ON INNER PRODUCTS DERIVED FROM THE STANDARD n-INNER PRODUCT ON AN INNER PRODUCT SPACE

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Abstract. In this paper, we study relations between inner products derived from the standard n-inner product defined on an inner product space. In particular, we are interested in knowing when orthogonality with respect to the original inner product is preserved by the derived inner product.

1. Introduction

Let X be a real vector space of dimension $d \geq n$ and let $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ such that for every $x_0, x_1, \dots, x_n, x_{n+1} \in X$ and $\alpha \in \mathbb{R}$ we have

(1) $\langle x_1, x_1 | x_2, \dots, x_n \rangle \geq 0$ and $\langle x, x | x_2, \dots, x_n \rangle = 0$ if and only if $x, x_2, \dots, x_n$ are linearly dependent;
(2) $\langle x, x | x_2, \dots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \dots, x_{i_n} \rangle$ for any permutation; $\{i_1, i_2, \dots, i_n\}$ of $(1, \dots, n)$;
(3) $\langle x_0, x_1 | x_2, \dots, x_n \rangle = \langle x_1, x_0 | x_2, \dots, x_n \rangle$;
(4) $\langle \alpha x_0, x_1 | x_2, \dots, x_n \rangle = \alpha \langle x_0, x_1 | x_2, \dots, x_n \rangle$;
(5) $\langle x_0 + x_{n+1}, x_1 | x_2, \dots, x_n \rangle = \langle x_0, x_1 | x_2, \dots, x_n \rangle + \langle x_{n+1}, x_1 | x_2, \dots, x_n \rangle$.

The function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$ is called n-inner product introduced by Misiak in 1989 [6]. Here, the pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is called n-inner product space. From n-inner product space we can derive n-norm, defined by

$$\|x_1, x_2, \dots, x_n\| = \langle x_1, x_1 | x_2, \dots, x_n \rangle^{\frac{1}{2}}.$$

Geometrically, $\|x_1, x_2, \dots, x_n\|$ represents the generalized volume of an n-dimensional parallelepiped spanned by $x_1, x_2, \dots, x_n$. Then,

$$\frac{\langle x_0, x_1 | x_2, \dots, x_n \rangle}{\|x_0, x_2, \dots, x_n\| \, \|x_1, x_2, \dots, x_n\|}$$

is the cosine of the angle between two parallelepipeds that are spanned by $x_0, x_2, \dots, x_n$ and $x_1, x_2, \dots, x_n$. See [2, 7] for more properties of n-inner products. The related result may also be found in [1, 8, 9].

If $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space of dimension $d \geq n$, we can define the standard n-inner product by

$$\langle x_0, x_1 | x_2, \dots, x_n \rangle = \begin{vmatrix} \langle x_0, x_1 \rangle & \langle x_0, x_2 \rangle & \dots & \langle x_0, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \dots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \dots & \langle x_n, x_n \rangle \end{vmatrix}.$$

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Meanwhile, if $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is n-inner product space, let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n linearly independent vectors in X. Then one may observe that

$$\langle x, y \rangle_A := \sum_{\{i_2, \ldots i_n\} \subset \{1,2,\ldots,n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n} \rangle$$

defines an inner product on X. Note that there are n terms in the above sum, as there are n subsets of $\{1, 2, \ldots, n\}$ consisting of $n-1$ elements. If $\dim X = d < \infty$, we can also define an inner product by the above formula using a set of d linearly independent vectors in X (see [3]).

Thus, starting from an inner product, we can define the standard n-inner product, and then from the n-inner product we can derive a new inner product. It is then interesting to investigate how the new inner product derived from standard n-inner product relates to the original inner product on $(X, \langle \cdot, \cdot \rangle)$. In particular we would like to know whether or not the new inner product preserves orthogonality. This, in general, depends on the set A that we choose in the definition of the new inner product. In the next sections, we present necessary and sufficient conditions for the set A to give the positive answer.

2. The n Dimensional Case

As indicated in [4] and [5], the n-dimensional case is special. So we shall first pay attention to the case where $\dim X = n$. Our results are the following theorems.

