\frac{2}{3}$ (otherwise we take the vector $-u$). By Lemma 3 the vector u has 3 elements with some positive value $a+\frac{2}{3}$ and 3 elements with some negative value $-b+\frac{2}{3}$. If $a=0$, then b cannot be 1 or 2, otherwise there are three elements of u with the sum equals 0. If $b \geq 3$, then $\|W^T u\|_\infty \geq 7$. If $a \geq 2$, then $\|W^T u\|_\infty \geq 8$. So, $a=1$, hence, $\|W^T u\|_\infty \geq 5$. Also we have $b=1$ or $b=2$, otherwise $\|W^T u\|_\infty \geq 7$.

2a. Let n be $3 \mod 9$. We also recall that n is n odd and there are at least 3 elements in u equal $\frac{5}{3}$ and at least 3 elements equal $-b+\frac{2}{3}$, where b is 1 or 2. We show that $\|W^T u\|_\infty \geq 6$.

Consider case $b=1$. Since $\sum_{i=1}^{n} u_i = 0$ and n is odd, there is an element $\frac{c}{3}$ of u for some even c. If $c \geq 8$ or $c \leq -16$ then $\|W^T u\|_\infty \geq 6$. If $c = 2, -4, -10$ then there are three elements in u having zero sum.

Consider case $b=2$. If all elements of u are $\frac{5}{3}$ or $-\frac{4}{3}$ then n is divisible by 9, which is not the case. So there is at least one element of u that differs from $\frac{5}{3}$ or $-\frac{4}{3}$. If there is element that is not less than $\frac{8}{3}$ or element that is not more than $-\frac{10}{3}$ or two elements that equals $-\frac{7}{3}$ then $\|W^T u\|_\infty \geq 6$. If in vector u there is an element $-\frac{1}{3}$ or two elements equals $\frac{2}{3}$ or two elements with values $\frac{2}{3}$ and $-\frac{7}{3}$ then there are three elements in u with zero sum. It remains to consider case when one element of u equals $\frac{c}{3}$, where c is 2 or -7, any other element is $\frac{5}{3}$ or $-\frac{4}{3}$. Denote by x the number of elements $\frac{5}{3}$ and by y the number of elements $-\frac{4}{3}$ in u. Then the sum of all elements equals $\frac{5x-4y+c}{3}$, where $x+y=n-1$. Since $\sum_{i=1}^{n} u_i = 0$ we have $x = \frac{4n-4-c}{9}$, i.e. $x = \frac{4n-6}{9}$ or $x = \frac{4n+3}{9}$. In both case $n = 6 \mod 9$. Hence, in all cases $\|W^T u\|_\infty \geq 6$ for $n = 3 \mod 9$.

2b. If $n = 0$ mod 9 take the vector

$$u^T = (\frac{5}{3}, \ldots, \frac{5}{3}, -\frac{4}{3}, \ldots, -\frac{4}{3}),$$

of length n, where $\frac{5}{3}$ is repeated $\frac{4n}{9}$ times. Hence, $\|W^T u\|_\infty = 5$.

2c. If $n = 6$ mod 9 take vector

$$u^T = (\frac{2}{3}, \frac{5}{3}, \ldots, \frac{5}{3}, -\frac{4}{3}, \ldots, -\frac{4}{3}),$$

of length n, where $\frac{5}{3}$ is repeated $\frac{4n-6}{9}$ times, $-\frac{4}{3}$ is repeated $\frac{5n-3}{9}$ times and $u_1 = \frac{2}{3}$. Hence, $\|W^T u\|_\infty = 5$.

