Transition Systems from Causal Reversible Prime Event Structures

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Abstract Two structurally different methods of associating transition system semantics to event structure models are distinguished in the literature. One of them is based on configurations (sets of already executed events), the other — on model residuals (not yet executed model fragments). Reversible computing is a new paradigm that has appeared recently and extends the traditional forwards-only mode of computation with the ability to execute in reverse, so that computation can run backwards as easily as forwards. In this paper, we deal with prime event structures extended with causal reversibility and provide bisimulation results for these two types of the transition systems of the model.

1 Introduction

In recent years, reversibility of computations has been extensively studied in quest for mechanisms to allow (partially) undoing some actions executed in the computational process that need to be canceled, for some reason (for example, in case of error). Therefore, reversible computations can be executed not only in the traditional forward direction as well as in the backward one, restoring past states and computing inputs from outputs. Reversibility in computing finds its applications in different fields including programming abstractions for reliable systems [16,32,37], program analysis and debugging [34], modelling biochemical simulations [30], hardware design and quantum computing [17]. To ensure that the applications are implemented and their use can bring long-lasting benefits, lucid formal presentations and solid theoretical foundations are created first.

In concurrency theory, a number of aspects of reversible computing have been studied, dealing with various concurrent models: parallel rewriting systems [1], cellular automata [27], process calculi [14,16,33,45], Petri nets [6,7,8,18,40,43,44], event structures [39,48,51], membrane systems [50], and etc. These studies led to the identification of three main methods to reverse concurrent processes: backtracking [15,44], causal reversibility [14,40,44], and out-of-causal reversibility [32,43,44,49], that differ in the order of executing actions in backward direction. Backtracking is generally understood as the ability to execute past actions in the exact reverse order in which they were executed. In a concurrent setting, reversing actions can occur in a more liberal way. Causal reversibility means that an action can be undone provided that all of its causal successors (if any) have been undone beforehand. Out-of-causal reversibility is a form of reversal, most characteristic of biochemical systems, does not preserve causes.

Event structures putted forward by Winskel in his PhD dissertation [52] are one of the central models of concurrent nondeterministic processes. Event structures were used for establishing relationships between different concurrent models [19,22,41,52], for defining denotational and operational semantics of algebraic calculi and specification languages of concurrent processes [12,23,28,29,31,52,53], for suggesting behavioral equivalences between processes [20], for modeling quantum strategies and games [54]. The association of transition systems with event structure models has proved to contribute to studying and solving various problems in the analysis and verification of concurrent systems. It is distinguished two methods of providing transition system semantics for event structures: a configuration-based and a residual-based method. In the first case (see [2,3,19,20,21,22,26,28,52,53] among others), states are understood as sets of events, called configurations, and state transitions are built by starting with the empty configuration and enlarging configurations by already executed events. In the second more 'structural' method [5,11,13,28,29,31,35,42], states are understood as event structures, and transitions are built by starting with the given event structure as an initial state and removing already executed (or conflicting) parts thereof in the course of an execution. In the literature, configuration-based transition systems seem to be predominantly used as the semantics of event structures, and residual-based transition systems are actively used in providing operational semantics of process calculi and in demonstrating the consistency of operational and denotational semantics. The two kinds of transition systems have occasionally been used alongside each other (see [28] as an example), but their general relationship has not been studied for a wide range of existing models. In a seminal paper, viz. [36], bisimulations between configuration-based and residual-based transition systems have been proved to exist for prime event structures [53]. The result of [36] has been extended in [9] to more complex event structure models with asymmetric conflict. Counterexamples illustrated that an isomorphism cannot be achieved with the various removal operators defined in [36,9] to obtain residuals. The paper [10] demonstrated that when using non-executable events, the operators can be tightened in such a way that isomorphisms, rather than just bisimulations, between the two types of transition systems belonging to a single event structure can be obtained, for a full spectrum of semantics (interleaving, step, pomset, multiset).

Reversible event structures extend event structures to represent reversible computational processes, capable of undoing executed actions by allowing configurations to evolve by removing events. In [48,51], Phillips et. al. have defined causal and out-of-causal reversible forms of prime [48], asymmetric [48,51], and general [51] event structures, and have shown a correspondence between their configurations and traditional ones when there are no reversible events. In [24], Graversen, et. al. have introduced categories of different classes of reversible event structures, including those mentioned above, and constructed functors between the categories. In [4], Aubert and Cristescu have provided a true concurrent semantics of a reversible extension of CCS, RCCS (without auto-concurrency, auto-conflict, or recursion), in terms of configuration structures. In [25], Graversen, et. al. have developed a category of reversible bundle event structures with symmetric conflict and used the causal subcategory to model semantics of another reversible extension of CCS, CCSK. They modified CCSK to control the reversibility with a rollback primitive, and gave, by exploiting the capacity for out-of-causal reversibility, semantics of this kind of CCSK in terms of reversible bundle event structures with asymmetic conflict. In [47], Event Identifier Logic (EIL) has been introduced in order to extend Hennessy-Milner logic with reverse modalities. EIL corresponds to hereditary history-preserving bisimulation equivalence within stable configuration structures. Moreover, natural sublogics of EIL correspond to coarser equivalences, several of them defined in terms of reversible events, sets of concurrent reversible events or pomsets of reversible events. In [46], these and other behavioral equivalences in the reversible setting were studied for the first time. Constructions associating causal reversible prime event structures to reversible occurrence nets and vice versa have been proposed within causal reversibility in [38], as well as within out-of-causal reversibility in [39]. In the latter case, causality is recovered from inhibitor arcs of a subclass of contextual Petri nets instead of the usual overlap between post and presets of transitions.

The aim of this paper is to identify two (configuration-based and residual-based) types of transition system semantics for causal reversible prime event structures and to understand how these types relate to each other, which can assist in the construction of algebraic calculi to describe reversible concurrent processes.

This paper is structured as follows. In Section 2, we start with recalling the syntax of prime and reversible prime event structures and their (step) semantics in terms of configurations and traces. In Section 3, we define a removal operator, which is useful for constructing model residuals, and demonstrate the correctness of the operator. In Section 4, we develop two types of transition system semantics for the models under consideration and establish bisimulation results between the semantics. In Section 5, we provide some concluding remarks. The paper also contains Appendix with the proofs of the lemmas and propositions presented here.

2 Reversing in Prime Event Structures

In this section, we first recall the notion of prime event structures (PESs) [52] labeled over the set $L = \{a, b, c, \ldots\}$ of actions, and then formulate the concept of reversible prime event structures (RPESs) [48] and consider their (step) semantics and properties.

The behavior of concurrent systems is formally modelled by event structure models where units of the behavior are represented by events. There are different ways to relate events. In prime event structures (PESs), the dependency between events, called causality, is given by a partial order, and the incompatibility is determined by a conflict relation. Two events which are neither in causal dependency nor in conflict are considered independent (concurrent).

Definition 1. *A (labeled)* prime event structure (PES) *(over the set L of actions) is a tuple $\mathcal{E} = (E, \leq, \sharp, l, C_0)$, where*

- *E is a countable set of* events*;*
- *$< \subseteq E \times E$ is an irreflexive partial order (the* causality relation*) satisfying the* principle of finite causes*: $\forall e \in E \diamond \lfloor e \rfloor = \{e' \in E \mid e' < e\}$ is finite;*
- *$\sharp \subseteq E \times E$ is an irreflexive and symmetric relation (the* conflict relation*) satisfying the* principle of hereditary conflict*: $\forall e, e', e'' \in E \diamond e < e'$ and $e \sharp e''$ then $e' \sharp e''$;*
- *$l : E \to L$ is a* labeling function*;*
- *$C_0 = \emptyset$ is the* initial configuration[(1)]*.*

So, the PES is a simple event-based model of concurrent and nondeterministic computations where events labeled over the set L of actions are considered as atomic, indivisible and instantaneous action occurrences, some of which can only be executed after another (i.e. there is a causal dependency represented by a partial order $\leq$ between the events) and some of which might not be executed together (i.e. there is a binary conflict $\sharp$ between the events). In addition, the principle of finite causes and the principle of conflict inheritance are required.

The PES progresses by executing events, thus moving from one state to another, starting from the initial state, which is an empty set. A state called a configuration is a set of events that have occurred. A subset of events $X \subseteq E$ is *left-closed under* $<$ iff for all $e \in X$ it holds that $\lfloor e \rfloor \subseteq X$; is *conflict-free* iff for all $e, e' \in X$ it holds that $\neg(e \sharp e')$, and we denote it with $CF(X)$. A subset $C \subseteq E$ is a *configuration* of $\mathcal{E}$ iff C is finite, left-closed under $<$ and conflict-free.

Reversible prime event structures (RPESs) [48,51] are based on a weaker form of PESs because conflict inheritance may not hold when adding reversibility to PESs. Also, in RPESs, some events are categorised as reversible, and two relations are added: the reverse causality relation and the prevention relation. The first one is a dependency relation in the backward direction: to reverse an event in the current configuration there must be other events on which the event reversibly depends. The second relation, on the contrary, identifies those events whose presence in the current configuration prevents the event being reversed.

(1) We add the initial configuration as an empty set to the PES definition but this does not affect the behavior of the structure in any way because in the classical definition, the PES progresses, moving from one configuration to another and starting from an empty set.

Definition 2. *A (labeled)* reversible prime event structure (RPES) *(over L) is a tuple $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$, where*

- *E is a countable set of* events*;*
- *$\sharp \subseteq E \times E$ is an irreflexive and symmetric relation (the* conflict relation*);*
- *$< \subseteq E \times E$ is an irreflexive partial order (the* causality relation*) satisfying: $\lfloor e \rfloor$ is finite and conflict-free, for every $e \in E$, and if $e < e'$ then $\neg(e \sharp e')$, for every $e, e' \in E$;*
- *$l : E \to L$ is a* labeling function*;*
- *$F \subseteq E$ are* reversible events *being denoted by $\underline{F} = \{\underline{e} : e \in F\}$;*
- *$\prec \subseteq E \times \underline{F}$ is the* reverse causality relation*, such that $a \prec \underline{a}$ for each $a \in F$, and also that $\{a : a \prec \underline{b}\}$ is finite and conflict-free for each $b \in F$;*
- *$\rhd \subseteq E \times \underline{F}$ is the* prevention relation *such that if $a \prec \underline{b}$ then $\neg(a \rhd \underline{b})$;*
- *$\ll$ is the transitive* sustained causation relation*: $a \ll b$ iff $a < b$ and if $a \in F$ then $b \rhd \underline{a}$. $\sharp$ is* hereditary w.r.t. the sustained causation *$\ll$: if $a \sharp b \ll c$ then $a \sharp c$;*
- *$C_0 \subseteq E$ is the* initial configuration *which is finite, left-closed under $<$ and conflict-free.*

It is straightforward to check that any PES is also an RPES with $F = \emptyset$ and $C_0 = \emptyset$. Then, any concept defined for RPESs apply to PESs as well.

Example 1. Consider the structure $\mathcal{E}_0 = (E_0, <_0, \sharp_0, l_0, F_0, \prec_0, \rhd_0, C_0^0)$, where $E_0 = \{a, b, c, d\}$; $<_0 = \{(b, c), (b, d)\}$; $\sharp_0 = \{(a, b), (b, a), (a, c), (c, a), (c, d), (d, c)\}$; l_0 is the identical function; $F_0 = \{a, b\}$; $\prec_0 = \{(a, \underline{a}), (b, \underline{b})\}$; $\rhd_0 = \{(c, \underline{b})\}$; $C_0^0 = \{a\}$. It is easy to make sure that the components of the structure $\mathcal{E}_0$ meet the requirements of the corresponding items of Definition 2. In particular, we see that $<_0 = \{(b, c), (b, d)\}$ and $(b, c), (b, d) \notin \sharp_0$, and, also, $\prec_0 = \{(a, \underline{a}), (b, \underline{b})\}$ and $(a, \underline{a}), (b, \underline{b}) \notin \rhd_0$. We emphasize that the initial configuration is not empty. Notice that $\sharp_0$ is not hereditary w.r.t. $<_0$ because $a \sharp_0 b <_0 d$ and $\neg(a \sharp_0 d)$. From Definition 2, we know that x and y are in the sustained causation relation iff x causes y, and x cannot be reversed as long as y is present. In $\mathcal{E}_0$, the pairs (b, c) and (b, d) are in the causality relation $<_0$, and the prevention relation $\rhd_0$ consists of the single pair $(c, \underline{b})$. Therefore, the only pair (b, c) belongs to the sustained causation $\ll_0$. It is easy to see that $\sharp_0$ is hereditary w.r.t. $\ll_0$. So, the structure $\mathcal{E}_0$ is indeed an RPES. ◇

The RPES progresses by executing events and/or by undoing previously executed events, thus moving from one configuration to another. The act of moving is a computation step. Reachable configurations are subsets of events which can be reached from the initial configuration by executing computation steps. A sequence of computation steps is a trace of the RPES.

Definition 3. *Given an RPES $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$ and $C \subseteq E$ such that C is finite and $CF(C)$,*

- *for $A \subseteq E$ and $B \subseteq F$, $A \cup \underline{B}$ is* enabled at C *if*
  - *$A \cap C = \emptyset$, $B \subseteq C$, $(C \cup A)$ is finite and $CF(C \cup A)$;*
  - *$\forall e \in A, \forall e' \in E$: if $e' < e$ then $e' \in (C \setminus B)$;*
  - *$\forall e \in B, \forall e' \in E$: if $e' \prec \underline{e}$ then $e' \in (C \setminus (B \setminus \{e\}))$;*
  - *$\forall e \in B, \forall e' \in E$: if $e' \rhd \underline{e}$ then $e' \notin (C \cup A)$.*

  If $A \cup \underline{B}$ is enabled at C then $C \stackrel{A \cup \underline{B}}{\longrightarrow} C' = (C \setminus B) \cup A$. We shall write $l(A \cup \underline{B}) = M$ iff M is a multiset over the set L of actions, defined as follows: $M(a) = \mid \{e \in (A \cup B) \mid l(e) = a\} \mid$ for all $a \in L$.
- *C is a* forwards reachable configuration of *$\mathcal{E}$ (from C_0) iff for all $i = 1, \ldots, n$ $(n \geq 0)$, there exists a set $A_i \subseteq E$ such that $C_{i-1} \stackrel{A_i \cup \emptyset_i}{\longrightarrow} C_i$ and $C_n = C$.*
- *C is a* (reachable) configuration of *$\mathcal{E}$ (from C_0) iff for all $i = 1, \ldots, n$ $(n \geq 0)$, there exist sets $A_i \subseteq E$ and $B_i \subseteq F$ such that $C_{i-1} \stackrel{A_i \cup \underline{B}_i}{\longrightarrow} C_i$ and $C_n = C$. In this case, $t = (A_1 \cup \underline{B}_1) \ldots (A_n \cup \underline{B}_n)$ $(n \geq 0)$ is a trace of $\mathcal{E}$ and $last(t) = C_n$. The set of (reachable) configurations of $\mathcal{E}$ is denoted by $Conf(\mathcal{E})$, and the set of traces of $\mathcal{E}$ — by $Trace(\mathcal{E})$. Clearly, any $C \in Conf(\mathcal{E})$ is conflict-free.*

– *Two traces* $t = (A_1 \cup \underline{B}_1)\dots(A_n \cup \underline{B}_n)$ $(n \geq 0)$ *and* $t' = (A'_1 \cup \underline{B'}_1)\dots(A'_m \cup \underline{B'}_m)$ $(m \geq 0)$ *of* $\mathcal{E}$ *are called to be* equivalent w.r.t. $\sim$ *(denoted* $t \sim t'$*) iff* $last(t) = last(t')$. *The equivalence class of* t *is denoted by* $[t]$.