Theorem 2.1. *Let $A = \{a_1, a_2, \ldots, a_n\} \subset X$ be an orthogonal set with $\|a_i\| = a$ for each $i = 1, 2, \ldots, n$. Then $\langle x, y \rangle_A = 0$ if and only if $\langle x, y \rangle = 0$ for every $x, y \in X$.*

Proof. For any subset $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$, we have

$$\langle x, y | a_{i_2}, a_{i_3}, \ldots, a_{i_n} \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, a_{i_2} \rangle & \ldots & \langle x, a_{i_n} \rangle \\ \langle a_{i_2}, y \rangle & \langle a_{i_2}, a_{i_2} \rangle & \ldots & \langle a_{i_2}, a_{i_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_n}, y \rangle & \langle a_{i_n}, a_{i_2} \rangle & \ldots & \langle a_{i_n}, a_{i_n} \rangle \end{vmatrix}_{n \times n}$$

$$= \langle x, y \rangle \begin{vmatrix} \|a_{i_2}\|^2 & 0 & \ldots & 0 \\ 0 & \|a_{i_3}\|^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \|a_{i_n}\|^2 \end{vmatrix}_{(n-1) \times (n-1)}$$

$$- \langle x, a_{i_2} \rangle \begin{vmatrix} \langle a_{i_2}, y \rangle & 0 & \ldots & 0 \\ \langle a_{i_3}, y \rangle & \|a_{i_3}\|^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_n}, y \rangle & 0 & \ldots & \|a_{i_n}\|^2 \end{vmatrix}_{(n-1) \times (n-1)}$$

$$+ \cdots + (-1)^{n-1} \langle x, a_{i_n} \rangle \begin{vmatrix} \langle a_{i_2}, y \rangle & \|a_{i_2}\|^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_{n-1}}, y \rangle & 0 & \ldots & \|a_{i_{n-1}}\|^2 \\ \langle a_{i_n}, y \rangle & 0 & \ldots & 0 \end{vmatrix}_{(n-1) \times (n-1)}$$

$$
\begin{aligned}
&= \langle x,y\rangle \prod_{j=2}^{n} \|a_{i_j}\|^2 - \langle x,a_{i_2}\rangle\langle a_{i_2},y\rangle \prod_{j=3}^{n} \|a_{i_j}\|^2 - \cdots - \langle x,a_{i_n}\rangle\langle a_{i_n},y\rangle \prod_{j=2}^{n-1} \|a_{i_j}\|^2 \\
&= \left[\langle x,y\rangle - \frac{\langle x,a_{i_2}\rangle\langle a_{i_2},y\rangle}{\|a_{i_2}\|^2} - \frac{\langle x,a_{i_3}\rangle\langle a_{i_3},y\rangle}{\|a_{i_3}\|^2} - \cdots - \frac{\langle x,a_{i_n}\rangle\langle a_{i_n},y\rangle}{\|a_{i_n}\|^2}\right] \prod_{j=2}^{n} \|a_{i_j}\|^2 .
\end{aligned}
$$

Let $\{1,2,\dots n\}\setminus\{i_2,\dots i_n\}=\{i_1\}$. By Parseval's identity, we have $\langle x,y\rangle = \sum_{j=1}^{n} \frac{\langle x,a_{i_j}\rangle\langle a_{i_j},y\rangle}{\|a_{i_j}\|^2}$ (since $\{i_1,i_2,\dots,i_n\}=\{1,2,\dots,n\}$ as sets). Hence it follows that

$$
\langle x,y|a_{i_2},a_{i_3},\dots,a_{i_n}\rangle = \frac{\langle x,a_{i_1}\rangle\langle a_{i_1},y\rangle}{\|a_{i_1}\|^2} \prod_{j=2}^{n} \|a_{i_j}\|^2 = \frac{\langle x,a_{i_1}\rangle\langle a_{i_1},y\rangle}{\|a_{i_1}\|^4} \prod_{i=1}^{n} \|a_i\|^2 .
$$

