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OPTIMAL GYROSCOPIC STABILIZATION OF VIBRATIONAL SYSTEM: ALGEBRAIC APPROACH

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Abstract. The paper deals with LTI vibrational systems with positive definite stiffness matrix K and symmetric damping matrix D. Gyroscopic stabilization means the existence of gyroscopic forces with a skew-symmetric matrix G, such that a closed loop system with damping matrix $D+G$ is asymptotically stable. The feature of characteristic polynomial in the case predetermines such stabilization as a low order control design. Assuming the necessary condition of gyroscopic stabilization is fulfilled, we pose the problem of achieving relative stability maximum using a stabilizer G. The stability maximum value is determined by a matrix D trace, but its reachability depends on the coincidence of all pole real parts with the corresponding minimal value, i.e. equality of characteristic and root polynomials. We illustrate a root polynomial technique application to optimal gyroscopic stabilizer design by examples of dimension 3–5.

Keywords: vibrational system, gyroscopic stabilizer, low order control, rightmost poles, relative stability, root polynomial.

1. Introduction

We consider one of the classes of LTI systems: multidimensional vibrational systems, which are described by a matrix differential equations of a form

$$X'' + DX' + KX = 0,$$

where $X(t)$ is a state variable vector, $D = D^T$ is indefinite damping matrix and $K = K^T$ is positive definite stiffness matrix.

The main problem is to achieve stability of such systems using gyroscopic forces. It lies in ODE theory area, which deals with the stability study of mechanical

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and other systems exposed to forces of various natures. A general classification (structure) of these forces was proposed in the works of Lord Kelvin (W. Thomson). Gyroscopic stabilization theorems for linear potential systems were proven by Thomson and Tait in 1879, and in the mid-twentieth century they were generalized to holonomic systems by N.G. Chetaev, [1]. Chetaev's studies are continued in a wide range of Russian school scientific directions from qualitative analysis in general theory to the consideration of complicated mechanical examples, see [2-4].

In outlining the problem, we will follow [5]. In general case vibrational systems are unstable; but if matrix D is also positive definite, then the system is asymptotically stable. The gyroscopic stabilization tends to stabilize a system without further dissipation or damping by adding gyroscopic forces GX' with some skew-symmetric matrix G. The reason is the 'negative damping' effect in models with vibrations which are induced by friction, [6]. This is similar to the effect on which Galileo Galilei have said in 1638 that «a glass of water may be made to emit a tone merely by the friction of the fingertip upon the rim of the glass»; then Benjamin Franklin in 1761 designed an 'armonica', where sound was produced by vibration of rotating glass bowls touched by moistened fingers of a performer, [7].

Definition 1. *Let us call the matrix $G = -G^T$ a gyroscopic stabilizer for a vibrational system, if the solution X of an equation*

$$X'' + (D + G)X' + KX = 0$$

is asymptotically stable. In this case this system is gyroscopically stabilizable.

A necessary condition for gyroscopic stabilization is well-known: both traces of matrices D and $K^{-1}D$ must be positive [8]; for short proof see [5]. In [8] the authors ask whether this condition is also sufficient. In the case $n = 2$ they proved it using Lyapunov matrix equation approach, and for $n > 2$ gyroscopic stabilizability was shown only under additional conditions. The sufficiency of this conditions in cases $n = 3, 4$ was proven in [5] using inverse eigenvector technique; the sufficiency hypothesis has been tested for dimensions $n > 4$ on a large number of examples. Prof. T. Damm in his lecture on gyroscopic stabilization has formulated several problems including a sufficiency proof for dimensions $n > 4$ (Lehrstuhl fuer Angewandte Mathematik, Universitaet Bayreuth, Wuerzburg, June 1, 2012).

2. Problem statement

The authors in [9–10] were finding maximal relative stability of LTI control systems through the appropriate value choice of a controller parameters vector $\bar{p}$. Characteristic polynomial $\chi(s)$ of a close loop system has coefficients depending on control parameters $\bar{p}$ – or, in our case, on elements of stabilizer matrix G. The difficulty of low order control design is that a number of control parameters or their entering into characteristic polynomial coefficients do not allow to achieve an arbitrary system pole location; therefore, one must search for an optimal one.