Example 2. First, recall the RPES $\mathcal{E}_0 = (E_0,\ <_0,\ \sharp_0,\ l_0,\ F_0,\ \prec_0,\ \rhd_0,\ C_0^0)$ (see Example 1) with the components: $E_0 = \{a, b, c, d\}$; $<_0 = \{(b, c), (b, d)\}$; $\sharp_0 = \{(a, b), (b, a), (a, c), (c, a), (c, d), (d, c)\}$; l_0 is the identical function; $F_0 = \{a, b\}$; $\prec_0 = \{(a, \underline{a}), (b, \underline{b})\}$; $\rhd_0 = \{(c, \underline{b})\}$; $C_0^0 = \{a\}$. Let us consider the possible computation steps from the initial configuration $C_0^0 = \{a\}$. Since the pair (a, b) $((a, c))$ is in the conflict relation $\sharp_0$, the events a and b (a and c) cannot occur together in any configuration. As the pairs (b, c) and (b, d) are in the causality relation $<_0$, the events c and d cannot occur before b has happened. Therefore, there is no forward computation step from the initial configuration $C_0^0 = \{a\}$, despite the fact that the events a and d are independent. However, the reverse step $\{a\} \stackrel{(\emptyset \cup \{\underline{a}\})}{\rightarrow} \emptyset$ is possible, because $\prec_0 = \{(a, \underline{a}), (b, \underline{b})\}$, i.e., the only reverse cause for the event a is the event itself, and $(\cdot, \underline{a}) \notin \rhd_0$ for all $\cdot \in \{b, c, d\}$, i.e., there is no event which could prevent the undoing of a. The next forward step $\emptyset \stackrel{(\{a\} \cup \emptyset)}{\rightarrow} \{a\}$ is permissible, since a has no cause. As b also has no cause, and, moreover, the pairs (b, c) and (b, d) in the causality relation $<_0$, we get the following forward steps: $\emptyset \stackrel{(\{b\} \cup \emptyset)}{\rightarrow} \{b\} \stackrel{(\{c\} \cup \emptyset)}{\rightarrow} \{b, c\}$ and $\emptyset \stackrel{(\{b\} \cup \emptyset)}{\rightarrow} \{b\} \stackrel{(\{d\} \cup \emptyset)}{\rightarrow} \{b, d\}$. Notice that the configurations $\{b, c\}$ and $\{b, d\}$ cannot be expanded, respectively, by the event d and by the event c, since c and d conflict. Due to $\prec_0 = \{(a, \underline{a}), (b, \underline{b})\}$, i.e., the only reverse cause for the event b is the event itself, and $\rhd_0 = \{(c, \underline{b})\}$, i.e., the event b can be undone only when c is not present, the following reverse steps are acceptable: $\{b\} \stackrel{(\emptyset \cup \{\underline{b}\})}{\rightarrow} \emptyset$ and $\{b, d\} \stackrel{(\emptyset \cup \{\underline{b}\})}{\rightarrow} \{d\}$. Since the events a and d are not in conflict, and a has no causes, we can move from $\{d\}$ to $\{a, d\}$ by executing the step $(\{a\} \cup \emptyset)$. As the only reverse cause for the event a is the event itself, and there is no event which could prevent the undoing of a, we can reverse a in $\{a, d\}$, obtaining again the configuration $\{d\}$. Therefore, the configurations of $\mathcal{E}_0$ are the following: $\emptyset$, $\{a\}$, $\{b\}$, $\{d\}$, $\{b, c\}$, $\{b, d\}$, $\{a, d\}$. So, we can reach the configurations $\{d\}$ and $\{a, d\}$ from the initial configuration $\{a\}$ with a combination of forward and reverse steps but we cannot reach them by doing only forward steps.

Second, consider the RPES $\mathcal{E}_1 = (E_1,\ <_1,\ \sharp_1,\ l_1,\ F_1,\ \prec_1,\ \rhd_1,\ C_0^1)$, where $E_1 = \{a, b\}$; $<_1 = \{(a, b)\}$; $\sharp_1 = \emptyset$; l_1 is the identical function; $F_1 = \{a\}$; $\prec_1 = \{(a, \underline{a})\}$; $\rhd_1 = \emptyset$; $C_0^1 = \emptyset$. As the only pair (a, b) is in the causality relation $<_1$, i.e., the event a has no cause and causes the event b, a can occur first and only after that b can happen. Then, we obtain the forward steps: $\emptyset \stackrel{(\{a\} \cup \emptyset)}{\rightarrow} \{a\} \stackrel{(\{b\} \cup \emptyset)}{\rightarrow} \{a, b\}$. The intended meaning of $a \prec_1 \underline{a}$ is that the event a can be undone if it has occurred and has not yet been undone. In this regard, the reverse step $\{a\} \stackrel{(\emptyset \cup \{\underline{a}\})}{\rightarrow} \emptyset$ is possible, thanks to $(b, \underline{a}) \notin \prec_1$ and $\rhd_1 = \emptyset$. Moreover, the event a can be undone in the configuration $\{a, b\}$ even though the event b is present because $(b, \underline{a}) \notin \rhd_1$. This means that we can move backwards from $\{a, b\}$ to $\{b\}$ by executing the step $(\emptyset \cup \{\underline{a}\})$. Therefore, the configurations of $\mathcal{E}_1$ are: $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$. We emphasize that we can reach the configuration $\{b\}$ from the initial configuration $\emptyset$ with a combination of forward and reverse steps but we cannot reach $\{b\}$ by doing only forward steps. The traces of $\mathcal{E}_1$ are:

$((\{a\} \cup \emptyset)(\emptyset \cup \{\underline{a}\}))^*$,
$((\{a\} \cup \emptyset)(\emptyset \cup \{\underline{a}\}))^*(\{a\} \cup \emptyset)$,
$((\{a\} \cup \emptyset)(\emptyset \cup \{\underline{a}\}))^*(\{a\} \cup \emptyset)(\{b\} \cup \emptyset)((\emptyset \cup \{\underline{a}\})(\{a\} \cup \emptyset))^*$,
$((\{a\} \cup \emptyset)(\emptyset \cup \{\underline{a}\}))^*(\{a\} \cup \emptyset)(\{b\} \cup \emptyset)((\emptyset \cup \{\underline{a}\})(\{a\} \cup \emptyset))^*(\emptyset \cup \{\underline{a}\})$.

Third, examine the RPES $\mathcal{E}_2 = (E_2,\ <_2,\ \sharp_2,\ l_2,\ F_2,\ \prec_2,\ \rhd_2,\ C_0^2)$, where $E_2 = \{a, b\}$; $<_2 = \emptyset$; $\sharp_2 = \emptyset$; l_2 is the identical function; $F_2 = \{a\}$; $\prec_2 = \{(a, \underline{a})\}$; $\rhd_2 = \{(b, \underline{a})\}$; $C_0^2 = \emptyset$. As the causality relation $<_2$ and the conflict relation $\sharp_2$ are empty, the events a and b are independent, and, therefore, they can take place in any order. This leads to the following forward steps: $\emptyset \overset{(\{a\}\cup\emptyset)}{\rightarrow} \{a\} \overset{(\{b\}\cup\emptyset)}{\rightarrow} \{a,b\}$ and $\emptyset \overset{(\{b\}\cup\emptyset)}{\rightarrow} \{b\} \overset{(\{a\}\cup\emptyset)}{\rightarrow} \{a,b\}$. Since $b \rhd_2 \underline{a}$, we conclude that b prevents the undoing of a, i.e. a cannot be undone if b is present. So, we can go back from $\{a\}$ to $\emptyset$ by executing the step $(\emptyset \cup \{\underline{a}\})$ and cannot move backwards from $\{a,b\}$. The configurations of $\mathcal{E}_2$ are: $\emptyset$, $\{a\}$, $\{b\}$, $\{a,b\}$. Moreover, the traces of $\mathcal{E}_2$ are:

$((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*$,
$((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*(\{a\}\cup\emptyset)$,
$((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*(\{a\}\cup\emptyset)(\{b\}\cup\emptyset)$,
$((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*(\{a,b\}\cup\emptyset)$,
$((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*(\{b\}\cup\emptyset)$,
$((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*(\{b\}\cup\emptyset)(\{a\}\cup\emptyset)$.

Fourth, check the RPES $\mathcal{E}_3 = (E_3,\ <_3,\ \sharp_3,\ l_3,\ F_3,\ \prec_3,\ \rhd_3,\ C_0^3)$, where $E_3 = \{a,b,c\}$; $<_3 = \emptyset$; $\sharp_3 = \{(a,c),(c,a)\}$; l_3 is the identical function; $F_3 = \{b\}$; $\prec_3 = \{(a,\underline{b}),\ (b,\underline{b})\}$; $\rhd_3 = \emptyset$; $C_0^3 = \{b\}$. As the causality relation $<_3$ is empty, all the events of $\mathcal{E}_3$ have no causes, and, in addition, the events a and b (and also b and c) are independent, because only the events a and c conflict. This means that the events a and b (and also b and c) can occur in any order, and the events a and c cannot occur together in any configuration. However, since the initial configuration C_0^3 already contains b, only the following forward steps are possible: $\{b\} \overset{(\{a\}\cup\emptyset)}{\rightarrow} \{a,b\}$ and $\{b\} \overset{(\{c\}\cup\emptyset)}{\rightarrow} \{b,c\}$. Notice that the configurations $\{a,b\}$ and $\{b,c\}$ cannot be extended, respectively, by the event c and by the event a, because a and c are in conflict. The meaning of the relation $\prec_3 = \{(a,\underline{b}),\ (b,\underline{b})\}$ is that the event b can be undone only if both the events a and b have already occurred. Then, we get: $\{a,b\} \overset{(\emptyset\cup\{\underline{b}\})}{\rightarrow} \{a\}$ and $\{b\} \overset{(\emptyset\cup\{\underline{b}\})}{\not\rightarrow}$, $\{b,c\} \overset{(\emptyset\cup\{\underline{b}\})}{\not\rightarrow}$. Therefore, it is easy to check that $\{a\}$, $\{b\}$, $\{a,b\}$, $\{b,c\}$ are configurations of $\mathcal{E}_3$. Also, we get the following traces of $\mathcal{E}_3$:

ϵ,
$(\{a\}\cup\emptyset)$,
$(\{c\}\cup\emptyset)$,
$(\{a\}\cup\emptyset)((\emptyset\cup\{\underline{b}\})(\{b\}\cup\emptyset))^*$,
$(\{a\}\cup\emptyset)((\emptyset\cup\{\underline{b}\})(\{b\}\cup\emptyset))^*(\emptyset\cup\{\underline{b}\})$.

Fifth, contemplate the RPES $\mathcal{E}_4 = (E_4,\ <_4,\ \sharp_4,\ l_4,\ F_4,\ \prec_4,\ \rhd_4,\ C_0^4)$, where $E_4 = \{a,b,c\}$; $<_4 = \{(a,b)\}$; $\sharp_4 = \{(a,c),(c,a),(b,c),(c,b)\}$; l_4 is the identical function; $F_4 = \{a,c\}$; $\prec_4 = \{(a,\underline{a}),(c,\underline{c})\}$; $\rhd_4 = \{(b,\underline{a})\}$; $C_0^4 = \emptyset$. We observe that the events a and b are in the causality relation, i.e., b cannot occur before a has happened, and the events a (b) and c conflict, i.e. the events a (b) and c cannot occur together in any configuration. It is not difficult to discover that the conflict relation $\sharp_4$ is hereditary w.r.t. $<_4$. The event a cannot be undone while the event b is present, since $b \rhd_4 \underline{a}$. Besides, the only reverse cause for the undoing of any reversible event of $\mathcal{E}_4$ is the event itself, because $F_4 = \{a,c\}$ and $\prec_4 = \{(a,\underline{a}),(c,\underline{c})\}$. Therefore, it is easy to see that the configurations of $\mathcal{E}_4$ are $\emptyset$, $\{a\}$, $\{a,b\}$, $\{c\}$. Furthermore, we have the following traces of $\mathcal{E}_4$:

$(((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*((\{c\}\cup\emptyset)(\emptyset\cup\{\underline{c}\}))^*)^*$,
$(((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*((\{c\}\cup\emptyset)(\emptyset\cup\{\underline{c}\}))^*)^*(\{a\}\cup\emptyset)$,
$(((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*((\{c\}\cup\emptyset)(\emptyset\cup\{\underline{c}\}))^*)^*(\{a\}\cup\emptyset)(\{b\}\cup\emptyset)$,
$(((\{a\}\cup\emptyset)(\emptyset\cup\{\underline{a}\}))^*((\{c\}\cup\emptyset)(\emptyset\cup\{\underline{c}\}))^*)^*(\{c\}\cup\emptyset)$.

Finally, consider the RPES $\mathcal{E}_5 = (E_5,\ <_5,\ \sharp_5,\ l_5,\ F_5,\ \prec_5,\ \rhd_5,\ C_0^5)$, where $E_5 = \{a,b,c\}$; $<_5 = \emptyset$; $\sharp_5 = \{(a,c),(c,a)\}$; l_4 is the identical function; $F_5 = \{b\}$; $\prec_5 = \{(b,\underline{b})\}$; $\rhd_5 = \emptyset$, $C_0^5 = \emptyset$. We see that a and c conflict, i.e. they cannot occur together in any configuration. As the causality relation $<_5$ is empty and the conflict relation $\sharp_5 = \{(a,c),(c,a)\}$, the events a and b (b and c) are independent, and, therefore, they can take place in any order. This leads to the following forward steps: $\emptyset \overset{(\{a\}\cup\emptyset)}{\rightarrow} \{a\} \overset{(\{b\}\cup\emptyset)}{\rightarrow} \{a,b\}$ and $\emptyset \overset{(\{b\}\cup\emptyset)}{\rightarrow} \{b\} \overset{(\{a\}\cup\emptyset)}{\rightarrow} \{a,b\}$ ($\emptyset \overset{(\{b\}\cup\emptyset)}{\rightarrow} \{b\} \overset{(\{c\}\cup\emptyset)}{\rightarrow} \{b,c\}$ and $\emptyset \overset{(\{c\}\cup\emptyset)}{\rightarrow} \{c\} \overset{(\{b\}\cup\emptyset)}{\rightarrow} \{b,c\}$). The only cause for the undoing of the event $b \in F_5$ is the event itself, because $\prec_5 = \{(b,\underline{b})\}$. Since $\rhd_5 = \emptyset$, the event b can be undone in any configuration where it occurs. It is easy to check that the configurations of $\mathcal{E}_5$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{b,c\}$. Also, we have the following traces of $\mathcal{E}_5$:

$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)(\{a\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)(\{c\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{a\}\cup\emptyset)((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{c\}\cup\emptyset)((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{a\}\cup\emptyset)((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{c\}\cup\emptyset)((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)(\{a\}\cup\emptyset)(\emptyset\cup\{\underline{b}\})((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)(\{c\}\cup\emptyset)(\emptyset\cup\{\underline{b}\})((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)(\{a\}\cup\emptyset)(\emptyset\cup\{\underline{b}\})((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)(\{c\}\cup\emptyset)(\emptyset\cup\{\underline{b}\})((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b\}\cup\emptyset)$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{a,b\}\cup\emptyset)((\emptyset\cup\{\underline{b}\})(\{b\}\cup\emptyset))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{a,b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\})((\emptyset\cup\{\underline{b}\})(\{b\}\cup\emptyset))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b,c\}\cup\emptyset)((\emptyset\cup\{\underline{b}\})(\{b\}\cup\emptyset))^*$,
$((\{b\}\cup\emptyset)(\emptyset\cup\{\underline{b}\}))^*(\{b,c\}\cup\emptyset)(\emptyset\cup\{\underline{b}\})((\emptyset\cup\{\underline{b}\})(\{b\}\cup\emptyset))^*$. ◇

The last two items of Definition 3 lead to the following auxiliary

Lemma 1. *Given an RPES $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$, it holds:*

(i) $\{last(t) \mid t \in Trace(\mathcal{E})\} = Conf(\mathcal{E})$;
(ii) *for any* $t \in Trace(\mathcal{E})$, *if* $t(A\cup\underline{B}) \in Trace(\mathcal{E})$ *then* $last(t) \stackrel{A\cup\underline{B}}{\rightarrow} last(t(A\cup\underline{B}))$;
(iii) *for any* $last(t) \in Conf(\mathcal{E})$, *if* $last(t) \stackrel{A\cup\underline{B}}{\rightarrow} last(t')$ *then* $t(A\cup\underline{B}) \in Trace(\mathcal{E})$ *and* $t(A\cup\underline{B}) \sim t'$.

RPESs are able to model such a peculiarity of reversible computation as causal-consistent reversibility which relates reversibility with causality: an event can be undone provided that all of its effects have been undone. This allows the system to get back to a past state, which could only be reached by forward computation. This notion of reversibility is natural in reliable concurrent systems since when an error occurs the system tries to go back to a past consistent state.

Definition 4. *An RPES $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$ is called*

- cause-respecting *if for any $e, e' \in E$, if $e < e'$ then $e \ll e'$;*
- causal *if for any $e \in E$ and $u \in F$ it holds: $e \prec \underline{u}$ iff $e = u$, and $e \rhd \underline{u}$ iff $u < e$.*

Informally, in the cause-respecting and causal RPES, causes can be only undone if their effects are not present in the current configuration. Clearly, if the RPES is causal, then it is cause-respecting as well.

Example 3. First, recall the RPES $\mathcal{E}_0$ (see Examples 1 and 2) with the components: $E_0 = \{a,b,c,d\}$; $<_0 = \{(b,c),(b,d)\}$; $\sharp_0 = \{(a,b),(b,a),(a,c),(c,a),(c,d),(d,c)\}$; l_0 is the identical function; $F_0 = \{a,b\}$; $\prec_0 = \{(a,\underline{a}),(b,\underline{b})\}$; $\rhd_0 = \{(c,\underline{b})\}$; $C_0^0 = \{a\}$. From Example 1 we know that $\ll_0 = \{b,c\}$, i.e., $(b,d) \notin \ll_0$. This is because $(d,\underline{b}) \notin \rhd_0$, although $(b,d) \in <_0$, $d \in E_0$, and $b \in F_0$. So, this RPES is neither cause-respecting nor causal.

Second, consider the RPES $\mathcal{E}_1$ (see Example 2) with the components: $E_1 = \{a,b\}$; $<_1 = \{(a,b)\}$; $\sharp_1 = \emptyset$; l_1 is the identical function; $F_1 = \{a\}$; $\prec_1 = \{(a,\underline{a})\}$; $\rhd_1 = \emptyset$; $C_0^1 = \emptyset$. It is easy to see that $\ll_1 = \emptyset$, since $<_1 = \{(a,b)\}$ and $(b,\underline{a}) \notin \rhd_1$. Then, we obtain $<_1 \neq \ll_1$. So, this RPES is neither cause-respecting nor causal.