$\square$

4.2. **Minimum of L_∞ norm of nowhere-zero integer eigenvectors with the first eigenvalue for block graphs of STSs.**

Theorem 6. *Let S be a Steiner triple system of order n, $n > 7$. If $n = 1$ mod 4 then $m(1, \Gamma_S) \leq 4$ and if $n = 3$ mod 4 then $m(1, \Gamma_S) \leq 5$.*

Proof. Let n be such that $n = 1 \bmod 4$. Suppose S is a STS with all triples containing point 1 being $\{1,2,3\}, \ldots, \{1, n-1, n\}$. We set vector u as follows: $u_1 = 0$, $u_i = 1$ for $2 \leq i \leq \frac{n+1}{2}$ and $u_i = -1$ for $\frac{n+3}{2} \leq i \leq n$. The vector $W_S^T u$ is a $\theta_1(J(n,3))$-eigenvector of Γ_S by Theorem 3. The elements of $W_S^T u$ are $u_i + u_j + u_k$ if $\{i,j,k\}$ is a triple of S. If $\{i,j,k\}$ contains 1, then by the choice of u, we see that $(W_S^T)_{i,j,k}$ is either 2 or -2. Otherwise, $u_i + u_j + u_k$ is the sum of three numbers with absolute value 1, therefore $\|W_S^T u\|_\infty = 3$.

Let n be such that $n = 3$ mod 4. Without restriction of generality, $\{1,2,3\}$ is a triple in S. Set the vector u as follows: $u_1 = -1$, $u_2 = 2$, $u_3 = -3$. We will arrange the remaining $\frac{n-5}{2}$ values -1 and $\frac{n-1}{2}$ values 1 in the remaining n positions of u according to the structure of STS S.

We consider an auxiliary graph on the vertices, which are points $\{4, \ldots, n\}$. The edges are pairs obtained from the all triples of S containing 2 or 3, excluding $\{1,2,3\}$, by removing the points 2 and 3. The edges are labeled "2" or "3" which is the point that completes the edge, i.e. pair of points, to a triple of S. From the definition of Steiner triple system, we see that the graph is the union of even length cycles that partition $\{4, \ldots, n\}$, where labels, i.e. 2 and 3, for any two incident edges are different.

For a vector u we set its remaining elements (with indices of the vertices of the auxiliary graph) to 1 and -1 as follows. We distinguish one cycle $i_1, \ldots, i_{2l}$. Choose a path i_1, i_2, i_3 in the cycle and set values $u_{i_1} = u_{i_2} = u_{i_3} = 1$. The remaining values of u in this cycle are alternating -1 and 1: $u_{i_4} = -1$, $u_{i_5} = 1$, $u_{i_6} = -1, \ldots$, $u_{i_{2l}} = -1$. For any other cycle $j_1, \ldots, j_{2m}$ we set the values of u in the alternating way: $u_{j_{2s+1}} = -u_{j_{2(s+1)}} = 1$, for $s = 0, \ldots, m-1$. By the choice of values of u on the points $\{4,\dots,n\}$, for any two adjacent vertices a, b in the auxiliary graph we have that $u_a+u_b=0$ or 2.

Since $\sum\limits_{i=1,\dots,n} u_i=0$, u is (-1)-eigenvector of K_n. Now consider the vector W^Tu, which is a $\theta_1(J(n,3))$-eigenvector of $J(n,3)$ by Theorem 3. For a triple $\{i,j,k\}$ of S such that $\{i,j,k\}\cap\{2,3\}=\emptyset$, we see that $u_i+u_j+u_k$ is a sum of elements with absolute values 1 (note that $u_1=1$ and $|u_i|=1, i=4\dots,n$). So $u_i+u_j+u_k$ is nonzero with absolute value less or equal to 3. Let $\{i,j,k\}$ be a triple of S, such that $k=2$ or 3 and $i,j\geq 4$. Since $u_2=2$, $u_3=-3$ and u_i+u_j is either 0 or 2, the sum $u_k+u_i+u_j$ is nonzero with absolute value less or equal to 4. For the last remaining case of triple $\{1,2,3\}$ in S we have $u_1+u_2+u_3=-2$. We see that $\|W_S^Tu\|_\infty=4$.

□

5. Nowhere zero flows for families of classic Steiner triple system

Throughout this section, we use terms of flows in Steiner triple systems in the sense of work [1] rather than terms of eigevectors.