Hurwitz stability function $H(\bar{p})$ is expressed directly through m system poles $z_1, ..., z_m$ (i.e. roots of a characteristic equation):

$$H(\bar{p}) = H(G) = maxRe(z_1, ..., z_m).$$

It is negative to relative stability of a system.

Thus Hurwitz function should be minimized by choosing optimal vector $\bar{p}$ value. As a rule, extrema and subextrema of function $H(\bar{p})$ and similar objective functions

[10] are achieved in those cases when system poles are accumulated in the maximum possible number on a right boundary of their location domain. This number depends on a vector $\bar{p}$ dimension (i.e. a number of free matrix stabilizer G elements); all variants of a mutual poles location on their right boundary are represented by critical root diagrams [9, 11] and divisibility of a characteristic polynomial by the corresponding root one. Using root polynomial technique, the authors examined, whether each pole location is realized and for what vector $\bar{p}$ value it is achieved. Minimization problems were solved in [9, 10] for the controlled plant composed of three massive bodies with elastic connections; its equations coincide with the vibrational system ones for $n = 3$. Here we consider the maximum possible relative stability of a vibrational system with a gyroscopic stabilizer and effective maximum finding technique by similar means as those used in [9,10]. This can be done in general as the method description. Therefore, here we will show applications of the latter to some examples of dimensions 3–5.

3. Theoretical preface

Turning from differential equations of a vibrational system to its Laplace operator form, we obtain a polynomial matrix equation: $L(s)X = (Is^2+(D+G)s+K)X = 0$. All the matrix properties here are as indicated above, X is Laplace transform of a state vector $X(t)$; I is the identity matrix. Hence the characteristic equation is

$$\chi(s) = det(Is^2 + (D + G)s + K) = 0.$$

System poles are denoted as above $z_1, ..., z_{2n}$. The following statement represents an analog of Vieta theorem:

Proposition 1. *(a) $z_1 + ... + z_{2n} = -trD$; (b) $z_1 \cdot ... \cdot z_{2n} = |K|$.*

Proof. (a) Let's find the polynomial $\chi(s)$ coefficient χ_{2n-1} of the monomial $\chi_{2n-1} \cdot s^{2n-1}$, it is negative to the pole sum. From the skew-symmetry of the matrix G it follows that $trD = tr(D + G)$. We expand the determinant by the 1^{st} row: $detL(s) = l_{11}L_{11} + l_{12}L_{12} + ... + l_{1n}L_{1n}$ (here L_{1k} is an element l_{1k} adjunct) and see, that for $k > 1$ the degrees in the variable s of all terms $l_{1k}L_{1k}$ do not exceed $2n - 2$ For the first term we get by induction

$$l_{11}L_{11} = (s^2 + d_{11}s)L_{11} = (s^2 + d_{11}s) \cdot (s^{2n-2} + D_{11}s^{2n-3} + ...),$$

where D_{11} is reduced matrix D without 1^{st} row and 1^{st} column. So, $trD = -z_1 - ... - z_{2n}$. The statement (b) is obvious. □

Corollary 1. *For Hurwitz function of a vibrational system with gyroscopic stabilization it holds the inequality $H(G) \geq -trD/2n$. The value $trD/2n$ is hypothetical supremum of a system relative stability.*

Proposition 2. *The polynomial $\chi(s)$ coefficient χ_1 at the variable s does not include stabilizer matrix G elements.*