Third, examine the RPES $\mathcal{E}_2 = (E_2, <_2, \sharp_2, l_2, F_2, \prec_2, \rhd_2, C_0^2)$ (see Example 2) with the components: $E_2 = \{a,b\}$; $<_2 = \emptyset$; $\sharp_2 = \emptyset$; l_2 is the identical function; $F_2 = \{a\}$; $\prec_2 = \{(a,\underline{a})\}$; $\rhd_2 = \{(b,\underline{a})\}$; $C_0^2 = \emptyset$. The RPES is cause-respecting, because the causality relation $<_2$ is empty, and, hence, for the only reversible event a of $\mathcal{E}_3$, the set of its effects is empty, which implies $<_2=\ll_2=\emptyset$. On the other hand, $\mathcal{E}_2$ is not causal, because there are the events a and b such that $b \rhd_2 \underline{a}$ and $a \not<_2 b$.

Fourth, treat the RPES $\mathcal{E}_3 = (E_3, <_3, \sharp_3, l_3, F_3, \prec_3, \rhd_3, C_0^3)$ (see Example 2) with the components: $E_3 = \{a, b, c\}$; $<_3=\emptyset$; $\sharp_3 = \{(a, c), (c, a)\}$; l_3 is the identical function; $F_3 = \{b\}$; $\prec_3= \{(a, \underline{b}),\ (b, \underline{b})\}$; $\rhd_3 = \emptyset$ and $C_0^3 = \{b\}$. Since for the only reversible event b, the set of its effects is empty, we conclude that the RPES is cause-respecting, whereas it is not causal because $(a, \underline{b}) \in \prec_3$ and $a \neq b$.

Fifth, contemplate the RPES $\mathcal{E}_4 = (E_4, <_4, \sharp_4, l_4, F_4, \prec_4, \rhd_4, C_0^4)$ (see Example 2) with the components: $E_4 = \{a, b, c\}$; $<_4= \{(a, b)\}$; $\sharp_4 = \{(a, c), (c, a), (b, c), (c, b)\}$; l_4 is the identical function; $F_4 = \{a, c\}$; $\prec_4= \{(a, \underline{a}), (c, \underline{c})\}$; $\rhd_4 = \{(b, \underline{a})\}$; $C_0^4 = \emptyset$. We see that $<_4=\ll_4$ because $<_4= \{(a, b)\}$ and $\rhd_4 = \{(b, \underline{a})\}$. Then, $\mathcal{E}_4$ is a cause-respecting RPES. Further, the only reverse cause for the undoing of any reversible event is the event itself, because $F_4 = \{a, c\}$ and $\prec_4= \{(a, \underline{a}), (c, \underline{c})\}$. In addition, the only events a and b are in the causality relation since $<_4= \{(a, b)\}$, and the only a cannot be undown while the only b is present since $\rhd_4 = \{(b, \underline{a})\}$. So, $\mathcal{E}_4$ is a causal RPES.

Finally, consider the RPES $\mathcal{E}_5 = (E_5, <_5, \sharp_5, l_5, F_5, \prec_5, \rhd_5, C_0^5)$ (see Example 2) with the components: $E_5 = \{a, b, c\}$; $<_5=\emptyset$; $\sharp_5 = \{(a, c), (c, a)\}$; l_4 is the identical function; $F_5 = \{b\}$; $\prec_5= \{(b, \underline{b})\}$; $\rhd_5 = \emptyset$, $C_0^5 = \emptyset$. The RPES is causal, and, therefore, cause-respecting. This is because $<_5=\emptyset$ and $\rhd_5 = \emptyset$, and the reverse cause for the undoing of the only reversible event is the event itself, since we have $F_5 = \{b\}$ and $\prec_5= \{(b, \underline{b})\}$. ◇

Any cause-respecting RPES with the empty initial configuration can be presented as a PES. On the other hand, any PES can be converted into a causal RPES with the empty initial configuration, once we specify which events are to be reversible. The following facts are slight modifications of Propositions 3.36 and 3.37 from [48].

Proposition 1.

(i) *If $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, \emptyset)$ is a cause-respecting RPES then $\phi(\mathcal{E}) = (E, <, \sharp, l, \emptyset)$ is a PES.*
(ii) *If $\mathcal{E} = (E, <, \sharp, l, \emptyset)$ is a PES and $F \subseteq E$ then $\varphi(\mathcal{E}, F) = (E, <, \sharp, l, F, \prec, \rhd, \emptyset)$ is a causal RPES, where $e \prec \underline{e}$ for any $e \in F$, and $e \rhd \underline{e'}$ for any $e \in E$ and $e' \in F$ such that $e' < e$. Moreover, $\phi(\varphi(\mathcal{E}, F)) = \mathcal{E}$.*

The following lemma states specific features of the configurations of the cause-respecting RPES, which are left-closed w.r.t. causality and forwards reachable.

Lemma 2. *Given a cause-respecting $\mathcal{E}$ and its configuration $C \in Conf(\mathcal{E})$, it holds:*

(i) *C is left-closed under $<$;*
(ii) *if C is reachable, then C is forwards reachable.*

Proof. See Appendix.

The next example explains the above lemma.

Example 4. Recall the non-cause-respecting RPES $\mathcal{E}_0$ (with $<_0= \{(b, c), (b, d)\}$) from Examples 1–3. We know that the configurations of $\mathcal{E}_0$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{d\}$, $\{b, c\}$, $\{b, d\}$, $\{a, d\}$. Clearly, the configurations $\{d\}$ and $\{a, d\}$ are not left-closed under $<_0$, since $(b, d) \in <_0$.

Consider the non-cause-respecting RPES $\mathcal{E}_1$ (with $<_1= \{(a, b)\}$) from Examples 2–3. The configurations of $\mathcal{E}_1$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$. We see that the configuration $\{b\}$ is not left-closed under $<_1$ because $<_1= \{(a, b)\}$.

It is easy to check that in the cause-respecting RPESs $\mathcal{E}_2$–$\mathcal{E}_5$ from Examples 2–3, all their configurations are left-closed under their causality relations. ◇

3 Residuals

The removal operator, the concept of which is based on deleting already executed configurations (traces) and events that conflict with the events presenting in the configurations (traces), is necessary for residual semantics.

Introduce the definition of the removal operator for RPESs by using their traces which allow us to simplify the definition.

Definition 5. *For a causal RPES $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$ and its trace $t = (A_1 \cup \underline{B}_1) \ldots (A_n \cup \underline{B}_n) \in Trace(\mathcal{E})$ $(n \geq 0)$, the residual $\mathcal{E} \setminus t$ of $\mathcal{E}$ after t under the removal operator $\setminus$ is defined by induction on $0 \leq i \leq n$ as follows:*

$i = 0$. $\mathcal{E} \setminus (t_0 = \epsilon) = \mathcal{E}$.

$i > 0$. $\mathcal{E} \setminus t_i = (E^i,\ <^i = <^{i-1} \cap (E^i \times E^i),\ \sharp^i = \sharp^{i-1} \cap (E^i \times E^i),\ l^i = l^{i-1} |_{E^i},\ F^i = F^{i-1} \cap E^i,$ $\prec^i = \prec^{i-1} \cap (E^i \times \underline{F}^i),\ \rhd^i = \rhd^{i-1} \cap\ (E^i \times \underline{F}^i),\ C_0^i)$, *with*

- $E^i = E^{i-1} \setminus (\widetilde{A}_i \cup \sharp^{i-1}(\widetilde{A}_i))$, *where*
  $\widetilde{A}_i = (A_i \backslash F^{i-1}) \cup (\{\widetilde{a} \in F^{i-1} \mid \exists a \in A_i \backslash F^{i-1} : \widetilde{a} <^{i-1} a\} = \lfloor (A_i \backslash F^{i-1}) \rfloor \cap F^{i-1})$,
  $\sharp^{i-1}(\widetilde{A}_i) = \{a \in E^{i-1} \mid \exists \widetilde{a} \in \widetilde{A}_i : a \sharp^{i-1} \widetilde{a}\}$;
- $C_0^i = (C_0^{i-1} \setminus B_i \cup A_i) \cap E^i$.

$\mathcal{E} \setminus t = \mathcal{E} \setminus t_n$.

The intuitive interpretation of the above definition is as follows. In the process of constructing the residual of the RPES after a trace, all the irreversible events occurred in the current computation step, their reversible causes and conflicting events thereof are removed, yielding a reduction of all the relations, the labelling function and the initial configuration in the residual. As the removal operator is intended to eliminate events that have been already executed, the irreversible events in the current step are removed because they can never be undone. In causal RPESs (where $e' < e$ iff $e \rhd \underline{e}'$ for all $e \in E$ and $e' \in F$), removing the reversible causes of the irreversible events presented in the current step is mandatory due to the fact that the presence of the irreversible effects forever prevents undoing their reversible causes, which in fact become irreversible and must be removed. At the same time, all the reversible events, if any, presented in the current step are retained, since they can be reversed in the following steps.

We illustrate the application of the removal operator to causal RPESs and their traces with

Example 5. First, consider the RPES $\mathcal{E}_4 = (E_4,\ <_4,\ \sharp_4,\ l_4,\ F_4,\ \prec_4,\ \rhd_4,\ C_0^4)$ from Examples 2–4, where $E_4 = \{a, b, c\}$; $<_4 = \{(a, b)\}$; $\sharp_4 = \{(a, c), (c, a), (b, c), (c, b)\}$; l_4 is the identical function; $F_4 = \{a, c\}$; $\prec_4 = \{(a, \underline{a}), (c, \underline{c})\}$; $\rhd_4 = \{(b, \underline{a})\}$, $C_0^4 = \emptyset$. We know that the configurations of $\mathcal{E}_4$ are: $\emptyset$, $\{a\}$, $\{c\}$, $\{a, b\}$, and the sequences $(\{a\} \cup \emptyset)$, $(\{a\} \cup \emptyset)(\emptyset \cup \{\underline{a}\})$, $(\{c\} \cup \emptyset)$, $(\{c\} \cup \emptyset)(\emptyset \cup \{\underline{c}\})$, $(\{a\} \cup \emptyset)(\{b\} \cup \emptyset)$ are traces of $\mathcal{E}_4$. From now on, set $x \in \{a, c\}$.

Applying the removal operator to the RPES $\mathcal{E}_4$ and the above traces, we obtain the following structures:

- $\dot{\mathcal{E}}_4 = \mathcal{E}_4 \setminus (A_1 = \{x\} \cup \underline{B}_1 = \emptyset) = (\dot{E} = E_4,\ \dot{<} = <_4,\ \dot{\sharp} = \sharp_4,\ \dot{l} = l_4,\ \dot{F} = F_4,\ \dot{\prec} = \prec_4,$ $\dot{\rhd} = \rhd_4,\ \dot{C}_0 = \{x\})$, because $(\widetilde{A}_1 \cup \sharp_4(\widetilde{A}_1)) = \emptyset$, due to $x \in F_4$, and $\dot{C}_0 = ((C_0^4 = \emptyset) \cup (A_1 = \{x\})) \cap (\dot{E} = \{a, b, c\}) = \{x\}$.
- $\ddot{\mathcal{E}}_4 = \mathcal{E}_4 \setminus (A_1 = \{x\} \cup \underline{B}_1 = \emptyset)(A_2 = \emptyset \cup \underline{B}_2 = \{\underline{x}\}) = (\ddot{E} = \dot{E},\ \ddot{<} = \dot{<},\ \ddot{\sharp} = \dot{\sharp},\ \ddot{l} = \dot{l},$ $\ddot{F} = \dot{F},\ \ddot{\prec} = \dot{\prec},\ \ddot{\rhd} = \dot{\rhd},\ \ddot{C}_0 = \emptyset)$, since $(\widetilde{A}_2 \cup \dot{\sharp}(\widetilde{A}_2)) = \emptyset$, thanks to $A_2 = \emptyset$, and $\ddot{C}_0 = (((\dot{C}_0 = \{x\}) \setminus (B_2 = \{x\})) \cap (\ddot{E} = \{a, b, c\})) = \emptyset$.
- $\dddot{\mathcal{E}}_4 = \mathcal{E}_4 \setminus (A_1 = \{a\} \cup \underline{B}_1 = \emptyset)(A_2 = \{b\} \cup \underline{B}_2 = \emptyset) = (\dddot{E} = \emptyset,\ \dddot{<} = \emptyset,\ \dddot{\sharp} = \emptyset,\ \dddot{l} = \emptyset,\ \dddot{F} = \emptyset,\ \dddot{\prec} = \emptyset,\ \dddot{\rhd} = \emptyset,\ \dddot{C}_0 = \emptyset)$, because $\widetilde{A}_2 = \{a, b\}$, due to $b \in A_2 \setminus \dot{F}$, $a \in \dot{F}$, and $a \dot{<} b$, and $\dot{\sharp}(\widetilde{A}_2) = \{c\}$, due to $a \sharp c$, and, moreover, $\dddot{C}_0 = (((\dot{C} = \{a\}) \cup (A_2 = \{b\})) \cap (\dddot{E} = \emptyset)) = \emptyset$.

Notice that the removal operator produces the same residuals after the different traces. For example, it is easy to see that $\mathcal{E}_4 \setminus \epsilon = \mathcal{E}_4 \setminus (\{a\} \cup \emptyset)(\emptyset \cup \{\underline{a}\}) = \mathcal{E}_4 \setminus (\{c\} \cup \emptyset)(\emptyset \cup \{\underline{c}\})$.

Second, examine the RPES $\mathcal{E}_5 = (E_5, <_5, \sharp_5, l_5, F_5, \prec_5, \rhd_5, C_0^5)$ from Examples 2–4, where $E_5 = \{a, b, c\}$; $<_5 = \emptyset$; $\sharp_5 = \{(a, c), (c, a)\}$; l_5 is the identical function; $F_5 = \{b\}$; $\prec_5 = \{(b, \underline{b})\}$; $\rhd_5 = \emptyset$, $C_0^5 = \emptyset$. We know that the configurations of $\mathcal{E}_5$ are: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, and the sequences $(\{x\} \cup \emptyset)$, $(\{b\} \cup \emptyset)$, $(\{b, x\} \cup \emptyset)$, $(\{b\} \cup \emptyset)(\{x\} \cup \emptyset)$, $(\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})$, $(\{x\} \cup \emptyset)(\{b\} \cup \emptyset)$, $(\{x\} \cup \emptyset)(\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})$ are traces of $\mathcal{E}_5$. Here and further, set $x \neq x' \in \{a, c\}$.