5.1. **Flows for some classic STSs.**

Lemma 4. [3, Lemma 1.4] *Let S be a resolvable STS of order n. If $n=1$ mod 4, S has a 2-flow, otherwise it has a 3-flow.*

Proposition 9. *1. Let S be the Steiner triple system formed by the supports of weight three codewords of the Hamming code of length 2^r-1, $r\geq 4$. Then we have that $m(2,\Gamma_S)=3$.*

2. Let S be the original Bose Steiner triple system of order $3p$ constructed from the latin square of Z_p, where p is an odd prime. Then S had 3 or 2-flow.

Proof. 1. If r is even, then the Hamming STS of order 2^r-1 is known to be resolvable, i.e. its blocks are parted into $\frac{2^r-2}{2}$ parallel classes [8]. In this case, following Lemma 4, we see that 3-flow exists.

When r is odd, then it is easy to see that 2^r is 2 modulo 6 by induction on r. Any Hamming STS is cyclic, i.e. it has an automorphism of order being equal to the order of S. By [4, Theorem 3.6] if the order of a STS S is 1 modulo 6 and S is cyclic, then a 3-flow for S exists.

Suppose a 2-flow v for Hamming STS S exists, so $W_Sv=0$ and the elements of the v are $+1$ and -1. This contradicts the fact that any row of W_S has exactly $\frac{n-1}{2}=\frac{2^r-1-1}{2}$ of ones, which is odd.

2. Steiner triple systems constructed by Bose method of order $9p$ were recently shown to be resolvable by Colbourn and Lusi [11]. The result follows from Lemma 4. □

5.2. **Flows for Assmuss-Mattson construction.**
Let us consider the Assmus-Mattson construction [6]. Given a Steiner triple system S of order n with pointset $\{1,\dots,n\}$ we denote by $\bar{i}$ the number $i+n$. For a function $\tau: S\rightarrow\{0,1\}$, we define the Assmuss-Mattson Steiner triple system of order $2n+1$ with point set $\{1,2,\dots,n,\bar{1},\bar{2},\dots,\bar{n},2n+1\}$:

$$\bar{S}=\bigcup_{\{i,j,k\}\in S,\,\tau(\{i,j,k\})=0} P(\{i,j,k\})\cup$$

$$\bigcup_{\{i,j,k\}\in S,\tau(\{i,j,k\})=1} P'(\{i,j,k\})\cup$$

$$\bigcup_{i=1,\ldots,n} \{i,\bar{i},2n+1\},$$

where

$$P(\{i,j,k\}) = \{\{i,j,k\},\{i,\bar{j},\bar{k}\},\{\bar{i},j,\bar{k}\},\{\bar{i},\bar{j},k\}\},$$

if $\tau(\{i,j,k\}) = 0$ and

$$P(\{i,j,k\}) = \{\{\bar{i},j,k\},\{\bar{i},\bar{j},\bar{k}\},\{i,j,\bar{k}\},\{i,\bar{j},k\}\}.$$

$\tau(\{i,j,k\}) = 1$ otherwise.

Our goal is to construct a zero-sum 5-flow in $S' = S^{2n+1,\tau}$, i.e. to find such an NZI vector v that $\sum\limits_{T\in S',i\in T} v_T = 0$ for any $i \in \{1,2,\ldots,n,\bar{1},\bar{2},\ldots,\bar{n},2n+1\}$.

For our following arguments we need several auxiliary statements.

Let a_1,a_2,a_3 be pairwise distinct elements from $\{1,\ldots,n\}$. We define a real-valued vector g indexed by $P(\{a_1,a_2,a_3\})$ depending on $\tau(\{a_1,a_2,a_3\})$. If $\tau(\{a_1,a_2,a_3\}) = 0$, define the elements of g (indexed by triples in P) in the following way:

$$g_{\{a_1,a_2,a_3\}} = g_{\{a_1,\bar{a}_2,\bar{a}_3\}} = 1, g_{\{\bar{a}_1,\bar{a}_2,a_3\}} = g_{\{\bar{a}_1,a_2,\bar{a}_3\}} = -1.$$

If $\tau(\{a_1,a_2,a_3\}) = 1$ define

$$g_{\{\bar{a}_1,a_2,a_3\}} = g_{\{\bar{a}_1,\bar{a}_2,\bar{a}_3\}} = -1, g_{\{a_1,a_2,\bar{a}_3\}} = g_{\{a_1,\bar{a}_2,a_3\}} = 1.$$

Consider the properties of the introduced vector g.