Summing the above expressions for $i_1 = 1,2,\dots,n$, we get

$$
\begin{aligned}
\langle x,y\rangle_A &= \sum_{\{i_2,\dots i_n\}\subset\{1,2,\dots,n\}} \langle x,y|a_{i_2},\dots,a_{i_n}\rangle \\
&= \left[\frac{\langle x,a_1\rangle\langle a_1,y\rangle}{\|a_1\|^4} + \frac{\langle x,a_2\rangle\langle a_2,y\rangle}{\|a_2\|^4} + \cdots + \frac{\langle x,a_n\rangle\langle a_n,y\rangle}{\|a_n\|^4}\right] \prod_{i=1}^{n} \|a_i\|^2 .
\end{aligned}
$$

However, we are assuming that $\|a_i\| = \alpha$ for each $i = 1,2,\dots,n$ and so we obtain

$$
\begin{aligned}
\langle x,y\rangle_A &= \left[\frac{\langle x,a_1\rangle\langle a_1,y\rangle}{\alpha^4} + \frac{\langle x,a_2\rangle\langle a_2,y\rangle}{\alpha^4} + \cdots + \frac{\langle x,a_n\rangle\langle a_n,y\rangle}{\alpha^4}\right] \prod_{i=1}^{n} \alpha^2 \\
&= \frac{\langle x,y\rangle}{\alpha^2}\alpha^{2n} = \alpha^{2(n-1)}\langle x,y\rangle.
\end{aligned}
$$

Since $\alpha \neq 0$, we conclude that $\langle x,y\rangle_A = 0$ if and only if $\langle x,y\rangle = 0$, which proves the theorem. □

Theorem 2.2. *Let $A = \{b_1,b_2,\dots,b_n\}$ be a set of n linearly independent vectors in X. Then $\langle x,y\rangle_A = \langle x,y\rangle$ if and only if A is an orthonormal basis for X.*

Proof. The sufficient part follows immediately from the previous theorem. For the necessary part, suppose that $\langle x,y\rangle_A = \langle x,y\rangle$ for every $x,y\in X$. To prove that A is an orthonormal basis for X, let us first compute (b_i,b_j) for $i\neq j$. We have

$$
\langle b_i,b_j\rangle = \langle b_i,b_j\rangle_A = \sum_{\{i_2,\dots i_n\}\subset\{1,2,\dots,n\}} \langle b_i,b_j|b_{i_2},\dots,b_{i_n}\rangle.
$$

For any $\{i_2,\dots,i_n\}\subset\{1,2,\dots,n\}$, observe that $b_i\in\{b_{i_2},\dots,b_{i_n}\}$ or $b_j\in\{b_{i_2},\dots,b_{i_n}\}$, because $\{b_{i_2},\dots,b_{i_n}\}$ consists of $n-1$ elements of A. Consequently , $\langle b_i,b_j|b_{i_2},\dots,b_{i_n}\rangle = 0$, because two rows or two columns in the determinant will be identical. Since this is true for any $\{i_2,\dots,i_n\}\subset\{1,2,\dots,n\}$, we conclude that

$$
\langle b_i,b_j\rangle = \sum_{\{i_2,\dots i_n\}\subset\{1,2,\dots,n\}} \langle b_i,b_j|b_{i_2},\dots,b_{i_n}\rangle = 0.
$$

Let us now compute $\langle b_i,b_i\rangle$ for $i = 1,2,\dots,n$. Notice that if $b_i\in\{b_{i_2},\dots,b_{i_n}\}$, then we have $\langle b_i,b_i|b_{i_2},\dots,b_{i_n}\rangle = 0$. Meanwhile, if $b_i\notin\{b_{i_2},\dots,b_{i_n}\}$ then —by the properties of the standard $n-$product— we have

$$\langle b_i, b_i | b_{i_2}, \dots, b_{i_n} \rangle = \langle b_1, b_1 | b_2, \dots, b_n \rangle = \begin{vmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_n \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_n, b_1 \rangle & \langle b_n, b_2 \rangle & \dots & \langle b_n, b_n \rangle \end{vmatrix} = \prod_{j=1}^{n} \|b_j\|^2 .$$

Hence, we obtain

$$\langle b_i, b_i \rangle_A = \sum_{\{i_2,\dots i_n\} \subset \{1,2,\dots,n\}} \langle b_i, b_i | b_{i_2}, \dots, b_{i_n} \rangle = \prod_{j=1}^{n} \|b_j\|^2 .$$

By our hypothesis, $\|b_i\|^2 = \|b_i\|_A^2 = \prod_{j=1}^{n} \|b_j\|^2$. This holds only if $\|b_i\| = 1$ for every $i = 1, 2, \dots, n$.