We construct the residuals of the RPES $\mathcal{E}_5$ after the above traces:

- $\breve{\mathcal{E}}_5 = \mathcal{E}_5 \setminus (A_1 = \{x\} \cup \underline{B}_1 = \emptyset) = (\breve{E} = \{b\}, \breve{<} = \emptyset, \breve{\sharp} = \emptyset, \breve{l} = l_5|_{\{b\}}, \breve{F} = \{b\}, \breve{\prec} = \{(b, \underline{b})\}, \breve{\rhd} = \emptyset, \breve{C}_0 = \emptyset)$, because $\widetilde{A}_1 = \{x\}$, due to $x \in A_1 \setminus F_5$, and $\sharp_5(\widetilde{A}_1) = \{x'\}$, due to $x \sharp_5 x'$, and, moreover, $\breve{C}_0 = (((C_0^5 = \emptyset) \cup (A_1 = \{x\})) \cap (\breve{E} = \{b\})) = \emptyset$.
- $\hat{\mathcal{E}}_5 = \mathcal{E}_5 \setminus (A_1 = \{b\} \cup \underline{B}_1 = \emptyset) = (\hat{E} = E_5, \hat{<} = <_5, \hat{\sharp} = \sharp_5, \hat{l} = l_5, \hat{F} = F_5, \hat{\prec} = \prec_5, \hat{\rhd} = \rhd_5, \hat{C}_0 = \{b\})$, since $(\widetilde{A}_1 \cup \sharp_5(\widetilde{A}_1)) = \emptyset$, thanks to $b \in F_5$, and $\hat{C}_0 = (((C_0^5 = \emptyset) \cup (A_1 = \{b\})) \cap (\hat{E} = \{a, b, c\})) = \{b\}$;
- $\check{\mathcal{E}}_5 = \mathcal{E}_5 \setminus (A_1 = \{x, b\} \cup \underline{B}_1 = \emptyset) = (\check{E} = \{b\}, \check{<} = \emptyset, \check{\sharp} = \emptyset, \check{l} = l_5|_{\{b\}}, \check{F} = \{b\}, \check{\prec} = \{(b, \underline{b})\}, \check{\rhd} = \emptyset, \check{C}_0 = \{b\})$, because $\widetilde{A}_1 = \{x\}$, due to $x \in A_1 \setminus F_5$, $b \in F_5$, and $\sharp_5(\widetilde{A}_1) = \{x'\}$, due to $x \sharp_5 x'$, and, moreover, $\check{C}_0 = (((C_0^5 = \emptyset) \cup (A_1 = \{x, b\})) \cap (\check{E} = \{b\})) = \{b\}$.
- $\dot{\mathcal{E}}_5 = (\mathcal{E}_5 \setminus (A_1 = \{b\} \cup \underline{B}_1 = \emptyset)(A_2 = \{x\} \cup \underline{B}_2 = \emptyset) = (\dot{E} = \{b\}, \dot{<} = \emptyset, \dot{\sharp} = \emptyset, \dot{l} = \hat{l}|_{\{b\}}, \dot{F} = \{b\}, \dot{\prec} = \{(b, \underline{b})\}, \dot{\rhd} = \emptyset, \dot{C}_0 = \{b\})$, because $\widetilde{A}_2 = \{x\}$, due to $x \in A_2 \setminus \hat{F}$, and $\hat{\sharp}(\widetilde{A}_2) = \{x'\}$, due to $x \,\hat{\sharp}\, x'$, and, moreover, $\dot{C}_0 = (((\hat{C}_0 = \{b\}) \cup (A_2 = \{x\})) \cap (\dot{E} = \{b\})) = \{b\}$.
- $\ddot{\mathcal{E}}_5 = \mathcal{E}_5 \setminus (A_1 = \{b\} \cup \underline{B}_1 = \emptyset)(A_2 = \emptyset \cup \underline{B}_2 = \{\underline{b}\}) = (\ddot{E} = \hat{E}, \ddot{<} = \hat{<}, \ddot{\sharp} = \hat{\sharp}, \ddot{l} = \hat{l}, \ddot{F} = \hat{F}, \ddot{\prec} = \hat{\prec}, \ddot{\rhd} = \hat{\rhd}, \ddot{C}_0 = \emptyset)$, since $(\widetilde{A}_2 \cup \hat{\sharp}(\widetilde{A}_2)) = \emptyset$, thanks to $A_2 = \emptyset$, and $\ddot{C}_0 = ((\hat{C}_0 = \{b\}) \setminus (B_2 = \{b\})) \cap (\ddot{E} = \{a, b, c\}) = \emptyset$;
- $\dddot{\mathcal{E}}_5 = \mathcal{E}_5 \setminus (A_1 = \{x\} \cup \underline{B}_1 = \emptyset)(A_2 = \{b\} \cup \underline{B}_2 = \emptyset) = (\dddot{E} = \breve{E}, \dddot{<} = \breve{<}, \dddot{\sharp} = \breve{\sharp}, \dddot{l} = \breve{l}, \dddot{F} = \breve{F}, \dddot{\prec} = \breve{\prec}, \dddot{\rhd} = \breve{\rhd}, \dddot{C}_0 = \{b\})$, because $(\widetilde{A}_2 \cup \breve{\sharp}(\widetilde{A}_2)) = \emptyset$, due to $b \in \breve{F}$, and $\dddot{C}_0 = (((\breve{C}_0 = \emptyset) \cup (A_2 = \{b\})) \cap (\dddot{E} = \{b\})) = \{b\}$;
- $\ddddot{\mathcal{E}}_5 = \mathcal{E}_5 \setminus (A_1 = \{x\} \cup \underline{B}_1 = \emptyset)(A_2 = \{b\} \cup \underline{B}_2 = \emptyset)(A_3 = \emptyset \cup \underline{B}_3 = \{\underline{b}\}) = (\ddddot{E} = \dddot{E}, \ddddot{<} = \dddot{<}, \ddddot{\sharp} = \dddot{\sharp}, \ddddot{l} = \dddot{l}, \ddddot{F} = \dddot{F}, \ddddot{\prec} = \dddot{\prec}, \ddddot{\rhd} = \dddot{\rhd}, \ddddot{C}_0 = \emptyset)$, since $(\widetilde{A}_3 \cup \dddot{\sharp}(\widetilde{A}_3)) = \emptyset$, thanks to $A_3 = \emptyset$, and $\ddddot{C}_0 = ((\dddot{C}_0 = \{b\}) \setminus (B_3 = \{b\})) \cap (\ddddot{E} = \{b\}) = \emptyset$.

It is not difficult to make sure that $\mathcal{E}_5 \setminus \epsilon = \mathcal{E}_5 \setminus (\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})$, $\mathcal{E}_5 \setminus (\{b\} \cup \emptyset)(\{a\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{b\} \cup \emptyset)(\{c\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{a\} \cup \emptyset)(\{b\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{c\} \cup \emptyset)(\{b\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{b, c\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{a, b\} \cup \emptyset)$, and $\mathcal{E}_5 \setminus (\{a\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{c\} \cup \emptyset) = \mathcal{E}_5 \setminus (\{a\} \cup \emptyset)(\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}) = \mathcal{E}_5 \setminus (\{c\} \cup \emptyset)(\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})$. ◇

Below are some technical facts specific to the removal operator for RPESs.

Lemma 3. *Given a causal RPES $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$, a trace $t = (A_1 \cup \underline{B_1}) \ldots (A_n \cup \underline{B_n})$ ($C_0 \overset{A_1 \cup \underline{B_1}}{\to} C_1 \ldots C_{n-1} \overset{A_n \cup \underline{B_n}}{\to} C_n$) ($n \geq 0$) of $\mathcal{E}$, and $\mathcal{E} \setminus t = (E^n, <^n, \sharp^n, l^n, F^n, \prec^n, \rhd^n, C_0^n)$, it holds:*

(i) $E^j \subseteq E^i$, $F^j \subseteq F^i$, $l^j \subseteq l^i$, $\nabla^j \subseteq \nabla^i$ ($\nabla \in \{<, \sharp, \prec, \rhd\}$)*, for all* $0 \leq i \leq j \leq n$;
(ii) $\mathcal{E} \setminus t_i$ *is a causal RPES, for all* $0 \leq i \leq n$;
(iii) $e \in C_i$, *if* $e \in \widetilde{A}_i$, *for all* $1 \leq i \leq n$;
(iv) $B_i \subseteq F^{i-1}$, *for all* $1 \leq i \leq n$;
(v) $A_i \subseteq E^{i-1}$, *for all* $1 \leq i \leq n$;
(vi) $e \notin F$, *if* $e \in A^i \setminus F^{i-1}$ *for some* $1 \leq i \leq n$;
(vii) $e \in C_n$, *if* $e \in A^i \setminus F^{i-1}$ *for some* $1 \leq i \leq n$;
(viii) $C_0^n = C_n \cap E^n$.

Proof. See Appendix.

We establish that residuals of the causal RPES are invariant with respect to equivalent traces.

Proposition 2. *Given a causal RPES $\mathcal{E}$ with traces $t, t' \in Trace(\mathcal{E})$ such that $[t] = [t']$, $\mathcal{E} \setminus t = \mathcal{E} \setminus t'$.*

Example 6. Consider the causal RPES $\mathcal{E}_5 = (E_5,\ <_5,\ \sharp_5,\ l_5,\ F_5,\ \prec_5,\ \rhd_5,\ C_0^5)$ from Examples 2–5, where $E_5 = \{a, b, c\}$; $<_5 = \emptyset$; $\sharp_5 = \{(a, c), (c, a)\}$; l_5 is the identical function; $F_5 = \{b\}$; $\prec_5 = \{(b, \underline{b})\}$; $\rhd_5 = \emptyset$, $C_0^5 = \emptyset$. We are also given that the configurations of $\mathcal{E}_5$ are the following: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, and the sequences $t = (\{a\} \cup \emptyset)$ and $t' = (\{b\} \cup \emptyset)(\{a\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})$ are traces of $\mathcal{E}_5$. Thanks to Definition 3, we obtain that $last(t) = (C_0^5 \setminus \emptyset) \cup \{a\} = \{a\}$ and $last(t') = (((((C_0^5 \setminus \emptyset) \cup \{b\}) \setminus \emptyset) \cup \{a\}) \setminus \{b\}) \cup \emptyset = \{a\}$. Therefore, $t \sim t'$ holds. From Example 5 we know that $\mathcal{E}_5 \setminus t = \breve{\mathcal{E}}_5$ with $\breve{E} = \{b\}$, $\breve{<} = \emptyset$, $\breve{\sharp} = \emptyset$, $\breve{l} = l_5|_{\{b\}}$, $\breve{F} = \{b\}$, $\breve{\prec} = \{(b, \underline{b})\}$, $\breve{\rhd} = \emptyset$, $\breve{C}_0 = \emptyset$; and $\mathcal{E}_5 \setminus (\{b\} \cup \emptyset)(\{a\} \cup \emptyset) = \dot{\mathcal{E}}_5$ with $\dot{E} = \{b\}$, $\dot{<} = \emptyset$, $\dot{\sharp} = \emptyset$, $\dot{l} = l_5|_{\{b\}}$, $\dot{F} = \{b\}$, $\dot{\prec} = \{(b, \underline{b})\}$, $\dot{\rhd} = \emptyset$, $\dot{C}_0 = \{b\}$. Due to Definition 5, we get $\mathcal{E}_5 \setminus t' = \dot{\mathcal{E}}_5 \setminus (A_3 = \emptyset \cup \underline{B}_3 = \{\underline{b}\}) = (\dot{E},\ \dot{<},\ \dot{\sharp},\ \dot{l},\ \dot{F},\ \dot{\prec},\ \dot{\rhd},\ \bar{C}_0) = \bar{\mathcal{E}}_5$, since $(\widetilde{A}_3 \cup \dot{\sharp}(\widetilde{A}_3)) = \emptyset$, due to $A_3 = \emptyset$, and $\bar{C}_0 = (((\dot{C}_0 = \{b\}) \setminus (B_3 = \{b\})) \cup (A_3 = \emptyset)) \cap (\dot{E} = \{b\}) = \emptyset$. This implies that $\breve{\mathcal{E}}_5 = \bar{\mathcal{E}}_5$. Therefore, we obtain $\mathcal{E}_5 \setminus t = \mathcal{E}_5 \setminus t'$.

We leave it up to the reader to check the rest of the traces of $\mathcal{E}_5$. ◇

Thanks to Proposition 2, we can assert that for any trace t of the RPES $\mathcal{E}$, the residual $\mathcal{E} \setminus t$ of $\mathcal{E}$ after t under the removal operator $\setminus$ can be represented as $\mathcal{E} \setminus [t]$. Besides, it is routine to verify that the application of this removal operator to the RPES $\varphi(\mathcal{E}, \emptyset)$ (see Proposition 1(ii)), where $\mathcal{E}$ is a PES, produces the same result as the application of the removal operator from [36](2) to $\mathcal{E}$.

The following two statements demonstrate compositional properties of the residual operator for causal RPESs.

Proposition 3. *Given a causal RPES $\mathcal{E}$ with a trace $t \in Trace(\mathcal{E})$ and its residual $\mathcal{E}' = \mathcal{E} \setminus [t]$ with a trace $t' \in Trace(\mathcal{E}')$, it holds that $tt' \in Trace(\mathcal{E})$, and, moreover, $\mathcal{E} \setminus [tt'] = \mathcal{E}' \setminus [t']$.*

Proof. See Appendix.

So, it turned out that the concatenation of any trace t of the causal RPES $\mathcal{E}$ and any trace t' of the residual $\mathcal{E} \setminus [t]$ is a trace of $\mathcal{E}$, and, moreover, the residuals $\mathcal{E} \setminus [tt']$ and $\mathcal{E} \setminus [t] \setminus [t']$ coincide.

Example 7. First, consider the cause-respecting and non-causal RPES $\mathcal{E}_2 = (E_2,\ <_2,\ \sharp_2,\ l_2,\ F_2,\ \prec_2,\ \rhd_2,\ C_0^2)$ from Examples 2–3, where $E_2 = \{a, b\}$; $<_2 = \emptyset$; $\sharp_2 = \emptyset$; l_2 is the identical function; $F_2 = \{a\}$; $\prec_2 = \{(a, \underline{a})\}$; $\rhd_2 = \{(b, \underline{a})\}$; $C_0^2 = \emptyset$. As was demonstrated in Example 2, the sequences $t = (\{a\} \cup \emptyset)$, $t' = (\{a\} \cup \emptyset)(\{b\} \cup \emptyset)$ are traces of $\mathcal{E}_2$. Construct the residuals of $\mathcal{E}_2$ after t and t' under the removal operator $\setminus$ as follows:

– $\mathcal{E}_2 \setminus [t = (A_1 = \{a\} \cup \underline{B}_1 = \emptyset)] = (\tilde{E} = E_2,\ \tilde{<} = <_2,\ \tilde{\sharp} = \sharp_2,\ \tilde{l} = l_2,\ \tilde{F} = F_2,\ \tilde{\prec} = \prec_2,\ \tilde{\rhd} = \rhd_2,\ \tilde{C}_0 = \{a\})$, because $(\widetilde{A}_1 \cup \sharp_2(\widetilde{A}_1)) = \emptyset$, due to $a \in F_2$, and $\tilde{C}_0 = ((C_0^2 = \emptyset) \cup (A_1 = \{a\})) \cap (\tilde{E} = \{a, b\}) = \{a\}$;
– $\mathcal{E}_2 \setminus [t' = (A_1 = \{a\} \cup \underline{B}_1 = \emptyset)(A_2 = \{b\} \cup \underline{B}_2 = \emptyset)] = (\tilde{E}' = \{a\},\ \tilde{<}' = \emptyset,\ \tilde{\sharp}' = \emptyset;\ \tilde{l}' = l_2|_{\{a\}};\ \tilde{F}' = \{a\};\ \tilde{\prec}' = \{(a, \underline{a})\};\ \tilde{\rhd}' = \emptyset,\ \tilde{C}_0' = \{a\})$, because $\widetilde{A}_2 = \{b\}$, due to $b \in A_2 \setminus \tilde{F}$, and $\tilde{C}_0' = ((\tilde{C}_0 = \{a\}) \cup (A_2 = \{b\})) \cap (\tilde{E} = \{a\}) = \{a\}$.

(2) For the PES $\mathcal{E} = (E, <, \sharp, l, C_0)$ and its configuration $C \in Conf(\mathcal{E})$, the residual $\mathcal{E} \setminus C$ of $\mathcal{E}$ after C under the removal operator $\setminus$ is defined as follows: $\mathcal{E} \setminus C = (E', \leq \cap (E' \times E'), \sharp \cap (E' \times E'), l\,|_{E'})$, with $E' = E \setminus (C \cup \sharp(C))$, where $\sharp(C)$ denotes the events conflicting with the events in C.

It is easy to see that $t'' = (\emptyset \cup \{\underline{a}\})$ is a trace of $\mathcal{E}_2 \setminus [t']$, whereas the sequence $t'' = (\{a\} \cup \emptyset)(\{b\} \cup \emptyset)(\emptyset \cup \{\underline{a}\})$ is not a trace of $\mathcal{E}_2$.

Using Examples 2 and 5, it is not difficult to make sure that Proposition 3 holds for the causal RPESs $\mathcal{E}_4$ and $\mathcal{E}_5$. ◇

It is stated below that any suffix t' of any trace tt' of the causal RPES $\mathcal{E}$ is a trace of the residual $\mathcal{E} \setminus [t]$.

Proposition 4. *Given a causal RPES $\mathcal{E}$ with traces $t', t't'' \in Trace(\mathcal{E})$, $t'' \in Trace(\mathcal{E} \setminus [t'])$ holds.*

Proof. See Appendix.

Example 8. Examine the non-causal RPES $\mathcal{E}_1$ from Examples 1–4, with $E_1 = \{a, b\}$; $<_1 = (a, b)$; $\sharp_1 = \emptyset$; l_1 is the identical function; $F_1 = \{a\}$; $\prec_1 = \{(a, \underline{a})\}$; $\rhd_1 = \emptyset$; $C_0^1 = \emptyset$; and $Conf(\mathcal{E}_1) = \{\emptyset, \{a\}, \{a, b\}, \{b\}\}$. Check the trace $t = (\{a\} \cup \emptyset)(\{b\} \cup \emptyset)(\emptyset \cup \{\underline{a}\})(\{a\} \cup \emptyset)$ of $\mathcal{E}_1$, and its prefix $t' = (\{a\} \cup \emptyset)(\{b\} \cup \emptyset)$ and postfix $t'' = (\emptyset \cup \{\underline{a}\})(\{a\} \cup \emptyset)$. Using Definition 5, we obtain the RPES $\mathcal{E}_1 \setminus [t'] = (E'_1 = \emptyset, <'_1 = \emptyset, \sharp'_1 = \emptyset, l'_1 = \emptyset, F'_1 = \emptyset, \prec'_1 = \emptyset, \rhd'_1 = \emptyset, C'^1_0 = \emptyset)$. It is clear that $Trace(\mathcal{E}_1 \setminus [t']) = \emptyset$. Therefore, we get $t'' \notin Trace(\mathcal{E}_1 \setminus [t'])$.

Using Examples 2 and 5, it is not difficult to check that Proposition 4 holds for the causal RPESs $\mathcal{E}_4$ and $\mathcal{E}_5$. ◇

4 Transition Systems $TC(\mathcal{E})$ and $TR(\mathcal{E})$ from the causal RPES $\mathcal{E}$

In this section, we first give some basic definitions concerning labeled transition systems. Then, we define the mappings $TC(\mathcal{E})$ and $TR(\mathcal{E})$, which associate two distinct kinds of transition systems – one whose states are configurations and one whose states are residuals – with the RPES $\mathcal{E}$ labeled over the set L of actions.