Proof. By an orthogonal transformation we reduce the matrix K to a diagonal form $SKS^T = diag(\kappa_1 ... \kappa_n) = K_1$, where equalities $\kappa_j = \alpha_j^2$ hold due to positive definiteness of matrix K. Since the determinant $\chi(s) = detL(s)$ remains the same, we get

$$\chi(s) = s^{2n} det(I + S(D + G)S^T s^{-1} + K_1 s^{-2}) = s^{2n} det(I + S(D_1 + G_1)S^T \zeta + K_1 \zeta^2),$$

where $\zeta = s^{-1}$ and matrix $G_1 = SGS^T = \left(g^1_{ij}\right)_{n\times n}$ is still skew-symmetric. Taking out α_k, $k = 1, ..., n$, as common factors of k^{th} row and k^{th} column, we note that the matrix $\left((g^1_{ij})/(\alpha_i \cdot \alpha_j)\right)_{n\times n}$ is still skew-symmetric. By Proposition 1 we conclude

$$\chi(s) = s^{2n}\kappa_1 \cdot ... \cdot \kappa_n \cdot det(I_2 + S(D_2 + G_2)S^T\zeta + I\zeta^2) =$$
$$= s^{2n} \cdot |K| \cdot (\zeta^{2n} + trD_2\zeta^{2n-1} + ...) = |K| + |K| \cdot trD_2 s + ...$$

□

Corollary 2. *Gyroscopic stabilization for any vibrational system dimension is a low order control design problem. This is clear from the fact that the two highest and two lowest characteristic polynomial coefficients do not include gyroscopic stabilizer G elements, and an arbitrarily given pole location of a closed loop system is not reachable.*

Proposition 3. *A characteristic polynomial $\chi(s)$ coefficients vector has zero derivatives with respect to gyroscopic stabilizer G elements at zero values of the latter.*

Proof. Since partial derivatives are to be found at the initial value $G_0 = (0)_{n\times n}$, we can consider without loss of generality the case when only two elements $g_{12} = -g_{21}$ of the matrix G are nonzero and the matrix D has a diagonal form. Let us clarify the characteristic polynomial dependence on stabilizer elements:

$$\chi(s) = det\left(Is^2 + \left(\begin{pmatrix} d_{1,1} & 0 & \cdots & 0 \\ 0 & d_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n,n}\end{pmatrix} + \begin{pmatrix} 0 & g_{1,2} & \cdots & 0 \\ -g_{1,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0\end{pmatrix}\right)s + \right.$$
$$\left. + \begin{pmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ k_{1,2} & k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & k_{n,n}\end{pmatrix}\right) =$$
$$= det\begin{pmatrix} 0 & g_{1,2}s & \cdots & 0 \\ k_{1,2} - g_{1,2}s & s^2 + d_{2,2}s + k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & s^2 + d_{n,n}s + k_{n,n}\end{pmatrix} +$$
$$+det\begin{pmatrix} s^2 + d_{1,1}s + k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ k_{1,2} - g_{1,2}s & s^2 + d_{2,2}s + k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & s^2 + d_{n,n}s + k_{n,n}\end{pmatrix} =$$
$$= det\begin{pmatrix} 0 & g_{1,2}s & \cdots & 0 \\ -g_{1,2}s & s^2 + d_{2,2}s + k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & k_{2,n} & \cdots & s^2 + d_{n,n}s + k_{n,n}\end{pmatrix} +$$
$$+det\begin{pmatrix} 0 & g_{1,2}s & \cdots & 0 \\ k_{1,2} & s^2 + d_{2,2}s + k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & s^2 + d_{n,n}s + k_{n,n}\end{pmatrix} +$$

$$+det\begin{pmatrix} 0 & k_{1,2} & \cdots & k_{1,n} \\ -g_{1,2}s & s^2+d_{2,2}s+k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & k_{2,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}+$$

$$+det\begin{pmatrix} s^2+d_{1,1}s+k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ k_{1,2} & s^2+d_{2,2}s+k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}=$$

$$=det\begin{pmatrix} 0 & g_{1,2}s & \cdots & 0 \\ -g_{1,2}s & s^2+d_{2,2}s+k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & k_{2,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}+$$

$$+det\begin{pmatrix} s^2+d_{1,1}s+k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ k_{1,2} & s^2+d_{2,2}s+k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}=$$