Proposition 10. *The following holds for vector g:*

$$\sum_{T'\in P(\{a_1,a_2,a_3\}),a_1\in T'} g_{T'} = 2, \tag{4}$$

$$\sum_{T'\in P(\{a_1,a_2,a_3\}),\bar{a}_1\in T'} g_{T'} = -2, \tag{5}$$

$$\sum_{T'\in P(\{a_1,a_2,a_3\}),s\in T'} g_{T'} = 0 \textit{ for } s \in \{a_2,a_3,\bar{a}_2,\bar{a}_3\}. \tag{6}$$

Proof. The proof is obtained by direct calculations. □

Clearly, by direct permutation of points a_1, a_2 and a_3 one may apply Proposition 10 for a_2 or a_3.

Lemma 5. *Let S be $STS(n)$, where $n \geq 49$. Then there is a function $h : S \to \{1,\ldots,n\}$ such that*

1. *for any $\{i,j,k\} \in S$, $h(\{i,j,k\}) \in \{i,j,k\}$,*
2. *for any $i \in \{1,\ldots,n\}$, $|\{T \in S : h(T) = i\}| \geq 4$.*

Proof. Consider the set A of all functions satisfying the first condition from Lemma. Each function maps any triple $T \in S$ to a point in T. Therefore, $|A| = 3^{\frac{n(n-1)}{6}}$. The next step is counting the number of functions from A not satisfying the second condition of Lemma for a fixed point i_0. Clearly, for any such function f, $|\{T \in S :$

$h(T) = i_0\}|$ equals 0, 1, 2 or 3. We conclude that the number of such functions is equal to

$$R = 2^{\frac{n-1}{2}} 3^{\frac{n(n-1)}{6} - \frac{n-1}{2}} + \binom{\frac{n-1}{2}}{1} 2^{\frac{n-1}{2}-1} 3^{\frac{n(n-1)}{6} - \frac{n-1}{2}} +$$

$$\binom{\frac{n-1}{2}}{2} 2^{\frac{n-1}{2}-2} 3^{\frac{n(n-1)}{6} - \frac{n-1}{2}} + \binom{\frac{n-1}{2}}{3} 2^{\frac{n-1}{2}-3} 3^{\frac{n(n-1)}{6} - \frac{n-1}{2}}.$$

Consequently, the number of functions in A not satisfying the second condition from Lemma (in at least one point) is not greater than nR. As a result, we have that the number of functions from A satisfying both conditions is at least $3^{\frac{n(n-1)}{6}} - nR$ which is equal to

$$2^{\frac{n-1}{2}} 3^{\frac{n(n-1)}{6} - \frac{n-1}{2}} \left((\frac{3}{2})^{\frac{n-1}{2}} - \frac{1}{384} n^4 - \frac{1}{128} n^3 - \frac{71}{384} n^2 - \frac{103}{128} n \right).$$

This expression is strictly positive for $n \geq 49$ and the proof is finished. □

Theorem 7. *Let $\overline{S}$ be a Assmuss-Mattson Steiner triple system of order N, $N \geq 99$. Then $\overline{S}$ admits a zero-sum 5-flow.*

Proof. Let $\overline{S}$ of order $N = 2n + 1$ be obtained from STS S of order n, $n \geq 49$ by Assmuss-Mattson construction.