A transition system $T = (S, \to, i)$ labeled over a set $\mathcal{L}$ of labels consists of a set of states S, a transition relation $\to \subseteq S \times \mathcal{L} \times S$, and an initial state $i \in S$. We call a relation $R \subseteq S \times S'$ a *bisimulation* between transition systems $T = (S, \to, i)$ and $T' = (S', \to', i')$ over $\mathcal{L}$ iff $(i, i') \in R$, and for all $(s, s') \in R$ and $l \in \mathcal{L}$: if $(s, l, s_1) \in \to$, then $(s', l, s'_1) \in \to'$ and $(s_1, s'_1) \in R$, for some $s'_1 \in S'$; and if $(s', l, s'_1) \in \to'$, then $(s, l, s_1) \in \to$ and $(s_1, s'_1) \in R$, for some $s_1 \in S$.

For a fixed set L of actions in RPESs, define the set $\mathbb{L} := \mathbb{N}_0^L$ (the set of multisets over L, or functions from L to the non-negative integers). The set $\mathbb{L}$ will be used as the set of labels in transition systems.

We are ready to define transition systems (labeled over $\mathbb{L}$) with configurations as states.

Definition 6. *For an RPES $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$ over L,*

$$TC(\mathcal{E}) \textit{ is a transition system } (Conf(\mathcal{E}), \rightharpoondown, C_0) \textit{ over } \mathbb{L},$$

where $C \overset{M}{\rightharpoondown} C'$ iff $C \overset{(A \cup \underline{B})}{\rightarrow} C'$ in $\mathcal{E}$ and $M = l(A \cup \underline{B})$.

Let us explain the above definition with

Example 9. First, consider the causal RPES $\mathcal{E}_4$ from Examples 2–7. We know from Example 2 that the configurations of $\mathcal{E}_4$ are $\emptyset$, $\{a\}$, $\{a, b\}$, $\{c\}$, and transitions between them exist. Using Definition 6, we obtain the configuration transition system $TC(\mathcal{E}_4)$ depicted in Fig. 1.

Second, contemplate the causal RPES $\mathcal{E}_5$ from Examples 2–7. In Example 2, we can see that $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$ are the configurations of $\mathcal{E}_5$, and we are given some explanations concerning possible transitions between the configurations. Using Definition 6, we construct the configuration transition system $TC(\mathcal{E}_5)$ shown in Fig. 2. ◇

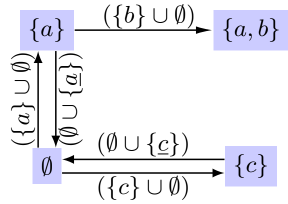

Figure 1. The configuration transition system $TC(\mathcal{E}_4)$

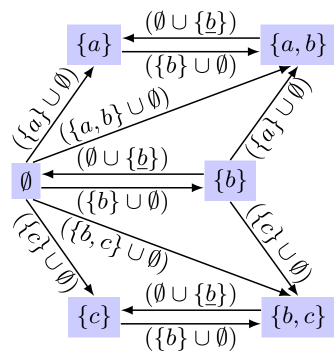

Figure 2. The configuration transition system $TC(\mathcal{E}_5)$

Definitions 3, 6 and Proposition 1(ii) lead to the following auxiliary

Lemma 4. *Given a PES* $\mathcal{E} = (E, <, \sharp, l, \emptyset)$ *over* L, $TC(\mathcal{E}) = TC(\varphi(\mathcal{E}, \emptyset) = (E, <, \sharp, l, \emptyset, \emptyset, \emptyset, \emptyset))$.

We next propose the definition of transition systems (labeled over $\mathbb{L}$) with RPESs as states.

Definition 7. *For an RPES* $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$ *over* L,

$$TR(\mathcal{E}) \text{ is a transition system } (Reach(\mathcal{E}), \rightharpoonup, \mathcal{E}) \text{ over } \mathbb{L},$$

where $\mathcal{F} \overset{M}{\rightharpoonup} \mathcal{F}'$ *iff* $\mathcal{F}' = \mathcal{F} \setminus [(A \cup \underline{B})]$ *and* $M = l(A \cup \underline{B})$, *and* $Reach(\mathcal{E}) = \{\mathcal{F} \mid \exists \mathcal{E}_0, \ldots, \mathcal{E}_k$ $(k \geq 0)$ *s.t.* $\mathcal{E}_0 = \mathcal{E} \setminus [\epsilon]$, $\mathcal{E}_k = \mathcal{F}$, *and* $\mathcal{E}_i \overset{l(A \cup \underline{B})}{\rightharpoonup} \mathcal{E}_{i+1}$ $(0 \leq i < k)\}$.

We illustrate the above definition with

Example 10. Consider the RPES $\mathcal{E}_4$ from from Examples 2–7. Using Definitions 5, 7 and Propositions 3, 4, we construct the residual transition system $TR(\mathcal{E}_4)$ of $\mathcal{E}_4$, depicted in Fig. 3. Obviously, the configuration transition system $TC(\mathcal{E}_4)$ (see Fig. 1) and the residual transition system $TR(\mathcal{E}_4)$ are isomorphic.

Contemplate the RPES $\mathcal{E}_5$ from from Examples 2–7. Before constructing the residual transition system $TR(\mathcal{E}_5)$, we first need to find out, using Definition 5 and Propositions 3, 4, the following:

- $\mathcal{E}_5 = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*$;
- $\mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)(\{a\} \cup \emptyset) = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)(\{c\} \cup \emptyset) = \widetilde{\mathcal{E}}_5 = \widehat{\mathcal{E}}_5 = \overset{\frown}{\mathcal{E}}_5$, where
  $\widetilde{\widetilde{\mathcal{E}}}_5 = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{a\} \cup \emptyset)((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset) = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{c\} \cup \emptyset)((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)$,
  $\widehat{\widetilde{\mathcal{E}}}_5 = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)(\{a\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset) = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)(\{c\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)$,

$\widehat{\mathcal{E}}_5 = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{a,b\} \cup \emptyset)((\emptyset \cup \{\underline{b}\})(\{b\} \cup \emptyset))^* = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b,c\} \cup \emptyset)((\emptyset \cup \{\underline{b}\})(\{b\} \cup \emptyset))^*$;

– $\mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{a\} \cup \emptyset)((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^* = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{c\} \cup \emptyset)((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^* = \overline{\mathcal{E}}_5 = \overline{\overline{\mathcal{E}}}_5$, where
$\overline{\mathcal{E}}_5 = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)(\{a\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^* = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b\} \cup \emptyset)(\{c\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*$,
$\overline{\overline{\mathcal{E}}}_5 = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{a,b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^* = \mathcal{E}_5 \setminus ((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*(\{b,c\} \cup \emptyset)(\emptyset \cup \{\underline{b}\})((\{b\} \cup \emptyset)(\emptyset \cup \{\underline{b}\}))^*$.

With Definition 7, we obtain the residual transition system $TR(\mathcal{E}_5)$ shown in Fig. 4, where $x \in \{a, c\}$. It is not difficult to verify that the configuration transition system $TC(\mathcal{E}_5)$ (see Fig. 2) and the residual transition system $TR(\mathcal{E}_5)$ are bisimilar and not isomorphic. ◇

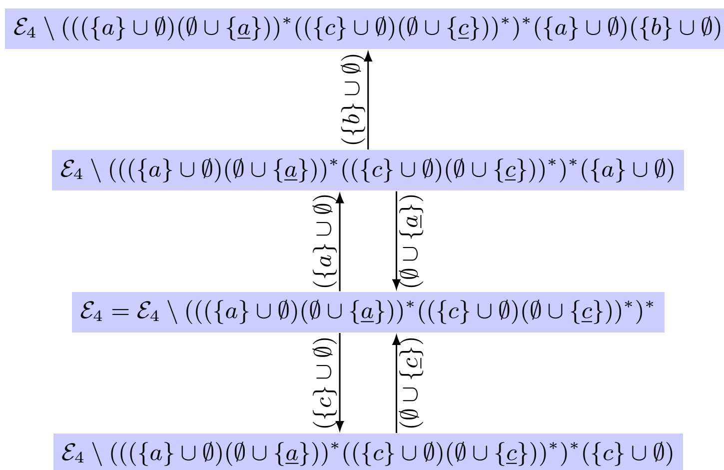

Figure 3. The residual transition system $TR(\mathcal{E}_4)$

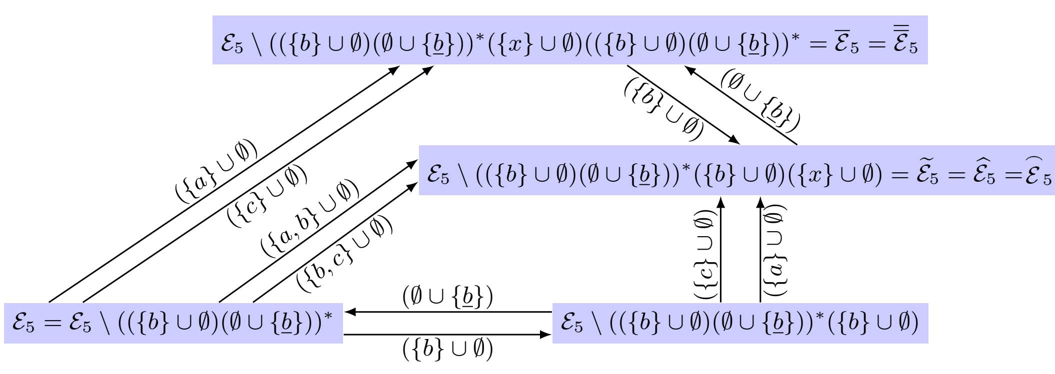

Figure 4. The residual transition system $TR(\mathcal{E}_5)$

We establish the relationships between the states and transitions of the configuration-based and residual-based transition systems of the RPES.

Proposition 5. *Given a causal RPES* $\mathcal{E} = (E, <, \sharp, l, F, \prec, \rhd, C_0)$ *over* L,

(i) *for any* $last(t) \in Conf(\mathcal{E})$*,* $\mathcal{E} \setminus [t] \in Reach(\mathcal{E})$*;*
(ii) *for any* $\mathcal{E}' \in Reach(\mathcal{E})$*, there is* $last(t) \in Conf(\mathcal{E})$ *such that* $\mathcal{E}' = \mathcal{E} \setminus [t]$*;*
(iii) *for any* $last(t), last(t') \in Conf(\mathcal{E})$*, if* $last(t) \overset{l(A \cup \underline{B})}{\rightharpoondown} last(t')$*, then* $\mathcal{E} \setminus [t] \overset{l(A \cup \underline{B})}{\rightharpoondown} \mathcal{E} \setminus [t']$*;*
(iv) *for any* $\mathcal{E}', \mathcal{E}'' \in Reach(\mathcal{E})$*, if* $\mathcal{E}' \overset{l(A \cup \underline{B})}{\rightharpoondown} \mathcal{E}''$*, then for any* $last(t) \in Conf(\mathcal{E})$ *such that* $\mathcal{E}' = \mathcal{E} \setminus [t]$*, there is* $last(t') \in Conf(\mathcal{E})$ *such that* $\mathcal{E}'' = \mathcal{E} \setminus [t']$*, and* $last(t) \overset{l(A \cup \underline{B})}{\rightharpoondown} last(t')$*.*

Proof. See Appendix.

Theorem 1. *Given a causal RPES* $\mathcal{E}$ *over* L*,* $TC(\mathcal{E})$ *and* $TR(\mathcal{E})$ *are bisimilar and in general not isomorphic.*

Proof. From Example 10 we know that, for the causal RPES $\mathcal{E}_5$, $TC(\mathcal{E}_5)$ and $TR(\mathcal{E}_5)$ are not isomorphic.

We shall check that $TC(\mathcal{E})$ and $TR(\mathcal{E})$ are bisimilar for an arbitrary causal RPES $\mathcal{E} = (E, <, \sharp, L, l, F, \prec, \rhd, C_0)$. Due to Lemma 1(i) and Propositions 5(i), (ii), we can define a relation $R \subseteq Conf(\mathcal{E}) \times Reach(\mathcal{E})$ as follows: $R = \{(last(t), \mathcal{E}') \mid last(t) \in Conf(\mathcal{E})$ and $\mathcal{E}' = \mathcal{E} \setminus [t] \in Reach(\mathcal{E})\}$.

We need to show that R is a bisimulation between $TC(\mathcal{E})$ and $TR(\mathcal{E})$. Clearly, we have that $C_0 = last(\epsilon) \in Conf(\mathcal{E})$ and $\mathcal{E} = \mathcal{E} \setminus [\epsilon] \in Reach(\mathcal{E})$. So, $(C_0, \mathcal{E}) \in R$ holds. Take an arbitrary $(last(t), \mathcal{E} \setminus [t]) \in R$. Suppose that $last(t) \overset{l(A \cup \underline{B})}{\rightharpoondown} C'$ in $TC(\mathcal{E})$ for some $C' \in Conf(\mathcal{E})$. By Lemma 1(i), there is $t' \in Trace(\mathcal{E})$ such that $C' = last(t')$. According to Proposition 5(iii), $\mathcal{E} \setminus [t] \overset{l(A \cup \underline{B})}{\rightharpoondown} \mathcal{E} \setminus [t']$ is true. Moreover, $(last(t'), \mathcal{E} \setminus [t']) \in R$ holds. In the opposite direction, assume that $\mathcal{E} \setminus [t] \overset{l(A \cup \underline{B})}{\rightharpoondown} \mathcal{E}'$ in $TR(\mathcal{E})$ for some $\mathcal{E}' \in Reach(\mathcal{E})$. Due to Propositions 5(iv), there is $last(t') \in Conf(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus [t']$ and $last(t) \overset{l(A \cup \underline{B})}{\rightharpoondown} last(t')$. This implies that $(last(t'), \mathcal{E}') \in R$ holds. Hence, R is indeed a bisimulation. □

Thanks to Proposition 1(ii), Lemma 4, and Theorem 1, we obtain the following

Corollary 1. *Given a PES* $\mathcal{E} = (E, <, \sharp, l, \emptyset)$ *over* L*,* $TC(\mathcal{E})$ *and* $TR(\varphi(\mathcal{E}, \emptyset) = (E, <, \sharp, l, \emptyset, \emptyset, \emptyset, \emptyset))$ *are bisimilar.*

5 Concluding Remarks

In this paper, we dealt with two different – configuration-based and residual-based – ways of giving (step) transition system semantics for causal reversible prime event structures which encompass prime event structures. For this purpose, we firstly defined (step) semantics from [48], which is based on configurations (traces) obtained by starting with the initial configuration and by executing events and/or undoing previously executed events, and, secondly, developed a removal operator which is useful for constructing residuals (model fragments) by retaining an appropriate amount of structure during the execution of the models. We also stated some correctness criteria for the removal operator. The meaning of the correctness properties is that the obtained residuals do not allow configurations (traces) that are disallowed by original structures. Also, in some sense, this signifies some compositionality properties of the removal operator. It has turned out that, in the context of PESs, the removal operator developed here produces the same residuals as the removal operator proposed in [36]. As our main result, we have obtained a (step) bisimulation between configuration-based and residual-based transition systems of the models. It is hoped that, in providing operational semantics of process algebraic calculi, the results obtained here may be as helpful for RPESs as for traditional (non-reversible) event structures (see among others [12,23]).

As for future work, we plan to broaden the list of studied models with flow, bundle, general event structures with symmetric and asymmetric conflict. Work on extending our approach to cause-respecting and out-of-causal reversible prime event structures is under way and has yielded promising intermediate results. Another future line of research is to generalize the model of reversible prime event structures with non-executable events (for example, by dropping the transitivity/acyclicity of causality, as well as the principles of finite causes) in order to obtain isomorphisms between the corresponding transition systems of the models as was done for PESs in the paper [9]. There, the authors have been able to argue that non-executable events are useful in comparative semantics, facilitating the elimination of non-fundamental inconsistencies between models. Furthermore, isomorphisms between two different types of transition systems are expected to allow one to relate those constructed on configurations and those derived from denotational semantics of process calculi in a tight way.

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Appendix

Proof of Lemma 2.

Let $\mathcal{E} = (E, <, \sharp, F, \prec, \triangleright, C_0)$ be a causal-respecting RPES, and $C \in \mathit{Conf}(\mathcal{E})$. Then, C_0 is left-closed under $<$ and conflict-free, due to Definition 2, and $CF(C)$ holds, due to Definition 3. The truth of item (i) follows from Proposition 3.38(1) [48]. The truth of item (ii) follows from Proposition 3.40(2) [48]. $\square$

Proof of Lemma 3.

Let $\mathcal{E} = (E, <, \sharp, F, \prec, \triangleright, C_0)$ be an RPES, and $C_0 \stackrel{A_1 \cup \underline{B_1}}{\rightarrow} C_1 \ldots C_{n-1} \stackrel{A_n \cup \underline{B_n}}{\rightarrow} C_n$ $(n \geq 0)$ in $\mathcal{E}$.

(i) Follows from Definition 5.

(ii) We have to show that $\mathcal{E} \setminus t_i$ is a causal RPES for all $0 \leq i \leq n$. We shall proceed by induction i.