To sum up, we have proved that $\langle x, y \rangle_A = \langle x, y \rangle$ if and only if A is an orthonormal basis for X. $\square$

3. The Higher Dimensional Case

Let us now consider the case where $n + 1 \le d = \dim\ X < \infty$. Let $A := \{a_1, a_2, \dots, a_d\}$ be a set of linearly independent vectors in X. (What happens if we use only n vectors will be discussed later, together with the case where $d = \infty$.) We define the following inner product on X:

$$\langle x, y \rangle_A := \sum_{\{i_2,\dots i_n\} \subset \{1,2,\dots,n\}} \langle x, y | a_{i_2}, \dots, a_{i_n} \rangle.$$

Note that there are $\binom{d}{n-1}$ terms in the above sum. Analogous to Theorem 2.1, we have the following theorem.

Theorem 3.1. *Let $A = \{a_1, a_2, \dots, a_d\} \subset X$ be an orthogonal set with $\|a_i\| = \alpha$ for each $i = 1, 2, \dots, d$. Then $\langle x, y \rangle_A = 0$ if and only if $\langle x, y \rangle = 0$ for every $x, y \in X$.*

Proof. Let $I_d := \{1, 2, \dots, d\}$. For any $I_0 := \{i_2, \dots, i_n\} \subset I_d$, let $I_1 := I_d \setminus I_0$. Then, we have

$$\begin{aligned} \langle x, y | a_{i_2}, \dots a_{i_n} \rangle &= \sum_{i \in I_1} \frac{\langle x, a_i \rangle \langle a_i, y \rangle}{\|a_i\|^2} \prod_{i \in I_0} \|a_i\|^2 \\ &= \left[\langle x, y \rangle - \sum_{i \in I_0} \frac{\langle x, a_i \rangle \langle a_i, y \rangle}{\|a_i\|^2} \right] \prod_{i \in I_0} \|a_i\|^2 . \end{aligned}$$

Summing over all subsets $I_0 \subset I_d$ and using the assumption that $\|a_i\| = \alpha$ for each $i = 1, 2, \dots n$, we obtain

$$\begin{aligned} \langle x, y \rangle_A &= \sum_{I_0 \subset I_d} \left[\langle x, y \rangle - \frac{\langle x, a_i \rangle \langle a_i, y \rangle}{\|a_i\|^2} \right] \alpha^{2(n-1)} \\ &= \sum_{I_0 \subset I_d} \alpha^{2(n-1)} \langle x, y \rangle - \sum_{I_0 \subset I_d} \sum_{i \in I_0} \alpha^{2(n-1)} \frac{\langle x, a_i \rangle \langle a_i, y \rangle}{\alpha^2} . \end{aligned}$$

The first sum on the right hand side is equal to $\binom{d}{n-1} \alpha^{2(n-1)} \langle x, y \rangle$. For the second sum, we observe that each expression $\langle x, a_i \rangle \langle a_i, y \rangle$ occurs precisely $\binom{d-1}{n-2}$ times for every $i \in I_d$.

Hence, by Parseval's identity, the second sum is equal $\binom{d-1}{n-2}\alpha^{2(n-1)}\langle x,y\rangle$. Therefore we get

$$\langle x,y\rangle_A=\left[\binom{d}{n-1}-\binom{d-1}{n-2}\right]\alpha^{2(n-1)}\langle x,y\rangle=\binom{d-1}{n-1}\alpha^{2(n-1)}\langle x,y\rangle,$$

which gives us the desired conclusion. □

Corollary 3.2. *If $A=\{a_1,a_2,\dots,a_d\}$ is an orthonormal basis for X, then $\langle x,y\rangle_A=\binom{d-1}{n-1}\langle x,y\rangle$ for every $x,y\in X$.*