$$=-g_{12}^2s^2detX_{n-2}(s)+\chi^0(s),$$

where $\chi^0(s)=det(Is^2+Ds+K)$,

$$X_{n-2}(s)=\begin{pmatrix} s^2+d_{3,3}s+k_{3,3} & \cdots & k_{3,n} \\ \vdots & \ddots & \vdots \\ k_{3,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}, \text{ and}$$

$$det\begin{pmatrix} 0 & g_{1,2}s & \cdots & 0 \\ k_{1,2} & s^2+d_{2,2}s+k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,n} & k_{2,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}+$$

$$+det\begin{pmatrix} 0 & k_{1,2} & \cdots & k_{1,n} \\ -g_{1,2}s & s^2+d_{2,2}s+k_{2,2} & \cdots & k_{2,n} \\ 0 & \vdots & \ddots & \vdots \\ 0 & k_{2,n} & \cdots & s^2+d_{n,n}s+k_{n,n} \end{pmatrix}=0.$$

So $\frac{\partial}{\partial g_{ij}}\chi_k=0$ for all the coefficients $\chi_k=0$ of the polynomial $\chi(s)$ at the starting point $G_0=(0)$.

□

Corollary 3. *The point $G_0=(0)_{n\times n}$ is critical point of Hurwits stability function $H(G)$.*

Proof. Since coefficients of a characteristic polynomial $\chi(s)$ are even functions of each stabilizer element g_{ij} at zero values of other ones, a dependence $z_k(g_{ij})$ of each of the poles $z_1,...,z_{2n}$. also turns out to be even. So is Hurwitz stability function $H(G)=maxRe(z_1,...,z_m)$. Since a derivative of an even function at zero point is equal to zero too, then in both cases – rightmost pole dependence on stabilizer elements g_{ij} is differentiable , as well as nondifferentiable (the latter occurs at multiple roots) – the point $G_0=(0)$ is critical for Hurwits function. □

Remark. By corollary 3 numerical minimization of Hurwitz function should take as starting points some non-zero values of a stabilizer G. Many numerical examples have shown that the point $G_0 = (0)$ turns out to be a local minimum for a function $H(G)$. In other words, a gyroscopic stabilizer with small coefficients does not improve a relative stability value of an open loop vibrational system.

4. Numerical examples

Everywhere below we consider the problem of optimal gyroscopic stabilization in coordinates, where the symmetric matrix D takes a diagonal form.

***Example* 1.** The necessary condition holds for the following coefficient matrices of the vibrational system of dimension 3:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, K = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, K^{-1}D = \begin{pmatrix} 3/2 & -1/2 & -1/4 \\ -1 & 1 & 1/2 \\ 1/2 & -1/2 & -3/4 \end{pmatrix}.$$

It is unstable, because the system poles z_k set contains a complex pair $0.3552 \pm 1.2882i$. Obviously $z_1 + ... + z_6 = -2$ and $maxRe(z_1, ..., z_6) \geq -\frac{1}{3}$. There exists a one-dimensional gyroscopic stabilizer $G = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}$, which provides a suboptimal result with $b \approx 1.658$:

$$maxRe(z_1, ..., z_6) \approx Re(z_3, ..., z_6) \approx -0.3241 \approx -\frac{1}{3}.$$

By introducing into the stabilizer the second parameter $c = g_{23} = -g_{32}$, we can try to achieve a theoretical maximum of relative stability $H_{opt}(G) = maxRe(z_k) = -1/3$. Let us seek all the poles location on the complex plane vertical: $Rez = -1/3$, i.e. characteristic polynomial $\chi(s)$ must be equal to the *root polynomial* $r(s)$(see [9, 10]):

$$\chi(s) = s^6 + 2s^5 + (b^2 + c^2 + 5)s^4 + (b^2 + 2c^2 + 6)s^3 + (2b^2 - 2bc + 2c^2 + 8)s^2 + 7s + 4 =$$
$$= (s^2 + \frac{2}{3}s + y_1) \cdot (s^2 + \frac{2}{3}s + y_2) \cdot (s^2 + \frac{2}{3}s + y_3),$$

here $y_k = \frac{1}{9} + Im^2 z_{2k-1,\,2k}$ are *root coordinates* [9]. We equate coefficients and solve the polynomial equations with respect to $b, c, y_{1,2,3}$. From the system we get the poles $z_{1,2} \approx -\frac{1}{3} \pm 0.6488i, z_{3,4} \approx -\frac{1}{3} \pm 1.4537i, z_{5,6} \approx -\frac{1}{3} \pm 1.8080i$. The correction c is rather small: ≈ -0.092; the parameter b is almost of the same value: $b \approx 1.569$.