$i = 0$. By Definition 5, we have $\mathcal{E} \setminus (t_0 = \epsilon) = \mathcal{E}$. So, $\mathcal{E} \setminus t_0$ is a causal RPES.

$i > 0$. Take arbitrary events $e \in E^i$ and $u \in F^i$ such that $e \prec^i \underline{u}$ or $e =^i u$ ($e \triangleright^i \underline{u}$ or $u <^i e$). Thanks to item (i), it is true that $e \in E^{i-1}$, $u \in F^{i-1}$, and, moreover, $e \prec^{i-1} \underline{u}$ iff $e =^{i-1} u$ ($e \triangleright^{i-1} \underline{u}$ iff $u <^{i-1} e$) because $\mathcal{E}_{i-1}$ is a causal RPES due to the induction hypothesis. As $e \in E^i$ and $u \in F^i \subseteq E^i$, we get $e \prec^i \underline{u}$ iff $e =^i u$ ($e \triangleright^i \underline{u}$ iff $u <^i e$), by Definition 5.

Therefore, $\mathcal{E}^i$ is a causal RPES for all $0 \leq i \leq n$.

(iii) Take an arbitrary $e \in \widetilde{A}_i$ for some $1 \leq i \leq n$. Two cases are admissible.

- $e \in A_i \setminus F^{i-1}$. As $(A_i \cup \underline{B_i})$ is enabled at C_{i-1}, we have that $A_i \cap C_{i-1} = \emptyset$, $B_i \subseteq C_{i-1}$, and, hence, $e \notin B_i$. So, $e \in C_i$ holds.
- $e \in F^{i-1}$ and $\exists a \in A_i \setminus F^{i-1} \colon e <^{i-1} a$. Thanks to item (i), it is true that $e < a$. As $(A_i \cup \underline{B_i})$ is enabled at C_{i-1} and $e < a$, we have that $e \in C_{i-1} \setminus B_i$, and, hence, $e \in C_i$.

(iv) We have to show that $B_i \subseteq F^{i-1}$ for all $1 \leq i \leq n$. We shall proceed by induction on i.

$i = 1$ We get $B_1 \subseteq F^0$ because $(A_1 \cup \underline{B_1})$ is enabled at C_0.

$i > 1$ By the induction hypothesis, it is true that $B_{i'} \subseteq F^{i'-1} \subseteq E^{i'-1}$ for all $1 \leq i' < i$. Suppose a contrary, i.e. $B_i \not\subseteq F^{i-1}$. This means that there is $x \in B_i \subseteq F^0 \subseteq E^0$ such that $x \notin F^{i-1}$. As $x \in F^0$ and $x \notin F^{i-1}$, there is the minimal $1 \leq k < i$ such that $x \in F^{k-1}$ and $x \notin E^k$, due to Definition 5. This implies that $x \in \widetilde{A}_k$ or $x \in \sharp^{k-1}(\widetilde{A}_k)$.

Assume $x \in \widetilde{A}_k$. Two cases are admissible.

- $x \in A_k \setminus F^{k-1}$. This contradicts $x \in F^{k-1}$.
- $x \in F^{k-1}$ and $\exists a \in A_k \setminus F^{k-1} \colon x <^{k-1} a$. By item (iii), we get $a \in C_k$. Thanks to item (i), it is true that $x < a$, and, hence, $a \triangleright \underline{x}$ because $\mathcal{E}$ is a causal RPES. Moreover, since $(A_i \cup \underline{B_i})$ is enabled at C_{i-1}, $x \in B_i$, and $a \triangleright \underline{x}$, $a \notin C_{i-1} \cup A_i$ holds. This implies $a \in B_l$ for some $k < l < i$. Due to $B_{i'} \subseteq F^{i'-1}$ for all $1 \leq i' < i$, we have $B_l \subseteq F^{l-1}$. By virtue of item (i), we get $a \in F^{k-1}$, contradicting $a \in A_k \setminus F^{k-1}$.

Suppose $x \in \sharp^{k-1}(\widetilde{A}_k)$. Then, there is an event $e \in \widetilde{A}_k$ such that $x \ \sharp^{k-1} e$. By item (i), we get $x \ \sharp \ e$. Thanks to $(A_i \cup \underline{B_i})$ being enabled at C_{i-1}, we have that $B_i \subseteq C_{i-1}$ and $CF(C_{i-1} \cup A_i)$. Then, $e \notin C_{i-1}$ holds, because $x \in C_{i-1}$ and $x \ \sharp \ e$. Two cases are possible.

- $e \in A_k \setminus F^{k-1}$. By item (iii), we have $e \in C_k$. Since $e \notin C_{i-1}$ and $k < i$, $e \in B_l$ holds for some $k < l < i$. Due to $B_{i'} \subseteq F^{i'-1}$ for all $1 \leq i' < i$, $B_l \subseteq F^{l-1}$ is true. Thanks to item (i), we get $e \in F^{k-1}$, contradicting $e \in A_k \setminus F^{k-1}$.
- $e \in F^{k-1}$ and $\exists a \in A_k \setminus F^{k-1} \colon e <^{k-1} a$. By item (iii), we get $e, a \in C_k$. Since $e \notin C_{i-1}$ and $k < i$, we get $e \in B_l$ for some $k < l < i$. Thanks to item (i), it is true that $e < a$, and, hence, $a \triangleright \underline{e}$ because $\mathcal{E}$ is a causal RPES. Due to $(A_l \cup \underline{B_l})$ being enabled at C_{l-1}, $e \in B_l$, and $a \triangleright \underline{e}$, we have $a \notin C_{l-1} \cup A_l$. As $a \in C_k$ and $k < l$, $a \in B_m$ holds for some $k < m < l$. Due to $B_{i'} \subseteq F^{i'-1}$ for all $1 \leq i' < i$, we have $B_m \subseteq F^{m-1}$. Thanks to item (i), we get $a \in F^{k-1}$, contradicting $a \in A_k \setminus F^{k-1}$.

(v) We have to check that $A_i \subseteq E^{i-1}$ for all $1 \le i \le n$. If $i = 1$, we get $A_1 \subseteq E^0$ because $(A_1 \cup \underline{B}_1)$ is enabled at C_0. Consider the case with $i > 1$.

Claim. If $e \notin C_{i-1}$, then $e \notin \widetilde{A}_k$ for all $1 \le k < i$.

Proof. Suppose a contrary, i.e. $e \notin C_{i-1}$ and $e \in \widetilde{A}_k$ for some $1 \le k < i$. Two cases are admissible.

- $e \in A_k \setminus F^{k-1}$. Due to item (iii), we have $e \in C_k$. Since $e \notin C_{i-1}$ and $k < i$, there is $k < l < i$ such that $e \in B_l$. By item (iv), we obtain $B_l \subseteq F^{l-1}$. Thanks to item (i), we get $e \in F^{k-1}$, contradicting $e \in A_k \setminus F^{k-1}$.
- $e \in F^{k-1}$ and $\exists a \in A_k \setminus F^{k-1}$: $e <^{k-1} a$. By item (iii), we get $e, a \in C_k$. Due to $e \notin C_{i-1}$ and $k < i$, we get $e \in B_l$ for some $k < l < i$. Thanks to item (i), it is true that $e < a$, and, moreover, $a \rhd \underline{e}$ because $\mathcal{E}$ is a causal RPES. Since $(A_l \cup \underline{B_l})$ is enabled at C_{l-1}, $e \in B_l$, and $a \rhd \underline{e}$, we have $a \notin C_{l-1} \cup A_l$. This implies $a \in B_m$ for some $k < m < l$. By item (iv), we obtain $B_m \subseteq F^{m-1}$. Thanks to item (i), we get $a \in F^{k-1}$, contradicting $a \in A_k \setminus F^{k-1}$. □

Suppose a contrary, i.e. $A_i \not\subseteq E^{i-1}$. This means that there is $x \in A_i$ such that $x \notin E^{i-1}$. As $(A_i \cup \underline{B_i})$ is enabled at C_{i-1}, it holds that $A_i \subseteq E^0$ and $A_i \cap C_{i-1} = \emptyset$. Therefore, it is true that $x \in E^0$ and $x \notin C_{i-1}$. Since $x \in E^0$ and $x \notin E^{i-1}$, there is $1 \le k < i$ such that $x \in \widetilde{A}_k$ or $x \in \sharp^{k-1}(\widetilde{A}_k)$, by Definition 5. The case with $x \in \widetilde{A}_k$ contradicts Claim. Suppose $x \in \sharp^{k-1}(\widetilde{A}_k)$. Then, there is an event $e \in \widetilde{A}_k$ such that $x \ \sharp^{k-1} e$. By item (i), we get $x \ \sharp \ e$. Since $(A_i \cup \underline{B_i})$ is enabled at C_{i-1}, $CF(C_{i-1} \cup A_i)$ holds. Hence, we get $e \notin C_{i-1}$ because $x \in A_i$ and $x \ \sharp \ e$. We get a contradiction with Claim.
Thus, $A_i \subseteq E^{i-1}$ for all $1 \le i \le n$.

(vi) For $i = 1$, the result is obvious. Consider the case with $i > 1$. Suppose a contrary, i.e. $e \in A_i \setminus F^{i-1}$ for some $1 < i \le n$, and $e \in F^0 \subseteq E^0$. Due to item (v), we get $e \in E^{i-1}$. Then, $e \in E^j$ is true for all $1 \le j < i$, by item (i). From Definition 5, we know that $F^j = F^{j-1} \cap E^j$ for all $1 \le j < i$. As $e \in F^0$ is given, we get $e \in F^{i-1}$, contradicting $e \in A_i \setminus F^{i-1}$.

(vii) Assume $e \in A^i \setminus F^{i-1}$ for some $1 \le i \le n$. Then, we get that $e \in C_i$, by item (iii), and $e \notin F$, by item (vi). Hence, we have $e \notin B_k \subseteq F$ for all $i < k \le n$. This implies that $e \in C_k$ for all $i < k \le n$, i.e. $e \in C_n$.

(viii) Follows from Definition 5 and item (i). □

Proof of Proposition 2.

As $[t] = [t']$ is true, $C_n = C'_m$ holds.

First, check $E^n = E^m$. As t (t') is a trace of $\mathcal{E}$, there exists a sequence $C_0 \overset{A_1 \cup \underline{B_1}}{\to} C_1 \ldots C_{n-1} \overset{A_n \cup \underline{B_n}}{\to} C_n$ $(n \ge 0)$ $(C_0 \overset{X_1 \cup \underline{Y_1}}{\to} C'_1 \ldots C'_{m-1} \overset{X_m \cup \underline{Y_m}}{\to} C'_m$ $(m \ge 0))$ in $\mathcal{E}$.

Claim. Given $1 \le j \le m$ and $\widetilde{x} \in \widetilde{X}^j = (X_j \setminus F^{j-1}) \ \cup \ \lfloor (X_j \setminus F^{j-1}) \rfloor \cap F^{j-1}$, it holds:

(i) $\widetilde{x} \in A_i \setminus F^{i-1}$ for some $1 \le i \le n$, if $\widetilde{x} \in X_j \setminus F^{j-1}$;
(ii) $\widetilde{x} \in \lfloor (A_i \setminus F^{i-1}) \rfloor \cap F^{i-1}$ for some $1 \le i \le n$, if $\widetilde{x} \in \lfloor (X_j \setminus F^{j-1}) \rfloor \cap F^{j-1}$.

Proof.

(i) Assume $\widetilde{x} \in X_j \setminus F^{j-1}$. Then, we get $\widetilde{x} \notin F$, according to Lemma 3(vi). Moreover, we have $\widetilde{x} \in C'_m = C_n$, by Lemma 3(vii). This implies that $\widetilde{x} \in A_i$ for some $1 \le i \le n$. As $\widetilde{x} \notin F$, $\widetilde{x} \in A_i \setminus F^{i-1}$ holds, according to Lemma 3(i).

(ii) Suppose $\widetilde{x} \in \lfloor (X_j \setminus F^{j-1}) \rfloor \cap F^{j-1}$. This means that $\widetilde{x} \in F^{j-1}$, and $\widetilde{x} <^{j-1} x$ for some $x \in X_j \setminus F^{j-1}$. Then, we have $\widetilde{x} \in F$, and $\widetilde{x} < x$, by Lemma 3(i). Moreover, we get that $x \in A_i \setminus F^{i-1}$ for some $1 \le i \le n$, due to item (i). Hence, this implies that $x \in E^{i-1}$ due to Lemma 3(v), and $x \in C'_m = C_n$, due to Lemma 3(vii). By virtue of Lemma 2(i), $\widetilde{x} \in C_n$ holds, because $\widetilde{x} < x$.

We next show $\widetilde{x} \in E^{i-1}$. Suppose a contrary, i.e. $\widetilde{x} \notin E^{i-1}$. Since $\widetilde{x} \in E$ and $\widetilde{x} \notin E^{i-1}$, we can find $1 \le k \le i-1$ such that $\widetilde{x} \in \widetilde{A}_k$ or $\widetilde{x} \in \sharp^{k-1}(\widetilde{A}_k)$, by Definition 5.

Assume $\widetilde{x} \in A_k \setminus F^{k-1}$. Due to Lemma 3(vi), we get the contradiction $\widetilde{x} \notin F$.

Suppose $\widetilde{x} \in \sharp^{k-1}(\widetilde{A}_k)$, i.e. $\widetilde{x} \in E^{k-1}$, and $\widetilde{x} \ \sharp^{k-1} \ \widetilde{y}$ for some $\widetilde{y} \in \widetilde{A}_k$. By Lemma 3(i), $\widetilde{x} \ \sharp \ \widetilde{y}$ holds. Due to Definition 5, we get $\widetilde{y} \in (A_k \setminus F^{k-1}) \ \cup \ (\lfloor (A_k \setminus F^{k-1}) \rfloor \cap F^{k-1})$.

- $\widetilde{y} \in A_k \setminus F^{k-1}$. Then, thanks to Lemma 3(vii), we get $\widetilde{y} \in C_n$, contradicting $\widetilde{x} \in C_n$.
- $\widetilde{y} \in F^{k-1}$ and $\exists y \in A_k \setminus F^{k-1}$: $\widetilde{y} <^{k-1} y$. Then, we have $\widetilde{y} < y$, by Lemma 3(i). Moreover, we get $y \in C_n$, due to Lemma 3(vii). By virtue of Lemma 2(i), $\widetilde{y} \in C_n$ is true, contradicting $\widetilde{x} \in C_n$.

So, we obtain $\widetilde{x} \in E^{i-1}$. Then, we have $\widetilde{x} \in F^{i-1} \subseteq E^{i-1}$. Moreover, it is true that $x, \widetilde{x} \in E^k$ for all $0 \le k \le i-1$, by Lemma 3(i). According to Definition 5, we get $\widetilde{x} <^{i-1} x$ because $\widetilde{x} <^0 x$. As $x \in A_i \setminus F^{i-1}$ for some $1 \le i \le n$, $\widetilde{x} \in \lfloor (A_i \setminus F^{i-1}) \rfloor \cap F^{i-1}$ holds for some $1 \le i \le n$.

Thus, $\widetilde{x} \in (A_i \setminus F^{i-1}) \ \cup \ (\lfloor (A_i \setminus F^{i-1}) \rfloor \cap F^{i-1})$ for some $1 \le i \le n$. □

Suppose a contrary, i.e. $E^n \ne E^m$. Consider the proof of the case when $e \in E^n$ and $e \notin E^m$ (the proof of the case when $e \in E^m$ and $e \notin E^n$ is symmetric). Since $e \in E^n$, we have $e \in E^i$, for all $0 \le i \le n$, by Lemma 3(i). As $e \in E$ and $e \notin E^m$, we can find $1 \le j \le m$ such that $e \in (\widetilde{X}^j \ \cup \ \sharp^{j-1}(\widetilde{X}^j))$, due to Definition 5.

- $e \in \widetilde{X}^j$. Thanks to Claim, there is $1 \le i \le n$ such that $e \in \widetilde{A}_i$, contradicting $e \in E^n$.
- $e \in \sharp^{j-1}(\widetilde{X}^j)$. Then, $e \in E^{j-1}$, and $e \ \sharp^{j-1} \ x$ for some $x \in \widetilde{X}^j$. By Lemma 3(i), we get $e \ \sharp \ x$. According to Claim, there is $1 \le i \le n$ such that $x \in \widetilde{A}_i$. Then, we get $x \in E^{i-1}$ due to Lemma 3(ii), if $x \in A_i \setminus F^{i-1}$, and $x \in F^{i-1} \subseteq E^{i-1}$ due to Definition 5, otherwise. By virtue of Lemma 3(i), we get that $e \in E^{\widetilde{i}}$ for all $0 \le \widetilde{i} \le n$, and $x \in E^{\widetilde{i}'}$ for all $0 \le \widetilde{i}' \le i-1$. Since $e \ \sharp \ x$, $e \ \sharp^{i-1} \ x$ holds, by Definition 5. Therefore, we obtain $e \in \sharp^{i-1}(\widetilde{A}_i)$, contradicting $e \in E^n$.

Thus, $E^n = E^m$.

From now on, we suppose that $k \in \{m, n\}$.