Remark 3.3. The converse of the above corollary does not hold. To give an example, let $d=\dim X=3$ and $n=2$. Let $A=\{a_1,a_2,a_3\}$ be linearly independent set in X. Suppose that $\langle x,y\rangle_A=2\langle x,y\rangle$ for every $x,y\in X$. We would like to check whether we would have $\|a_i\|=1$ for $i=1,2,3$ and $\langle a_i,a_j\rangle=0$ for $i\neq j$. Notice that

$$\langle x,y\rangle_A=\sum_{i=1}^{3}\langle x,y|a_i\rangle=\sum_{i=1}^{3}\begin{vmatrix}\langle x,y\rangle & \langle x,a_i\rangle\\ \langle a_i,y\rangle & \langle a_i,a_i\rangle\end{vmatrix}$$

for every $x,y\in X$. From the hypothesis, we have $\langle a_i,a_j\rangle_A=2\langle a_i,a_j\rangle$ for $i,j=1,2,3$, which may be rewritten as

$$\begin{aligned}\|a_1\|^2\|a_2\|^2-\langle a_1,a_2\rangle^2+\|a_1\|^2\|a_3\|^2-\langle a_1,a_3\rangle^2&=2\|a_1\|^2\\ \|a_1\|^2\|a_2\|^2-\langle a_1,a_2\rangle^2+\|a_2\|^2\|a_3\|^2-\langle a_2,a_3\rangle^2&=2\|a_2\|^2\\ \|a_1\|^2\|a_3\|^2-\langle a_1,a_3\rangle^2+\|a_2\|^2\|a_3\|^2-\langle a_2,a_3\rangle^2&=2\|a_3\|^2\\ \langle a_1,a_2\rangle\|a_3\|^2-\langle a_1,a_3\rangle\langle a_2,a_3\rangle&=2\langle a_1,a_2\rangle\\ \langle a_1,a_3\rangle\|a_2\|^2-\langle a_1,a_2\rangle\langle a_2,a_3\rangle&=2\langle a_1,a_3\rangle\\ \langle a_2,a_3\rangle\|a_1\|^2-\langle a_1,a_2\rangle\langle a_1,a_3\rangle&=2\langle a_2,a_3\rangle.\end{aligned}$$

Let $A:=\|a_1\|,B:=\|a_2\|,C:=\|a_3\|,D:=\langle a_1,a_2\rangle,E:=\langle a_1,a_3\rangle,F:=\langle a_2,a_3\rangle$. Then

$$\begin{aligned}A^2B^2-D^2+A^2C^2-E^2&=2A^2\\ A^2B^2-D^2+B^2C^2-F^2&=2B^2\\ A^2C^2-E^2+B^2C^2-F^2&=2C^2\\ DC^2-EF&=2D\\ EB^2-DF&=2E\\ FA^2-DE&=2F.\end{aligned}$$

Observe that $A=B=C=1,D=E=F=0$ satisfy the above equations simultaneously. We shall see that there are other possible solutions with $D,E,F\neq 0$.

Multiplying both sides in the last three equations by D, E, and F (respectively) and rearranging the terms, we obtain

$$D^2 + E^2 = A^2B^2 + A^2C^2 - 2A^2 \tag{1}$$
$$D^2 + F^2 = A^2B^2 + B^2C^2 - 2B^2 \tag{2}$$
$$E^2 + F^2 = A^2C^2 + B^2C^2 - 2C^2 \tag{3}$$
$$C^2D^2 - 2D^2 = DEF \tag{4}$$
$$B^2E^2 - 2E^2 = DEF \tag{5}$$
$$A^2F^2 - 2F^2 = DEF. \tag{6}$$

From $(1), (2)$, and (3), we get

$$D^2 = A^2B^2 - A^2 - B^2 + C^2 \tag{7}$$
$$E^2 = A^2C^2 - A^2 + B^2 - C^2 \tag{8}$$
$$F^2 = B^2C^2 + A^2 - B^2 - C^2. \tag{9}$$

From $(4), (5)$, and (6), we get

$$(C^2 - 2)D^2 = (B^2 - 2)E^2 = (A^2 - 2)F^2 = DEF. \tag{10}$$

Substituting $(7), (8)$, and (9) into (10), we obtain

$$\begin{aligned}(B^2 - C^2)(A^2 + B^2 + C^2 - 4) &= 0\\ (A^2 - C^2)(A^2 + B^2 + C^2 - 4) &= 0\\ (A^2 - B^2)(A^2 + B^2 + C^2 - 4) &= 0.\end{aligned}$$

Now one may check that $A = B = C = \frac{4}{3}, D+E+F = -\frac{2}{3}$ satisfy the above equations simultaneously. This tells us that A is not necessarily an orthonormal basis.