We start from the all-zero vector v, indexed by the triples of $\bar{S}$ and update it with the course of the proof. At the end of the proof, v will be 5-flow for STS $\bar{S}$. Consider the set $B = \{T \in S : 1 \in T\}$ and $T_0 \in S \setminus B$. Without loss of generality we have that $B = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \ldots, \{1, n-1, n\}\}$ and $T_0 = \{2, 4, 6\}$. The triples of $\bar{S}$ are parted into the following three sets of triples: $P(T_0) \cup \bigcup_{T' \in B} P(T')$, $\bigcup_{T' \in S \setminus (T_0 \cup B)} P(T')$ and $\bigcup_{i=1,\ldots,n} \{i, \bar{i}, 2n + 1\}$, . We consequently define the vector v on these sets.

Let us consider an auxiliary vector w indexed by triples from $B \cup \{T_0\}$. For i, $1 \leq i \leq \frac{n-1}{2}$, and even $\frac{n-1}{2}$, we define $w_{\{2,4,6\}} = -2$ and

$$w_{\{1,2i,2i+1\}} = \begin{cases} 1, & 1 \leq i \leq \frac{n-1}{4} + 1 \\ -1, & \text{otherwise.} \end{cases}$$

And for i, $1 \leq i \leq \frac{n-1}{2}$, and odd $\frac{n-1}{2}$, we define $w_{\{2,4,6\}} = -2$ and

$$w_{\{1,2i,2i+1\}} = \begin{cases} 1, & 1 \leq i \leq \frac{n-3}{4} \\ -1, & \frac{n+1}{4} \leq i \leq \frac{n-3}{2} \\ 2, & i = \frac{n-1}{2}. \end{cases}$$

The choice of w implies:

$$\sum_{T \in B_0 \cup T} w_T = 0. \tag{7}$$

Let us now define the elements of v on the triples arising from $B \cup \{T_0\}$ in Assmuss-Mattson recursive approach. For $T \in B \cup \{T_0\}$, put $v_{T'} = w_T$ for all $T' \in P(T)$.

Let us define $\alpha_i = \sum_{T' \in P(T_0) \cup \bigcup_{T \in B} P(T)} v_T$ for $i \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$. From the definition of $P(T)$ we see that for any $i \in T$ the points i or $\bar{i}$ are in exactly two triples of $P(T)$. We see that

$$\alpha_i = \alpha_{\bar{i}} = \sum_{T' \in P(T_0) \cup \bigcup_{T \in B} P(T)} v_T$$

for the vector v. Moreover, we see that the absolute values of the elements of vector w are 1 or 2, so we have that $|v_T| = 1$ or 2 for v_T in the sum above , so for all $i \in \{1, \dots, n\}$ we have that

$$\alpha_i \in \{\pm 2, \pm 4\}. \tag{8}$$

The equality (7) gives the following (note that so far, some values of v are still zeros):

$$\sum_{i \in \{1, \dots, n\}} \alpha_i = 0. \tag{9}$$

By Lemma 5 there is a function h defined on S such that for any $\{i, j, k\} \in S$, $h(\{i, j, k\}) \in \{i, j, k\}$, and for any $i \in [n]$, $|\{T \in S : h(T) = i\}| \geq 4$. Take any point $j \in [n] \setminus \{1\}$. Clearly, $B \cup T_0$ covers every point at most twice. Consequently,

$$|M_j| \geq 2,$$

where $M_j = \{T \in S \setminus (B \cup T_0) : h(T) = j\}$. The next step is to apply Proposition 10 in order to define v on $P(T)$ and $P'(T)$ for $T \in M_j$. If the size of M_j is even then we divide M_j into two sets M_j^1 and M_j^2 of equal cardinality. After that we define values of v as follows. For a triple T in M_j^1, judging by the value of τ, we set the values $v_{T'}$ to be $+1$ and -1 for all triples in $T' \in P(T)$ as the values of vector g in Proposition 10 with $a_1 = j$. For a triple T in M_j^2 we set the values $v_{T'}$ to be $+1$ and -1 for all triples in $T' \in P(T)$ as the values of vector $-g$ in Proposition 10 with $a_1 = j$.