A decrease in the matrix D trace allows achieving the maximum stability according to prop. 1 by using one more stabilizer coordinate $a = g_{12} = -g_{21}$. E.g. let vibrational system matrices be the following:

$$D = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, G = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Here the polynomial equation $\chi(s) = r(s)$ leads to 5 equations for coefficients at degrees $s^0 - s^4$, because the coefficients at s^5 and s^6 are equal a priori. Therefore, one can choose some free variable and assign different values to it, finding five other unknowns and obtaining one-dimensional manifold, at each point of which the system relative stability is maximal(see [12]). For example, we can set one parameter $c = -0.2833$ and find the rest ones: $y_1 \approx 0.11565, y_2 \approx 0.9292, y_3 \approx 37.2233,$

$a \approx -3.9337, b \approx -4.2092$. All the poles are located on one complex vertical $Rez = -trD/6 = -1/60$: $Imz_{1,2} \approx \pm 0.3397i, Imz_{3,4} \approx \pm 0.9638i, Imz_{5,6} \approx \pm 6.1011i$.

***Example* 2.** Consider a four-dimensional vibrational system; as above, coordinates are assumed to be those in which the matrix D is diagonal:

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, K = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

It is unstable, because its eight poles contain the rightmost pair $z_{1,2} \approx 0.4262 \pm 0.8087i$.

The necessary condition holds: $trD = trK^{-1}D = 1$. Gyroscopic stabilizer in general case is represented by the matrix $G = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$.

So the characteristic polynomial has the form:

$$\chi(s) = s^8 + s^7 + (5 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2)s^6 + (5 - 2a^2 + d^2 + e^2 + 3f^2)s^5 + ...$$
$$+[6 + a(3a - 2b - 2c + 4d - 2e + 2f) + b(3b - 2c - 4d + 2e - 2f)+$$
$$+c(3c - 2e - 2f) + 4d(d - e + f) + e(3e - 2f) + 3f^2]s^2 + 4s + 4.$$

By analogy with example 1, a three-parametric stabilizer $G = \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & d & 0 \\ -b & -d & 0 & 0 \\ -c & 0 & 0 & 0 \end{pmatrix}$ provides the requirement $Re(z) = -trD/8$ for four complex pole pairs location. It allow us to reach the theoretical maximum of relative stability $H_{opt}(G) = maxRe(z_1, ..., z_8) = -\frac{1}{8}$ at the values $b \approx -0.3901, c \approx -1.5436$ and $d \approx -1.3153$. The system pole pairs have imaginary components

$$Im(z_k) \approx \pm 0.7051, \pm 0.8684, \pm 1.3062, \pm 2.4228.$$

The number of variable stabilizer parameters allows us to set a denser root location. E.g. let us set the damping matrix $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$, which satisfies the necessary conditions, and the requirement of triple complex pole pairs location on the rightmost vertical $Re(z) = -1/8$, i.e. the root polynomial is $r(s) = (s^2 + 0.25s + 0.015625 + y^2)^3$. By prop.1 the fourth complex pair must be located on the same vertical too. Equating to zero the element a and the remainder of division $rem(\chi(s), r(s))$, we obtain six polynomial equations with respect to the rest stabilizer elements and triple pole imaginary part y. Their solution gives the values

$$b \approx 0.3836, c \approx -0.0877, d \approx 1.2909, e \approx -0.4512, f \approx -2.0864, y \approx 0.8698.$$

Hence 'almost triple' pole pair $z_{1,...,6} \approx -0.125 \pm 0.87i$ is located approximately on the same vertical with the forth one $z_{7,8} \approx -0.125 \pm 2.9446i$.