Let $\nabla \in \{<, \sharp\}$ and $\widetilde{\nabla} \in \{\prec, \rhd\}$. By Definition 5, it holds that $\nabla^k = \nabla^0 \cap (E^1 \times E^1) \cap \ldots \cap (E^k \times E^k)$, $l^k = l^0|_{E^1 \cap \ldots \cap E^k}$, $F^k = F^0 \cap E^1 \cap \ldots \cap E^k$, and $\widetilde{\nabla}^k = \nabla^0 \cap (E^1 \times \underline{F}^1) \cap \ldots \cap (E^k \times \underline{F}^k)$. Due to Lemma 3(i), it is true that $\nabla^k = \nabla^0 \ \cap \ (E^k \times E^k)$, $l^k = l^0|_{E^k}$, $F^k = F^0 \ \cap \ E^k$, and $\widetilde{\nabla}^k = \nabla^0 \ \cap \ (E^k \times \underline{F}^k)$. Since $E^n = E^m$, we obtain that $\nabla^n = \nabla^m$, $l^n = l^m$, $F^n = F^m$, and $\widetilde{\nabla}^n = \widetilde{\nabla}^m$.

According to Lemma 3(viii), we have $C_0^n = C_n \cap E^n$ and $C_0'^m = C'_m \cap E^m$. Since $C_n = C'_m$ and $E^n = E^m$, we have $C_0^n = C_0'^m$. □

Proof of Proposition 3.

Let $t \in Trace(\mathcal{E})$ and $t' \in Trace(\mathcal{E}' = \mathcal{E} \setminus [t])$. First, verify the validity of $tt' \in Trace(\mathcal{E})$. As $t \in Trace(\mathcal{E})$, there exists a sequence $C_0 \stackrel{A_1 \cup B_1}{\rightarrow} C_1 \ \ldots \ C_{n-1} \stackrel{A_n \cup B_n}{\rightarrow} C_n$ ($n \ge 0$) in $\mathcal{E}$. If $n = 0$, then we have $\mathcal{E} = \mathcal{E} \setminus \epsilon$, due to Definition 5. This implies that $t' \in Trace(\mathcal{E})$. Consider the case when $n > 0$. Since $t' \in Trace(\mathcal{E}')$, there exists a sequence $C'_0 \stackrel{A'_1 \cup B'_1}{\rightarrow} C'_1 \ \ldots \ C'_{m-1} \stackrel{A'_m \cup B'_m}{\rightarrow} C'_m$ ($m \ge 0$) in $\mathcal{E}'$. We shall proceed by inductions on m.

$m = 0$. Clearly, $t\epsilon \in Trace(\mathcal{E})$.

$m > 0$. By the induction hypothesis, we have $tt'_{m-1} \in Trace(\mathcal{E})$. Then, there exists a sequence $C_0 \stackrel{A_1 \cup B_1}{\rightarrow} C_1 \ \ldots \ C_{n-1} \stackrel{A_n \cup B_n}{\rightarrow} C_n \stackrel{A_{n+1} = A'_1 \cup B_{n+1} = B'_1}{\rightarrow} C_{n+1} \ \ldots \ C_{n+m-2} \stackrel{A_{n+m-1} = A'_{m-1} \cup B_{n+m-1} = B'_{m-1}}{\rightarrow} C_{n+m-1}$ in $\mathcal{E}$. Here, $C_{n+m-1} = (\ldots (C_n = [(\ldots (C_0 \setminus B_1 \cup A_1) \setminus \ldots) \setminus B_n \cup A_n] \setminus B'_1 \cup A'_1) \setminus \ldots) \setminus B'_{m-1} \cup A'_{m-1}$. Moreover, we get

$C'_{m-1} = (\ldots(C'_0 \setminus B'_1 \cup A'_1)\setminus \ldots)\setminus B'_{m-1} \cup A'_{m-1}$, where $C'_0 = C_n \cap E^n$, by Lemma 3(viii). So, $C'_{m-1} \subseteq C_{n+m-1}$ holds.

Check the validity of $tt'_m \in Trace(\mathcal{E})$. First we need to show that $(A'_m \cup \underline{B'_m})$ is enabled at C_{n+m-1} in $\mathcal{E}$.

- As $(A'_m \cup \underline{B'_m})$ is enabled at C'_{m-1}, it holds that $A'_m \subseteq E^n$ and $B'_m \subseteq F^n$ such that $A'_m \cap C'_{m-1} = \emptyset$, $B'_m \subseteq C'_{m-1}$, $CF(C'_{m-1} \cup A'_m)$. Due to Lemma 3(i), it is true that $A'_m \subseteq E^0$ and $B'_m \subseteq F^0$. As $C'_0 = C_n \cap E^n$, by Lemma 3(viii), and $A'_m \subseteq E^n$, it holds that $C_n \cap A'_m = C'_0 \cap A'_m$. Since $A'_m \cap C'_{m-1} = \emptyset$, we get $A'_m \cap C_{n+m-1} = \emptyset$. Moreover, it is true that $B'_m \subseteq C'_{m-1} \subseteq C_{n+m-1}$. Check that $CF(C_{n+m-1} \cup A'_m)$. Take arbitrary events $x, y \in C_{n+m-1} \cup A'_m$. Suppose a contrary, i.e. $x \sharp y$. Notice that $C_n = C_n \setminus E^n \cup C_n \cap E^n$. As $CF(C_{n+m-1})$ and $CF(C'_{m-1} \cup A'_m)$, we can assume that $x \in C_n \setminus E^n$ and $y \in A'_m \subseteq E^n$. Since $x \in E_0$ and $x \notin E^n$, there is $1 \le j \le n$ such that $x \in \widetilde{A}_j$ or $x \in \sharp^{j-1}(\widetilde{A}_j)$, due to Definition 5.
  - $x \in \widetilde{A}_j$. This implies that $x \in E^{j-1}$, by Lemma 3(v), if $x \in A_j \setminus F^{j-1}$, and by Definition 5, otherwise. Moreover, by virtue of Lemma 3(i), we obtain that $y \in E^{j'}$ for all $0 \le j' \le n$, because $y \in E^n$. Thanks to Definition 5, we have $x \sharp^{j-1} y$, i.e. $y \in \sharp^{j-1}(\widetilde{A}_j)$, and, hence, we get the contradiction $y \notin E^j$.
  - $x \in \sharp^{j-1}(\widetilde{A}_j)$. This means that there is $x' \in \widetilde{A}_j$ such that $x \sharp^{j-1} x'$. If $x' \in A_j \setminus F^{j-1}$, then, by Lemma 3(vii), we get $x' \in C_n$, contradicting $x \in C_n$. If $x' \in \lfloor (A_j \setminus F^{j-1}) \rfloor \cap F^{j-1}$, then there is $x'' \in A_j \setminus F^{j-1}$ such that $x' <^{j-1} x''$. Hence, we get $x \sharp x' < x''$, due to Lemma 3(i), and $x'' \in C_n$, due to Lemma 3(vii). Thanks to $\mathcal{E}$ being a causal RPES, $x' \ll x''$ is true. As $\sharp$ is hereditary w.r.t. $\ll$, we have $x \sharp x''$, contradicting $x, x'' \in C_n$.
- Take arbitrary $e \in A'_m \subseteq E'$ and $e' \in E$ such that $e' < e$. We have to show that $e' \in C_{n+m-1} \setminus B'_m$. Consider two possible cases.
  - $e' \in E'$. Since $e \in E'$ and $e' < e$, we have that $e' <' e$ in $\mathcal{E}'$, by Definition 5. Hence, $e' \in C'_{m-1} \setminus B'_m$, as $A'_m \cup \underline{B'_m}$ is enabled at C'_{m-1} in $\mathcal{E}'$. So, we may conclude that $e' \in C_{n+m-1} \setminus B'_m$, because $C'_{m-1} \subseteq C_{n+m-1}$.
  - $e' \notin E'$. By Definition 5, there is $1 \le i \le n$ such that $e' \in \widetilde{A}_i$ or $e' \in \sharp^{i-1}(\widetilde{A}_i)$. Check the validity of $e' \in \widetilde{A}_i$. Suppose a contrary, i.e. $e' \in \sharp^{i-1}(\widetilde{A}_i)$. Then, we can find an event $a \in \widetilde{A}_i$ such that $e' \sharp^{i-1} a$. This means that $a, e' \in E^{i-1}$, by Definition 5, and $e' \sharp a$, by Lemma 3(i). Due to $\mathcal{E}$ being a causal RPES, it holds that $e' \ll e$, because $e' < e$. As $\sharp$ is hereditary w.r.t. $\ll$, we have $a \sharp e$. Thanks to Lemma 3(i), $e \in E^{i'}$ is true for all $0 \le i' \le n$, because $e \in E'$. By Definition 5, we obtain $a \sharp^{i-1} e$, i.e. $e \in \sharp^{i-1}(\widetilde{A}_i)$, and, hence, we get the contradiction $e \notin E^i$.

    So, we obtain that $e' \in \widetilde{A}_i$ for some $1 \le i \le n$. Two cases are admissible.
    * $e' \in A_i \setminus F^{i-1}$. As $tt'_{m-1} \in Trace(\mathcal{E})$ and $1 \le i \le n+m-1$, we have $e' \in C_{n+m-1}$, due to Lemma 3(vii).
    * $e' \in F^{i-1}$ and $\exists a \in A_i \setminus F^{i-1}$: $e' <^{i-1} a$. As $tt'_{m-1} \in Trace(\mathcal{E})$ and $1 \le i \le n+m-1$, we get $a \in C_{n+m-1}$, due to Lemma 3(vii). Hence, we have $e' \in C_{n+m-1}$, thanks to Lemma 2(i).

    Due to $A'_m \cup \underline{B'_m}$ being enabled at C'_{m-1} in $\mathcal{E}'$, i.e. $B'_m \subseteq C'_{m-1} \subseteq E'$, it is true that $e' \in C_{n+m-1} \setminus B'_m$, because $e' \notin E'$.
- Take arbitrary $e \in B'_m \subseteq E'$ and $e' \in E$ such that $e' \prec \underline{e}$. We have to show that $e' \in C_{n+m-1} \setminus (B'_m \setminus \{e\})$. As $e' \prec \underline{e}$, $e' = e$ is true, due to $\mathcal{E}$ being a causal RPES. Then, we get $e' \in E'$. Moreover, we have $e = e'$ in $\mathcal{E}'$, by Definition 5. Hence, $e' \prec' \underline{e}$ in $\mathcal{E}'$, thanks to $\mathcal{E}'$ being a causal RPES, according to Lemma 3(ii). Since $e \in B'_m$, it holds that $e' \in C'_{m-1} \setminus (B'_m \setminus \{e\})$ because $A'_m \cup \underline{B'_m}$ is enabled at C'_{m-1} in $\mathcal{E}'$. Due to $C'_{m-1} \subseteq C_{n+m-1}$, we have $e' \in C_{n+m-1} \setminus (B'_m \setminus \{e\})$.
- Take arbitrary $e \in B'_m \subseteq F' \subseteq E'$ and $e' \in E$ such that $e' \rhd \underline{e}$. We shall show that $e' \notin C_{n+m-1} \cup A'_m$. Consider two possible cases.
  - $e' \in E'$. Then, we have $e, e' \in E^{i'}$ for all $0 \le i' \le n$, by Lemma 3(i). Since $e' \rhd \underline{e}$, we get $e' \rhd' \underline{e}$, due to Definition 5. Hence, $e' \notin C'_{m-1} \cup A'_m$, because

$A'_m \cup \underline{B'_m}$ is enabled at C'_{m-1}. As $e' \in E'$ and $C'_0 = C_n \cap E'$, we may conclude that $e' \notin C_{n+m-1} \cup A'_m$.

– $e' \notin E'$. By Definition 5, there is $1 \le i \le n$ such that $e' \in \widetilde{A}_i$ or $e' \in \sharp^{i-1}(\widetilde{A}_i)$. Check that $e' \in \sharp^{i-1}(\widetilde{A}_i)$. Suppose a contrary, i.e. $e' \in \widetilde{A}_i$. Two cases are admissible.

  * $e' \in A_i \backslash F^{i-1}$. Then, we have that $e' \in E^{i-1}$, by Lemma 3(v), and $e \in F^{i-1}$, by Lemma 3(i). Hence, we get that $e \in F^{i'}$ and $e' \in E^{i'}$ for all $0 \le i' \le i-1$, by virtue of Lemma 3(i). Due to Definition 5, it holds that $e' \rhd^{i-1} \underline{e}$ because $e' \rhd \underline{e}$. This implies that $e <^{i-1} e'$ is true, as $\mathcal{E}'$ is a causal RPES according to Lemma 3(ii). Therefore, by Definition 5, we obtain $e \in \widetilde{A}_i$, contradicting $e \in E'$.
  * $e' \in F^{i-1}$ and $\exists a \in A_i \setminus F^{i-1}$: $e' <^{i-1} a$. By Lemma 3(i), we have $e' < a$. Due to $\mathcal{E}$ being a causal RPES, it holds that $e < e'$ because $e' \rhd \underline{e}$, and, moreover, $e < a$ because $e \ll e' \ll a$ and $\ll$ is a transitive relation. Moreover, $a \in E^{i-1}$, by Lemma 3(v), and $e \in F^{i-1} \subseteq E^{i-1}$, by Lemma 3(i). Hence, we get that $e \in F^{i'}$ and $a \in E^{i'}$ for all $0 \le i' \le i-1$, by virtue of Lemma 3(i). Then, thanks to Definition 5, we obtain $e <^{i-1} a$. Therefore, it is true that $e \in \widetilde{A}_i$, contradicting $e \in E'$.

  Therefore, $e' \in \sharp^{i-1}(\widetilde{A}_i)$ holds. Then, there is an event $a \in \widetilde{A}_i$ such that $e' \sharp^{i-1} a$. By Lemma 3(i), we get $e' \sharp a$. Check two possible cases.

  * $a \in A_i \setminus F^{i-1}$. As $tt'_{m-1} \in Trace(\mathcal{E})$ and $1 \le i \le n+m-1$, we obtain $a \in C_{n+m-1}$, due to Lemma 3(vii). Due to $CF(C_{n+m-1})$, $e' \notin C_{n+m-1}$ holds.
  * $a \in F^{i-1}$ and $\exists a' \in A_i \setminus F^{i-1}$: $a <^{i-1} a'$. As $tt'_{m-1} \in Trace(\mathcal{E})$ and $1 \le i \le n+m-1$, we have $a' \in C_{n+m-1}$, due to Lemma 3(vii). By Lemma 3(i), $a < a'$. Moreover, $a \ll a'$ holds, due to $\mathcal{E}$ being a causal RPES. As $\sharp$ is hereditary w.r.t $\ll$, we have $e' \sharp a'$ because $e' \sharp a$. Due to $CF(C_{n+m-1})$, we get $e' \notin C_{n+m-1}$.

  Moreover, $e' \notin A'_m \subseteq E'$ is true because $e' \notin E'$. Therefore, $e' \notin C_{n+m-1} \cup A'_m$.

Thus, $(A'_m \cup \underline{B'_m})$ is enabled at C_{n+m-1} in $\mathcal{E}$. This means that $tt' \in Trace(\mathcal{E})$ is true.

Using Definition 5, it is easy to see that $\mathcal{E} \setminus [tt'] = (\mathcal{E} \setminus [t]) \setminus [t'] = \mathcal{E}' \setminus [t']$. □

Proof of Proposition 4.

Let $t't'' \in Trace(\mathcal{E})$, where $t' = (A_1 \cup \underline{B_1}) \ldots (A_k \cup \underline{B_k})$ and $t'' = (A_{k+1} \cup \underline{B_{k+1}}) \ldots (A_n \cup \underline{B_n})$. Then, there exists a sequence $C_0 \xrightarrow{(A_1 \cup \underline{B_1})} C_1 \ \ldots\ C_{k-1} \xrightarrow{(A_k \cup \underline{B_k})} C_k \xrightarrow{(A_{k+1} \cup \underline{B_{k+1}})} C_{k+1} \ \ldots\ C_{n-1} \xrightarrow{(A_n \cup \underline{B_n})} C_n$ $(n \ge k \ge 0)$ in $\mathcal{E}$. Verify $t'' \in Trace(\mathcal{E}' = \mathcal{E} \setminus [t'])$.

If $k = 0$ then we have $\mathcal{E}' = \mathcal{E} \setminus \epsilon = \mathcal{E}$, due to Definition 5. Hence, $t'' = Trace(\mathcal{E}')$.

Consider the case when $k > 0$. Suppose that $t'' = (A_{k+1} \cup \underline{B_{k+1}}) \ldots (A_n \cup \underline{B_n}) = (A'_1 \cup \underline{B'_1}) \ldots (A'_m \cup \underline{B'_m})$, where $m = n - k \ge 0$. We shall proceed by inductions on m.

$m = 0$. Clearly, $t'' = \epsilon \in Trace(\mathcal{E}')$.