Due to definition of $v_{T'}$ from $g(T')$ we have:

$$\sum_{T' \in P(T): j \in T', T \in M_j} v_{T'} = \sum_{T' \in P(T): j \in T', T \in M_j^1} g_{T'} - \sum_{T' \in P(T): j \in T, T \in M_j^2} g_{T'} =$$
$$\sum_{T \in M_j^1} \sum_{T' \in P(T): j \in T'} g(T') - \sum_{T \in M_j^2,} \sum_{T' \in P(T): j \in T'} g(T');$$

Taking into account (4) and because $|M_j^1| = |M_j^2|$, we see that

$$\sum_{T \in M_j^1} \sum_{T' \in P(T): j \in T'} g(T') - \sum_{T \in M_j^2,} \sum_{T' \in P(T): j \in T'} g(T') = \sum_{T \in M_j^1} 2 - \sum_{T \in M_j^2} 2 = 0.$$

The same holds for the point $\bar{j}$ as we use (5) to obtain the following:

$$\sum_{T' \in P(T): \bar{j} \in T', T \in M_j} v_{T'} = \sum_{T \in M_j^1} \sum_{T' \in P(T): \bar{j} \in T'} g(T') - \sum_{T \in M_j^2,} \sum_{T' \in P(T): \bar{j} \in T'} g(T') =$$
$$\sum_{T \in M_j^1} -2 + \sum_{T \in M_j^2} 2 = 0.$$

Thus, we have:

$$\sum_{T':T'\in P(T),T\in M_j,j\in T'} v_{T'} = \sum_{T':T'\in P(T),T\in M_j,\bar{j}\in T'} v_{T'} = 0. \tag{10}$$

Moreover, according to (6) we see that for the "projection"of any point different from j is zero. For any $s \in T \cup \{\bar{i} : i \in T\} \setminus \{j, \bar{j}\}$ we have that:

$$\sum_{T':T'\in P(T),T\in M_j,\bar{s}\in T'} v_{T'} = \sum_{T':T'\in P(T),T\in M_j,\bar{s}\in T'} g_{T'} = 0. \tag{11}$$

In the case of odd size of the set M_j, we divide M_j into three non-intersecting subsets M_j^1, M_j^2 and M_j^3 respectively of sizes $\frac{|M_j|+1}{2}$, $\frac{|M_j|-3}{2}$ and 1. We repeat the procedure (as for the case when M_j was of even size) for the first two sets with vectors g and $-g$ correspondingly. For the last one-element set we do the same but with the vector $-2g$. Similarly to the case of even size of M_j, we have equalities (10) and (11). We repeat the arguments for sets M_t for all remaining $t \in \{1, \dots, n\} \setminus \{1, j\}$.

So far we have defined the elements of v indexed by all triples in $\bar{S} \setminus \{\{i, \bar{i}, 2n+1\}, i = 1, \dots, n\}$ and they are nonzeros. Moreover, from (10), (11) and (8) for any $i \in \{1, \dots, n\}$

$$\sum_{T'\in \bar{S}\setminus\{\{i,\bar{i},2n+1\},i=1,\dots,n\},i\in\bar{S}} v_{T'} = \sum_{T'\in \bar{S}\setminus\{\{i,\bar{i},2n+1\},i=1,\dots,n\},\bar{i}\in\bar{S}} v_{T'} = \alpha_i \in \{\pm 2, \pm 4\}. \tag{12}$$

The last step that finishes the proof is to define v on the set of triples

$$\bigcup_{i=1,2,\dots,n} \{i, \bar{i}, 2n+1\}$$

in the following way:

$$v_{\{i,\bar{i},2n+1\}} = -\alpha_i \text{ for } i \in \{1, \dots, n\}. \tag{13}$$

From (9) we see that

$$\sum_{\{i,\bar{i},2n+1\},l=1,\dots,n} v_{\{i,\bar{i},2n+1\}} = \sum_{i=1,\dots,n} \alpha_i = 0. \tag{14}$$

Summing up the above we have the following.