Reducing the value trD and shifting closer to the necessary condition boundary do not affect the calculation method. Let us set $D = diag(1; 1; 1; -2.8)$, then we'll get rightmost vertical $Re(z) = -0.025$, and the solution of the equation system turns out to be $b \approx 1.4545, c \approx 2.4437, d \approx 0.498, e \approx -0.2076, f \approx 0.1972,$ $y \approx 0.8784$. Almost triple pole pair $z_{1,\dots,6} \approx -0.025 \pm 0.78i$ is located on the same vertical as the forth pair $z_{7,8} \approx -0.025 \pm 2.9471i$.

***Example* 3.** Let five-dimensional vibrational system be defined by the following matrices D and K:

$$D = \begin{pmatrix} 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, K = \begin{pmatrix} 2 & 1 & 0.5 & 0 & 0 \\ 1 & 2 & 1 & 0.5 & 0 \\ 0.5 & 1 & 2 & 1 & 0.5 \\ 0 & 0.5 & 1 & 2 & 1 \\ 0 & 0 & 0.5 & 1 & 2 \end{pmatrix}.$$

So the necessary stabilizability conditions are satisfied: $trD = 1, trK^{-1}D \approx 1.0707$.

As in example 2, we require four complex pole pairs to be located on the vertical $Re(z) = -trD/10 = -0.1$ (the fifth pair must be located approximately there too), and we assume one of them to be double:

$$r(s) = (s^2 + 0.2s + y_1)^2(s^2 + 0.2s + y_2)(s^2 + 0.2s + y_3), \quad y_k = 0.01 + Im^2 z_{2k-1,2k}.$$

Equating to zero the remainder of division $rem(\chi(s), r(s))$ will lead to eight polynomial equations, which will be enough for unknowns $y_{1,2,3}$ and five variable stabilizer parameters:

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & k & l \\ 0 & 0 & -k & 0 & m \\ -d & -g & -l & -m & 0 \end{pmatrix}.$$

The solution yields $d \approx -0.9817, g \approx 0.8544, k \approx 1.4066, l \approx 0.6231, m \approx 1.2527$, and the poles are $z_{1-4} \approx -0.1000 \pm 1.5335i$ (double), $z_{5,6} \approx -0.1000 \pm 0.5633i$, $z_{7,8} \approx -0.1000 \pm 0.8699i$; the fifth pair is $z_{9,10} \approx -0.1000 \pm 2.6034i$.

5. Conclusion

All aforesaid allows one to appreciate how diverse and meaningful issues raise by the problem of gyroscopic stabilization of vibrational systems. Studies in this field, apparently, can give new ideas and approaches to the development of control system design in general. From the point of view of control theory, the gyroscopic stabilization problem gives an important example of a low order design, regardless of a control parameter dimension.

From the point of view of the topic itself, along with the problem of gyroscopic stabilizability, one can state the problem of an optimal gyroscopic controller design.

A few technical details are noteworthy. Since all solutions turn out to be vibrational (in other words, all poles form complex pairs), we may not include one of the brackets in the root polynomial — the poles included in it will necessarily fall on the same vertical; this slightly reduces the order of the computational task. In example 2 one could notice that the vibrational frequencies (or poles imaginary parts) depend weakly on decreasing matrix D trace; a similar effect also appears in the five-dimensional system.

Along with possibilities of an algebraic approach (including using of root polynomials) we have also seen some impossibilities, first of all, difficulties in numerical solving of polynomial equation systems. Not for all potentially possible mutual pole locations we managed to find the corresponding matrix G; this seems not so much to the inability to reach the required pole location, but rather to the lack of reliable solving methods for polynomial equation systems. Since the last task is 'static' (without free parameters), we hope for its successful solution as a software method. Due to this, the method described in the paper can be widely applicable to low-order controllers design for vibration systems of various dimensions and in different conditions.

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