We now come to the case where $d = \dim\ X = \infty$. We assume that X is separable and $B := \{a_i : i = 1, 2, 3, \dots\}$ is an orthogonal basis for X. Thus for every $x, y \in X$, we have Parseval's identity $\langle x, y\rangle = \sum_{i=1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}$. Next, let $A := \{a_1, a_2, \dots, a_n\}$, where the vectors a_i's are the first n vectors in B. We define

$$\langle x, y\rangle_A := \sum_{\{i_2,\dots,i_n\}\subset\{1,2,\dots,n\}} \langle x, y|a_{i_2}, \dots, a_{i_n}\rangle$$

for every $x, y \in X$. Then we have the following theorem.

Theorem 3.4. *For every $x, y \in X$, we have*

$$\langle x, y\rangle_A = \left[\sum_{i=1}^{n} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^4} + n \sum_{i=n+1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right] \prod_{j=1}^{n} \|a_j\|^2 .$$

In particular, if $\|a_i\| = \alpha$ for $i = 1, 2, \dots, n$ then

$$\langle x, y\rangle_A = \alpha^{2(n-1)} \left[\sum_{i=1}^{n} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\alpha^2} + n \sum_{i=n+1}^{\infty} \frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right].$$

Proof. The proof is similar to the proof of Theorem 2.1 but this time we have

$$\langle x, y|a_{i_2}, \ldots, a_{i_n}\rangle = \left[\frac{\langle x, a_{i_1}\rangle\langle a_{i_1}, y\rangle}{\|a_{i_1}\|^4} + \frac{1}{\|a_{i_1}\|^2}\sum_{i=n+1}^{\infty}\frac{\langle x, a_{i_1}\rangle\langle a_{i_1}, y\rangle}{\|a_i\|^2}\right]\prod_{j=1}^{n}\|a_j\|^2,$$

where $\{i_1\} = \{1, 2, \ldots, n\} \setminus \{i_2, \ldots, i_n\}$. Summing all these expression for all subsets $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$, we obtain

$$\langle x, y\rangle_A = \left[\sum_{i=1}^{n}\frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^4} + \sum_{i=1}^{n}\frac{1}{\|a_i\|^2}\sum_{i=n+1}^{\infty}\frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right]\prod_{j=1}^{n}\|a_j\|^2$$

In particular, if $\|a_i\| = \alpha$ for $i = 1, 2, \ldots, n$, then we have

$$\langle x, y\rangle_A = \alpha^{2(n-1)}\left[\sum_{i=1}^{n}\frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\alpha^2} + n\sum_{i=n+1}^{\infty}\frac{\langle x, a_i\rangle\langle a_i, y\rangle}{\|a_i\|^2}\right]$$

as claimed. □

Remark 3.5. Note that if $\|a_i\| = 1$ for $i = 1, 2, 3, \ldots$ (that is, B is an orthonormal basis for X), then the conclusion in the above theorem tells us that

$$\langle x, y\rangle_A = \sum_{i=1}^{n}\langle x, a_i\rangle\langle a_i, y\rangle + n\sum_{i=n+1}^{\infty}\langle x, a_i\rangle\langle a_i, y\rangle$$

for every $x, y \in X$.

Corollary 3.6. *Suppose that $\|a_i\| = \alpha$ for $i = 1, 2, \ldots, n$. For every $x, y \in X$, let $x := x_A + x_A^{\perp}$ and $y := y_A + y_A^{\perp}$ where x_A and y_A are the orthogonal projections of x and y on span A (respectively), and $x_A^{\perp}$ and $y_A^{\perp}$ are their complements (respectively). If $\langle x_A, y_A\rangle_A = 0$ and $\langle x_A^{\perp}, y_A^{\perp}\rangle = 0$, then $\langle x, y\rangle = 0$. Conversely, if $\langle x_A, y_A\rangle = 0$ and $\langle x_A^{\perp}, y_A^{\perp}\rangle = 0$, then $\langle x, y\rangle_A = 0$.*

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