1. All values of v are nonzeros with absolute values not greater than 4.
2. From (12) and (13), for any $i \in \{1, \dots, n\}$ $\sum\limits_{T'\in\bar{S},i\in T'} v_{T'} = \sum\limits_{T'\in\bar{S},\bar{i}\in T'} v_{T'} = 0$.
3. From (14), $\sum\limits_{T'\in\bar{S},2n+1\in T'} v_{T'} = 0$.

In other words, v is a 5-flow for $\bar{S}$. □

6. Completely regular codes in the block graphs of STSs

Let S be a Steiner triple system in the classical sense of order n. The following are examples of completely regular codes in the block graph of S.

Covering radius $\rho = 1$ and eigenvalue $\theta_1(\Gamma)$:

Construction 1. $\{B \in S : i \in B\}$, where i is any fixed point $\{1, \dots, n\}$.

Construction 2. Any Steiner subsystem of S having order $\frac{n-1}{2}$.

Construction 3. $\{B \in S : i \in B\} \cup S'$, where S' is any Steiner subsystem of S having order $\frac{n-1}{2}$, such that i is a point of S but not a point of S'.

Covering radius $\rho = 1$ and eigenvalue $\theta_2(\Gamma)$:

Construction 4 (see Proposition 3). Any 1-subdesign of S.

Covering radius $\rho = 2$:

Construction 5. Any Steiner subsystem of S of order less then $\frac{n-1}{2}$.

Remark 2. Actually, Construction 5 lists all completely regular codes in the block graphs with $\rho = 2$. This can be shown, for example, using the technique from [22, Theorem 4 and Lemma 2] that utilizes the fact that all such codes naturally arise from subsets of the vertices of the clique graph of the block graph. This is beyond the scope of the current study, so we skip the details here.

The block graph of projective (Hamming) Steiner triple system of order $2^r - 1$ is isomorphic to the Grassmann graph $J_2(r, 2)$. The completely regular codes with $\rho = 1$ and the first eigenvalue in these graphs are known as Cameron-Liebler line classes. These objects were characterized in [16] as follows: these are Constructions 1-3 or their opposite codes. Judging by this fact for the most "symmetric" Steiner triple system, we propose the following:

Problem 2. Find any other completely regular codes with $\rho = 1$ and the first eigenvalue in the block graphs of Steiner triple systems of order $n, n \geq 13$ or prove that no such codes exist.

All Steiner triple systems of orders 13 and 15 are enumerated and there are 2 and 80 isomorphism classes of such Steiner triple systems respectively [12].

Theorem 8. *Let S be a Steiner triple system of order 13 or 15. Then all completely regular codes with $\rho = 1$ in Γ_S and eigenvalue $\theta_1(\Gamma_S)$ are codes from Constructions 1-3.*

Proof. For a given Steiner triple system of order n, the number of codes from Construction 2 equals the number of Steiner subsystems of order $\frac{n-1}{2}$. Using a well-known result of [15] the number of such subsystems equals the $2^{n-r} - 1$, where r is the binary rank of Steiner triple system. We recall that the rank is the dimension of the subspace, spanned by the characteristic vectors of triples in the point set. We see that there are exactly $(2^{n-r} - 1)$ and $(2^{n-r} - 1)\frac{n+1}{2}$ codes given by Constructions 2 and 3 respectively.

We conclude that for a given Steiner triple system, the number of codes from Constructions 1, 2, 3 are as follows:

$$n + (2^{n-r} - 1)(\frac{n+3}{2}). \tag{15}$$

We use integer based computer search for completely regular codes with $\rho = 1$, described in [21]. Given the intersection array, a computer linear programming solver outputs the number of completely regular codes having this intersection array. For all considered Steiner triple systems of orders $n = 13$ and 15 of any given rank r, the solver output the number of completely regular codes with eigenvalue $\theta_1(\Gamma_S)$ being equal to (15), thus we have the required. $\square$

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