$m > 0$. By the induction hypothesis, we have $t''_{m-1} \in Trace(\mathcal{E}')$. Then, there exists a sequence $C'_0 \xrightarrow{A'_1 \cup \underline{B'_1}} C'_1 \ \ldots\ C'_{m-2} \xrightarrow{A'_{m-1} \cup \underline{B'_{m-1}}} C'_{m-1}$ in $\mathcal{E}'$. Here, $C'_{m-1} = (\ldots(C'_0 \setminus B'_1 \cup A'_1) \setminus \ldots) \setminus B'_{m-1} \cup A'_{m-1}$, where $C'_0 = C_k \cap E^k$, by Lemma 3(viii). Moreover, we know that $C_{k+m-1} = (\ldots(C_k = [(\ldots(C_0 \setminus B_1 \cup A_1) \setminus \ldots) \setminus B_k \cup A_k] \setminus B'_1 \cup A'_1) \setminus \ldots) \setminus B'_{m-1} \cup A'_{m-1}$. So, $C'_{m-1} \subseteq C_{k+m-1}$ is true.

We first show that $(A'_m \cup \underline{B'_m})$ is enabled at C'_{m-1} in $\mathcal{E}'$.

- As $(A'_m = A_{k+m} \cup \underline{B'_m = B_{k+m}})$ is enabled at C_{k+m-1} in $\mathcal{E}$, we have that $A'_m \subseteq E$ and $B'_m \subseteq F$ such that $A'_m \cap C_{k+m-1} = \emptyset$, $B'_m \subseteq C_{k+m-1}$, and $CF(C_{k+m-1} \cup A'_m)$. First, check $A'_m \subseteq E'$. Suppose a contrary, i.e. there is $x \in A'_m$ such that $x \notin E'$. Since $A'_m \subseteq E$, we can find $1 \le j \le k$ such that $x \in \widetilde{A}^j$ or $x \in \sharp^{j-1}\widetilde{A}^j$, by Definition 5.
  Suppose $x \in \widetilde{A}^j$. Two cases are admissible.

– $x \in A^j \setminus F^{j-1}$. Since $t't''_{m-1} \in Trace(\mathcal{E})$ and $1 \le j \le k+m-1$, by Lemma 3(vii), we get $x \in C_{k+m-1}$, contradicting $A'_m \cap C_{k+m-1} = \emptyset$.
– $x \in F^{j-1}$ and $\exists a \in A^j \setminus F^{j-1}$: $x <^{j-1} a$. Thanks to Lemma 3(i), $x < a$ is true. As $t't''_{m-1} \in Trace(\mathcal{E})$ and $1 \le j \le k+m-1$, we have $a \in C_{k+m-1}$, by Lemma 3(vii). According to Lemma 2(i), we have $x \in C_{k+m-1}$, contradicting $A'_m \cap C_{k+m-1} = \emptyset$.

Assume $x \in \sharp^{j-1}(\widetilde{A}^j)$. Then, there is an event $a \in \widetilde{A}^j$ such that $x \sharp^{j-1} a$. By Lemma 3(i), we get $x \sharp a$. Check two possible cases.
– $a \in A^j \setminus F^{j-1}$. Since $t't''_{m-1} \in Trace(\mathcal{E})$ and $1 \le j \le k+m-1$, by Lemma 3(vii) we have $a \in C_{k+m-1}$, contradicting $CF(C_{k+m-1} \cup A'_m)$.
– $a \in F^{j-1}$ and $\exists b \in A^j \setminus F^{j-1}$: $a <^{j-1} b$. Thanks to Lemma 3(i), $a < b$ is true. As $t't''_{m-1} \in Trace(\mathcal{E})$ and $1 \le j \le k+m-1$, we get $b \in C_{k+m-1}$, due to Lemma 3(vii). According to Lemma 2(i), we have $a \in C_{k+m-1}$, contradicting $CF(C_{k+m-1} \cup A'_m)$.

So, $A'_m \subseteq E'$ holds.

Second, verify the truth of $B'_m \subseteq F'$. Suppose a contrary, i.e. there is $y \in B'_m$ and $y \notin F'$. Since $B'_m \subseteq F$, we can find $1 \le j \le k$ such that $y \notin E^j$, i.e., $y \in \widetilde{A}^j$ or $y \in \sharp^{j-1}\widetilde{A}^j$, by Definition 5.

Suppose $y \in \widetilde{A}^j$. Two cases are admissible.
– $y \in A^j \setminus F^{j-1}$. Then, by Lemma 3(vi), we get $y \notin F$, contradicting $B'_m \subseteq F$.
– $y \in F^{j-1}$ and $\exists a \in A^j \setminus F^{j-1}$: $y <^{j-1} a$. Then, we obtain that $y < a$, by Lemma 3(i), $a \in C_j$, by Lemma 3(iii), and $a \notin F$, by Lemma 3(vi). Due to $\mathcal{E}$ being a causal RPES, $a \rhd \underline{y}$ is true. As $(A_{k+m} \cup \underline{B_{k+m}})$ is enabled at C_{k+m-1}, we obtain $a \notin C_{k+m-1} \cup A_{k+m}$. Hence, we get $j < k+m-1$. Since $a \in C_j$, it is true that $a \in B_p$, for some $j < p \le k+m-1$, contradicting $a \notin F$.

Assume $y \in \sharp^{j-1}(\widetilde{A}^j)$. Then, there is an event $a \in \widetilde{A}^j$ such that $y \sharp^{j-1} a$. By Lemma 3(i), we get $y \sharp a$. Check two possible cases.
– $a \in A_j \setminus F^{j-1}$. As $t't''_{m-1} \in Trace(\mathcal{E})$ and $1 \le j \le k$, we get $a \in C_{k+m-1}$, by Lemma 3(vii). Due to $CF(C_{k+m-1})$, it holds that $y \notin C_{k+m-1}$, contradicting $B'_m \subseteq C_{k+m-1}$.
– $a \in F^{j-1}$ and $\exists a' \in A_j \setminus F^{j-1}$: $a <^{j-1} a'$. Then, we have $a < a'$, by Lemma 3(i). As $t't''_{m-1} \in Trace(\mathcal{E})$ and $1 \le j \le k$, we get $a' \in C_{k+m-1}$, due to Lemma 3(vii). According to Lemma 2(i), $a \in C_{k+m-1}$ is true. Due to $CF(C_{k+m-1})$, we get $y \notin C_{k+m-1}$, contradicting $B'_m \subseteq C_{k+m-1}$.

Therefore, $B'_m \subseteq F'$.

Third, we obtain $A'_m \cap C'_{m-1} = \emptyset$ because $A'_m \cap C_{k+m-1} = \emptyset$ and $C'_{m-1} \subseteq C_{k+m-1}$.

Fourth, check the truth of $B'_m \subseteq C'_{m-1}$. Suppose a contrary, i.e. there is $b \in B'_m \subseteq F' \subseteq E'$ such that $b \notin C'_{m-1}$. As $(A'_m \cup \underline{B'_m})$ is enabled at C_{k+m-1}, $b \in C_{k+m-1}$ is true. Notice that $C_k = (C_k \setminus E') \cup (C_k \cap E')$. Then, we get the following: $b \in C_k$ because $b \in C_{k+m-1}$, and $b \notin C_k \cap E'$ because $b \notin C'_{m-1}$. Hence, $b \in C_k \setminus E'$, contradicting $b \in E'$. So, $B'_m \subseteq C'_{m-1}$ holds.

Finally, due to $CF(C_{k+m-1} \cup A'_m)$ and $C'_{m-1} \subseteq C_{k+m-1}$, $CF(C'_{m-1} \cup A'_m)$ is true.

- Take arbitrary $e \in A'_m$ and $e' \in E'$ such that $e' <' e$. Then, we get $e' < e$, by Lemma 3(i). We have to show that $e' \in C'_{m-1} \setminus B'_m$. Assume a contrary, i.e. $e' \notin C'_{m-1} \setminus B'_m$. Thanks to $(A'_m \cup \underline{B'_m})$ being enabled at C_{k+m-1}, we get $e' \in C_{k+m-1} \setminus B'_m$. Notice that $C_k = (C_k \setminus E') \cup (C_k \cap E')$. Then, we obtain the following: $e' \in C_k$ because $e' \in C_{k+m-1} \setminus B'_m$, and $e' \notin C_k \cap E'$ because $e' \notin C'_{m-1} \setminus B'_m$. Hence, $e' \in C_k \setminus E'$, contradicting $e' \in E'$. So, $e' \in C'_{m-1} \setminus B'_m$.
- Take arbitrary $e \in B'_m$ and $e' \in E'$ such that $e' \prec' \underline{e}$. Then, we get $e' \prec e$, by Lemma 3(i). We have to show that $e' \in C'_{m-1} \setminus (B'_m \setminus \{e\})$. Assume a contrary, i.e. $e' \notin C'_{m-1} \setminus (B'_m \setminus \{e\})$. Since $(A'_m \cup \underline{B'_m})$ is enabled at C_{k+m-1}, we get $e' \in C_{k+m-1} \setminus (B'_m \setminus \{e\})$. Then, we obtain the following: $e' \in C_k$ because $e' \in C_{k+m-1} \setminus (B'_m \setminus \{e\})$, and $e' \notin C_k \cap E'$ because $e' \notin C'_{m-1} \setminus (B'_m \setminus \{e\})$. Hence, $e' \in C_k \setminus E'$, contradicting $e' \in E'$. So, $e' \in C'_{m-1} \setminus (B'_m \setminus \{e\})$.

- Take arbitrary $e \in B'_m$ and $e' \in E'$ such that $e' \rhd' \underline{e}$. Then, we get $e' \rhd e$, by Lemma 3(i). We shall show that $e' \notin C'_{m-1} \cup A'_m$. Since $(A'_m \cup \underline{B'_m})$ is enabled at C_{k+m-1}, we obtain that $e' \notin C_{k+m-1} \cup A_{k+m}$. As $C'_{m-1} \subseteq C_{k+m-1}$, we get $e' \notin C'_{m-1} \cup A'_m$.

Thus, $(A'_m \cup \underline{B'_m})$ is enabled at C'_{m-1} in $\mathcal{E}'$. This means that $t'' \in \mathit{Trace}(\mathcal{E}')$ holds. $\square$

Proof of Proposition 5.

(i) Take an arbitrary $\mathit{last}(t = (A_1 \cup \underline{B_1}) \dots (A_n \cup \underline{B_n})) \in \mathit{Conf}(\mathcal{E})$ $(n \geq 0)$. Let $t_0 = \epsilon$, and $t_i = (A_1 \cup \underline{B_1}) \dots (A_i \cup \underline{B_i})$ for all $1 \leq i \leq n$. Clearly, $t_i \in \mathit{Trace}(\mathcal{E})$ for all $0 \leq i \leq n$. According to Lemma 3(ii) and Proposition 2, $\mathcal{E} \setminus [t_i]$ is a causal RPES for all $0 \leq i \leq n$. Due to Proposition 4, we have that $(A_{i+1} \cup \underline{B_{i+1}}) \in \mathit{Trace}(\mathcal{E} \setminus [t_i])$ for all $0 \leq i < n$. Thanks to Definition 5, it is easy to see that $\mathcal{E} = \mathcal{E} \setminus [t_0]$, and $(\mathcal{E} \setminus [t_i]) \setminus [(A_{i+1} \cup \underline{B_{i+1}})] = \mathcal{E} \setminus [t_{i+1}]$ for all $0 \leq i < n$. Then, we can write $\mathcal{E} \setminus [t_i] \overset{M = l(A_{i+1} \cup \underline{B_{i+1}})}{\rightharpoonup} \mathcal{E}[t_{i+1}]$ for all $0 \leq i < n$. Therefore, we obtain $\mathcal{E} \setminus [t_n = t] \in \mathit{Reach}(\mathcal{E})$, by Definition 7.

(ii) Take an arbitrary $\mathcal{E}' \in \mathit{Reach}(\mathcal{E})$. Due to Definition 7, there exist $\mathcal{E}_0, \dots, \mathcal{E}_n$ $(n \geq 0)$ such that $\mathcal{E}_0 = \mathcal{E} \setminus [\epsilon]$, $\mathcal{E}_n = \mathcal{E}'$, and $\mathcal{E}_i \overset{l(A_{i+1} \cup \underline{B_{i+1}})}{\rightharpoonup} \mathcal{E}_{i+1}$ for all $0 \leq i < n$. Then, by the definition of the relation $\rightharpoonup$, we obtain $\mathcal{E}_{i+1} = \mathcal{E}_i \setminus [(A_{i+1} \cup \underline{B_{i+1}})]$, where $(A_{i+1} \cup \underline{B_{i+1}}) \in \mathit{Trace}(\mathcal{E}_i)$, for all $0 \leq i < n$. Let $t_i = (A_1 \cup \underline{B_1}) \dots (A_i \cup \underline{B_i})$ for all $0 \leq i \leq n$. We proceed by induction on $0 \leq i \leq n$ to show that $t_i \in \mathit{Trace}(\mathcal{E})$ and $\mathcal{E}_i = \mathcal{E} \setminus [t_i]$.

$i = 0$. Clearly, $t_0 = \epsilon \in \mathit{Trace}(\mathcal{E})$ and $\mathcal{E}_0 = \mathcal{E} \setminus [\epsilon]$.

$i > 0$. By the induction hypothesis, we have that $t_{i-1} \in \mathit{Trace}(\mathcal{E})$ and $\mathcal{E}_{i-1} = \mathcal{E} \setminus [t_{i-1}]$. Since $(A_i \cup \underline{B_i}) \in \mathit{Trace}(\mathcal{E}_{i-1})$, it follows from Proposition 3 that $t_{i-1}(A_i \cup \underline{B_i}) = t_i \in \mathit{Trace}(\mathcal{E})$ and $\mathcal{E} \setminus [t_i] = \mathcal{E}_{i-1} \setminus [(A_i \cup \underline{B_i})] = \mathcal{E}_i$.

Hence, we obtain $t_n \in \mathit{Trace}(\mathcal{E})$ and $\mathcal{E}' = \mathcal{E}_n = \mathcal{E} \setminus [t_n]$. Moreover, thanks to Lemma 1(i), $\mathit{last}(t_n) \in \mathit{Conf}(\mathcal{E})$ holds.

(iii) Take arbitrary $\mathit{last}(t), \mathit{last}(t') \in \mathit{Conf}(\mathcal{E})$ such that $\mathit{last}(t) \overset{l(A \cup \underline{B})}{\rightarrow} \mathit{last}(t')$. By virtue of item (i), it holds that $\mathcal{E} \setminus [t], \mathcal{E} \setminus [t'] \in \mathit{Reach}(\mathcal{E})$. Due to the definition of the relation $\rightarrow$, we have that $\mathit{last}(t) \overset{(A \cup \underline{B})}{\rightarrow} \mathit{last}(t')$ in $\mathcal{E}$. According to Lemma 1(iii), it holds that $t(A \cup \underline{B}) \in \mathit{Trace}(\mathcal{E})$ and $t' \sim t(A \cup \underline{B})$. Hence, $[t'] = [t(A \cup \underline{B})]$ is true. By Proposition 2, we get $\mathcal{E} \setminus [t'] = \mathcal{E} \setminus [t(A \cup \underline{B})]$. Moreover, due to Lemma 3(ii), we have that $\mathcal{E} \setminus [t]$ and $\mathcal{E} \setminus [t']$ are causal RPESs. By virtue of Proposition 4, we obtain $(A \cup \underline{B}) \in \mathit{Trace}(\mathcal{E} \setminus [t])$. Thanks to Proposition 3, we may conclude that $\mathcal{E} \setminus [t'] = (\mathcal{E} \setminus [t]) \setminus (A \cup \underline{B})$. Thus, $\mathcal{E} \setminus [t] \overset{l(A \cup \underline{B})}{\rightharpoonup} \mathcal{E} \setminus [t']$, by the definition of the relation $\rightharpoonup$.

(iv) Take arbitrary $\mathcal{E}', \mathcal{E}'' \in \mathit{Reach}(\mathcal{E})$ such that $\mathcal{E}' \overset{l(A \cup \underline{B})}{\rightharpoonup} \mathcal{E}''$. By vitue of item (ii), there is $\mathit{last}(\widetilde{t}) \in \mathit{Conf}(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus [\widetilde{t}]$. Take an arbitrary $\mathit{last}(t) \in \mathit{Conf}(\mathcal{E})$ such that $\mathcal{E}' = \mathcal{E} \setminus [t]$. Then, $t \in \mathit{Trace}(\mathcal{E})$ holds, by Lemma 1(i). Thanks to the definition of the relation $\rightharpoonup$, we have that $\mathcal{E}'' = \mathcal{E}' \setminus [(A \cup \underline{B})]$, where $(A \cup \underline{B}) \in \mathit{Trace}(\mathcal{E}')$. According to Proposition 3, it is true that $t' = t(A \cup \underline{B}) \in \mathit{Trace}(\mathcal{E})$ and $\mathcal{E} \setminus [t'] = (\mathcal{E} \setminus [t]) \setminus [(A \cup \underline{B})] = \mathcal{E}''$. Then, we obtain that $\mathit{last}(t') \in \mathit{Conf}(\mathcal{E})$, by Lemma 1(i), and, moreover, $\mathit{last}(t) \overset{(A \cup \underline{B})}{\rightarrow} \mathit{last}(t')$, by Lemma 1(ii). So, it holds that $\mathit{last}(t) \overset{l(A \cup \underline{B})}{\rightarrow} \mathit{last}(t')$ in $\mathcal{E}$, due to the definition of the relation $\rightarrow$. $\square$